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Combinatorics of regular A_2-crystals ☆

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Abstract

We show that a connected regular A_2-crystal (the crystal graph of a highest weight integrable module over $U_q(sl_3)$) can be produced from two half-grids by replicating them and gluing together in a certain way. Also some extensions and related aspects are discussed.

Keywords: Simply-laced algebra; Crystals of representations

1. Introduction

The notion of crystal graphs, or *crystals*, introduced by Kashiwara [7,8] embraces a certain class of edge-colored digraphs (directed graphs) in which each monochromatic subgraph consists of pairwise disjoint paths. An important subclass in it is formed by crystals related to representations of symmetrizable quantum Kac–Moody algebras, so-called *regular* crystals.

Stembridge [12] studied the problem of characterizing regular crystals in "local terms." (A characterization involving global properties is given, e.g., by Littelmann [10,11].) He pointed out a list of graph-theoretic axioms for the crystals, with finite monochromatic paths, that are related to representations of quantum simply-laced algebras (i.e., with a Cartan matrix whose off-diagonal entries are 0 or –1), called regular *simply-laced crystals*. Each of these axioms concerns

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a 2-colored subgraph of the digraph. In particular, an n-colored digraph is a regular simply-laced crystal if and only if each maximal 2-colored subgraph in it is such. (For a general result of this sort on crystals having a unique maximal vertex, see [6].) This shows an importance of a proper study of 2-colored crystals.

This paper is the first in our series of works devoted to a combinatorial study of crystals of representations, and to related topics. Here we consider regular 2-colored simply-laced crystals for the Cartan matrix $A_2 = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. For brevity, when such a crystal is connected and all monochromatic paths in it are finite, we refer to it as an *RA*2*-graph* (abbreviating "regular crystal graph of type A_2"). Adapting the characterization in [12] to the 2-colored case in question, we define an RA2-graph by imposing four simple axioms (A1)–(A4). The central role play axioms (A3) and (A4) (combinatorial analogs of Serre relations) which concern interrelations of the two partial operators on vertices associated with edges of color 1 and 2.

Our main structural theorem asserts that each RA2-graph can be produced, by use of replicating and gluing together in a certain way, from two RA2-graphs of a very special form, viewed as triangular halves of two-dimensional square grids. As a result, the combinatorial structure of these objects becomes rather transparent, which enables us to obtain additional results on combinatorial, polyhedral and other properties of RA2-graphs and their extensions. For example, it turns out that an RA2-graph is the Hasse diagram of a finite lattice. Also analysing the proof of the theorem, we give a structural characterization for the interesting class of crystals, called *WA*2*-graphs*, that satisfy axioms (A1)–(A3) but not necessarily (A4) (the latter axiom is known as the Verma relation of degree 4).

The paper is organized as follows. Section 2 gives basic definitions and exhibits some elementary properties of RA2-graphs. The above-mentioned structural theorem on RA2-graphs (Theorem 3.1) is stated in Section 3 and proved in Section 4. Section 5 demonstrates consequences from the theorem and from the method of proof. In particular, we show there that in the definition of RA2-graphs the Verma relation axiom (A4) can be replaced by the requirement that the digraph has exactly one vertex with zero indegree (Proposition 5.3), and describe the structure of WA2-graphs (Proposition 5.2). Section 6 extends our construction and results to infinite analogs of RA2-graphs (containing infinite monochromatic paths). The largest among these possesses the property that it contains any finite or infinite RA2-graph as an interval subgraph, and therefore, can be considered as a universal representative in the given class of crystals. The concluding Section 7 describes natural embeddings of RA2-graphs in $\mathbb{Z}^4$ and $\mathbb{Z}^3$ and discusses polyhedral aspects. In particular, we explain there that the regular A_2-crystals can be characterized via an alternative, numerical, model.

Our combinatorial approach is further developed in [2,3] where the structure of more complicated crystals is studied.

2. RA2-graphs

Let K be a digraph with vertex set V and with edge set E partitioned into two subsets E_1, E_2. We say that an edge in E_i has *color* i and for brevity call it an i*-edge*. Unless explicitly stated otherwise, any digraph in question is assumed to be (weakly) connected, i.e., it is not representable as the disjoint union of two nonempty digraphs. An *RA*2*-graph* is defined by imposing on $K = (V, E_1, E_2)$ four axioms (A1)–(A4) stated below. (These are a reformulation of axioms (P1)–(P6), (P5′), (P6′) in [12] that characterize regular n-colored simply-laced crystals; in our case $n = 2$.)

The first axiom concerns the structure of monochromatic subgraphs (V, E_i).

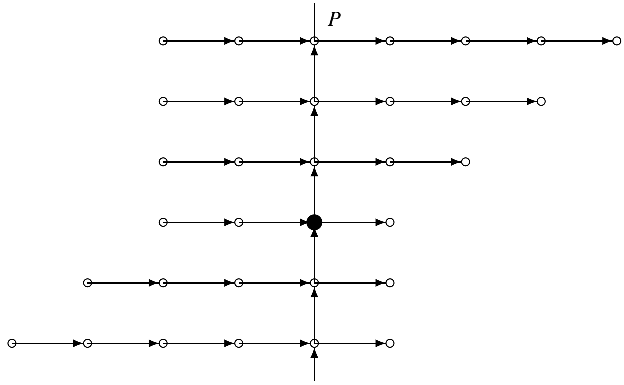


Fig. 1. A example of changing the lengths t_1 and h_1 along line P with color 2. The thick dot indicates the critical vertex in P.

(A1) For $i = 1, 2$, each connected component of (V, E_i) is a finite simple (directed) path, i.e., a sequence $(v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each e_i is an edge going from v_{i-1} to v_i.

In particular, each vertex v has at most one outgoing 1-edge and at most one incoming 1-edge, and similarly for 2-edges. For convenience, we refer to a *maximal* monochromatic path in K, with color i on the edges, as an i*-line*. The i-line passing through a given vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\mathrm{in}}(v)$, and its part from v to the last vertex by $P_i^{\mathrm{out}}(v)$. The lengths of $P_i^{\mathrm{in}}(v)$ and of $P_i^{\mathrm{out}}(v)$ (i.e., the numbers of edges in these paths) are denoted by $t_i(v)$ and $h_i(v)$, respectively.

The second axiom tells us how these lengths can change when one traverses an edge of the other color.

(A2) Each i-line P $(i = 1, 2)$ contains a vertex r satisfying the following property: for any edge (u, v) (from a vertex u to a vertex v) in $P_i^{\mathrm{in}}(r)$, one holds $t_{3-i}(v) = t_{3-i}(u) - 1$ and $h_{3-i}(v) = h_{3-i}(u)$, and for any edge (u', v') in $P_i^{\mathrm{out}}(r)$, one holds $t_{3-i}(v') = t_{3-i}(u')$ and $h_{3-i}(v') = h_{3-i}(u') + 1$.

Such a vertex r (which is unique) is called the *critical* vertex of the given line P. Axiom (A2) is illustrated in Fig. 1.

The digraphs defined by axioms (A1), (A2) constitute a subclass of the class of so-called (locally finite) *semi-normal* A_2*-crystals*. (In the definition of the latter, axiom (A2) is replaced by a weaker axiom: for an i-edge (u, v), either $t_{3-i}(v) = t_{3-i}(u) - 1$ and $h_{3-i}(v) = h_{3-i}(u)$, or $t_{3-i}(v) = t_{3-i}(u)$ and $h_{3-i}(v) = h_{3-i}(u) + 1$.) We will refer to a digraph satisfying (A1) and (A2) as an *SA*2*-graph*.

The quadruple $(t_1(v), h_1(v), t_2(v), h_2(v))$ giving an important information about a vertex v is called the *length-tuple* of v and denoted by $\tau(v)$. Also it is useful to associate with a vertex v the pair of integers $\sigma(v) := (h_1(v) - t_1(v), h_2(v) - t_2(v))$. From axiom (A2) it follows that $\sigma(v) - \sigma(u)$ is equal to $(-2, 1) =: \alpha$ for all 1-edges (u, v). In its turn, for each 2-edge (u, v), such a difference is equal to $(1, -2) =: \beta$. So, under the map $\sigma : V \to \mathbb{R}^2$, each 1-edge (2-edge) becomes a parallel translation of the same vector α (respectively β). Since α, β are linearly independent, we have the following property (which is known for any connected simply-laced crystal having a unique zero indegree vertex, see [12]).

Corollary 2.1. *An SA2-graph K is graded for each color i, which means that for any cycle ignoring the orientation of edges, the number of i-edges in one direction is equal to the number of i-edges in the other direction.*

This implies that K is acyclic and has no parallel edges.

Remark 1. SA2-graphs have a rather loose structure, in contrast to RA2-graphs. In particular, the fact that K is locally finite (in the sense that all monochromatic paths are finite, as required in axiom (A1)) does not guarantee the finiteness of the set of vertices even if the list of different length-tuples of vertices is finite. (We shall see later, in Remark 3 in Section 5, that a similar behavior is possible even if axiom (A3) is added.) Indeed, any SA2-graph K containing an undirected cycle can easily be transformed into an infinite SA2-graph with the same list of length-tuples and without undirected cycles (the "universal covering" over K). Also one can combine SA2-graphs as follows. Suppose a vertex v of an SA2-graph K_1 and a vertex v' of an SA2-graph K_2 have equal length-tuples (one may take as K_2 a copy of K_1). Choose an edge in K_1 incident with v and the corresponding edge in K_2 incident with v'; let for definiteness these edges be incoming 1-edges (u, v) and (u', v'). Then the digraph obtained by replacing these edges by (u, v') and (u', v) is also an SA2-graph (when it is connected).

To formulate two remaining axioms for RA2-graphs, we need some definitions and notation.

For an SA2-graph K, the edges with color i ($i = 1, 2$) are naturally associated with operator F_i acting on the corresponding subset of vertices. So, for a 1-edge (2-edge) (u, v), we write $v = F_1(u)$ and $u = F_1^{-1}(v)$ (respectively $v = F_2(u)$ and $u = F_2^{-1}(v)$). Using this notation, one can express any vertex via another one (since K is connected). For example, the expression $F_1^{-1} F_2^2 F_1(v)$ determines the vertex w obtained from a vertex v by traversing 1-edge (v, v'), followed by traversing 2-edges (v', u) and (u, u'), followed by traversing 1-edge (w, u') in backward direction. Emphasize that every time we use an expression with F_1 or F_2 in what follows, this automatically indicates that all involved edges do exist in K.

For an edge $e = (u, v)$ with color i, we assign *label* $\ell(e) := 0$ if $h_{3-i}(u) = h_{3-i}(v)$, and label $\ell(e) := 1$ otherwise. Axiom (A2) shows that the labels are monotonically nondecreasing along any i-line P. In terms of labels, the critical vertex in P is just the vertex where the incoming i-edge, if exists, is labeled 0 and the outgoing i-edge, if exists, is labeled 1.

In further illustrations we will draw 1-edges by horizontal arrows directed to the right, and 2-edges by vertical arrows directed up.

The third axiom describes situations when the operators F_1 and F_2 commute.

(A3) (a) If a vertex u has two outgoing edges (u, v), (u, v') and if $\ell(u, v) = 0$, then $\ell(u, v') = 1$ and $F_2 F_1(u) = F_1 F_2(u)$. Symmetrically: (b) if a vertex v has two incoming edges (u, v), (u', v) and if $\ell(u, v) = 1$, then $\ell(u', v) = 0$ and $F_2^{-1} F_1^{-1}(v) = F_1^{-1} F_2^{-1}(v)$.

Let us say that vertices $\tilde{u}, \tilde{v}, \tilde{u}', \tilde{v}'$ *form a square* if, up to renaming them, $\tilde{v} = F_1(\tilde{u})$, $\tilde{u}' = F_2(\tilde{u})$ and $\tilde{v}' = F_1(\tilde{u}') = F_2(\tilde{v})$. The opposite 1-edges $(\tilde{u}, \tilde{v})$ and $(\tilde{u}', \tilde{v}')$ in this square have equal labels, in view of the obvious relations $t_2(\tilde{u}') = t_2(\tilde{u}) + 1$ and $t_2(\tilde{v}') = t_2(\tilde{v}) + 1$, and similarly for the opposite 2-edges $(\tilde{u}, \tilde{u}')$ and $(\tilde{v}, \tilde{v}')$. Therefore, (a) in (A3) implies $\ell(v, w) = 1$ and $\ell(v', w) = 0$, where $w := F_2F_1(u)$, and (b) implies $\ell(w, u) = 0$ and $\ell(w, u') = 1$, where $w := F_2^{-1}F_1^{-1}(v)$. The picture illustrates both cases in (A3) (where edge (u, v) has color 1).

From (A3) it follows that

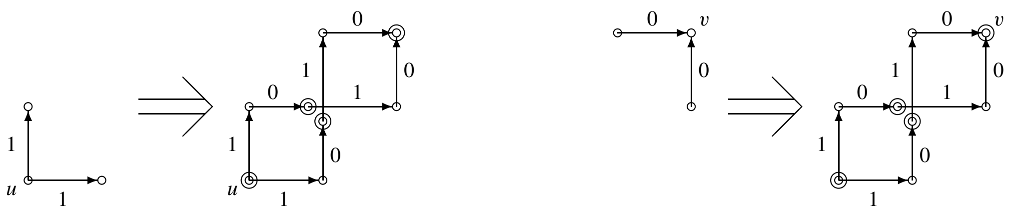
(1) if v is the critical vertex in an i-line, then v is simultaneously the critical vertex in the $(3-i)$-line passing through v.

Indeed, let for definiteness $i = 1$ and assume that v has outgoing 2-edge (v, w). Suppose this edge is labeled 0. Then $t_1(v) > t_1(w)$ implies that v has incoming 1-edge (u, v). It is labeled 0 (since v is the critical vertex in $P_1(v)$). This means that $h_2(u) = h_2(v) > 0$, and therefore, u has outgoing 2-edge (u, u'). Axiom (A3) implies $w = F_1(u')$ and $\ell(u, u') = 1$. But the latter contradicts the fact that (v, w) is labeled 0. Thus, $\ell(v, w) = 1$. Arguing similarly and using (b) in (A3), one shows that if v has incoming 2-edge, then this edge is labeled 0.

Therefore, we can speak of critical vertices without indicating the line color. The final axiom (the Verma relation of degree 4) points out situations when F_1 and F_2 "remotely commute."

(A4) (i) If a vertex u has two outgoing edges both labeled 1, then $F_1F_2^2F_1(u) = F_2F_1^2F_2(u)$. Symmetrically: (ii) if v has two incoming edges both labeled 0, then $F_1^{-1}(F_2^{-1})^2F_1^{-1}(v) = F_2^{-1}(F_1^{-1})^2F_2^{-1}(v)$.

Note that in case (i), we have $F_2F_1(u) \neq F_1F_2(u)$ (otherwise the vertices u, $v := F_1(u)$, $v' := F_2(u)$ and $w := F_2(v)$ would form a square; then both edges (v, w) and (v', w) have label 1, contrary to (A3)(b)). Similarly, $F_2^{-1}F_1^{-1}(v) \neq F_1^{-1}F_2^{-1}(v)$ in case (ii). The picture below illustrates axiom (A4). Here also the labels for all involved edges are indicated and the critical vertices are surrounded by circles. (These labels and critical vertices are determined uniquely, which is not difficult to show by use of (A3) and (A4). These facts will be seen from the analysis in the next section as well.)



Observe that the digraph obtained by reversing the orientation of all edges of K, while preserving their colors, again satisfies axioms (A1)–(A4) (and the label of each edge changes). This RA2-graph is called *dual* to K and denoted by K^*.

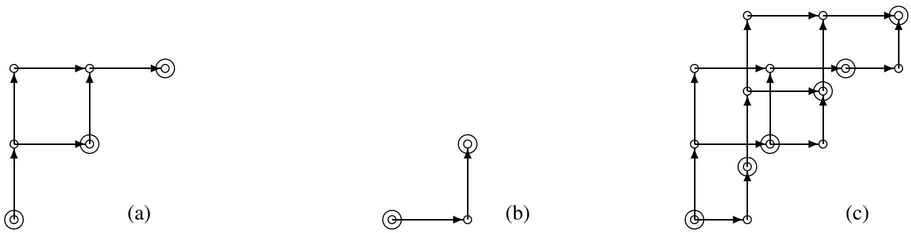


Fig. 2. (a) $K(2,0)$, (b) $K(0,1)$, (c) $K(2,0) \bowtie K(0,1)$.

3. Structural theorem

In this section we formulate the main theorem on the combinatorial structure of RA2-graphs (defined by axioms (A1)–(A4) above). According to it, each RA2-graph can be produced from two elementary RA2-graphs by use of a certain operation of replicating and gluing together. First of all we introduce this operation in a general form.

Consider arbitrary graphs or digraphs $G=(V,E)$ and $H=(V',E')$. Let S be a distinguished subset of vertices of G, and T a distinguished subset of vertices of H. Take $|T|$ disjoint copies of G, denoted as G_t ($t \in T$), and $|S|$ disjoint copies of H, denoted as H_s ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex s in G_t is identified with the vertex t in H_s. The resulting graph consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges is denoted by $(G,S) \bowtie (H,T)$.

In our case the role of G and H is played by special 2-colored digraphs $K(a,0)$ and $K(0,b)$ depending on parameters $a,b \in \mathbb{Z}_+$, each of which being a triangular part of the Cartesian product of two paths. More precisely, the vertices of $K(a,0)$ correspond to the pairs (i,j) for $i,j \in \mathbb{Z}$ with $0 \leqslant i \leqslant j \leqslant a$, and the vertices of $K(0,b)$ correspond to the pairs (i,j) for $0 \leqslant j \leqslant i \leqslant b$. The edges with color 1 in these graphs correspond to all possible pairs of the form $((i,j),(i+1,j))$, and the edges with color 2 to the pairs of the form $((i,j),(i,j+1))$. It is easy to check that $K(a,0)$ satisfies axioms (A1)–(A4) and that the *diagonal* $\{(i,i)\colon\ i=0,\dots,a\}$ is exactly the set of critical vertices in it. Similarly, $K(0,b)$ is an RA2-graph in which the set of critical vertices coincides with the diagonal $\{(i,i)\colon\ i=0,\dots,b\}$. These diagonals are just considered as the distinguished subsets S and T in these digraphs.

We refer to the digraph formed by use of the operation $\bowtie$ in this case as the *diagonal-product* of $K(a,0)$ and $K(0,b)$, and for brevity denote it by $K(a,0) \bowtie K(0,b)$. This digraph is 2-colored, where the edge colors are inherited from $K(a,0)$ and $K(0,b)$. The case $a=2$ and $b=1$ is shown in the picture; here the critical vertices are marked with circles. The trivial (degenerate) RA2-graph $K(0,0)$ consists of a single vertex; clearly $K(a,0) \bowtie K(0,0) = K(a,0)$ and $K(0,0) \bowtie K(0,b) = K(0,b)$ for any a, b.

It will be convenient for us to refer to a subgraph of an RA2-graph K isomorphic to $K(a,0)$ (respecting colors and labels of the edges), including $K(a,0)$ itself, as a *left sail* of size a. Symmetrically, a subgraph of K isomorphic to $K(0,b)$ is called a *right sail* of size b. In a left or right sail we specify, besides the diagonal, the 1-*side* (the largest 1-line) and the 2-*side* (the largest 2-line).

It is a relatively easy exercise to verify validity of axioms (A1)–(A4) for $K(a,0) \bowtie K(0,b)$ with any $a, b \in \mathbb{Z}_+$, i.e. a digraph of this sort is always an RA2-graph. We assert that the converse also takes place.

Theorem 3.1. *Every RA2-graph K is representable as $K(a,0) \bowtie K(0,b)$ for some $a, b \in \mathbb{Z}_+$.*

In particular, K is finite.

4. Proof of the theorem

The proof of Theorem 3.1 falls into several claims.

Claim 1. (i) *For any edge (u,v) with color i and label 0, there exists edge (w,u) with color $(3-i)$ and this edge has label 1. Symmetrically:* (ii) *for any i-edge (u,v) labeled 1, there exists $(3-i)$-edge (v,w) and this edge has label 0.*

Proof. (i) For an i-edge (u,v) labeled 0, one has $t_{3-i}(u) > t_{3-i}(v)$ (by axiom (A2)). Therefore, u has incoming $(3-i)$-edge (w,u). Suppose $\ell(w,u) = 0$. Then $h_i(w) = h_i(u) > 1$. So w has outgoing i-edge (w,w'). By axiom (A3) applied to the pair (w,u), (w,w'), the vertices w, w', u, v form a square, and $\ell(w,w') = 1$. But the edge (u,v) opposite to (w,w') in this square is labeled 0. This contradiction shows that (w,u) must be labeled 1.

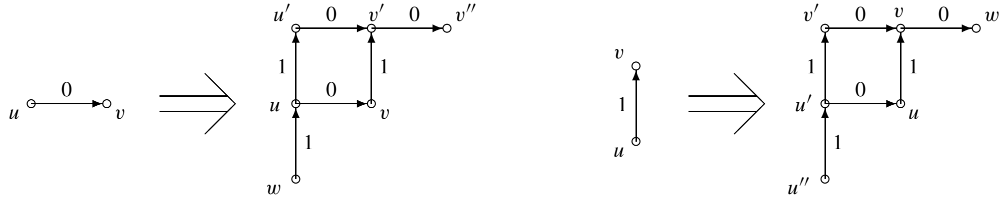
Part (ii) in this claim follows from part (i) applied to the dual RA2-graph K^*. □

Claim 2. (i) *Let (u,v) be an i-edge labeled 0 and let $h_{3-i}(v) > 0$. Then there exist $(3-i)$-edges (u,u'), (v,v') labeled 1 and i-edges (u',v'), (v',v'') labeled 0. Symmetrically:* (ii) *if (u,v) is an i-edge labeled 1 and if $t_{3-i}(u) > 0$, then there exist $(3-i)$-edges (u',u), (v',v) labeled 0 and i-edges (u'',u'), (u',v') labeled 1.*

Proof. (i) For an i-edge (u,v) labeled 0, one has $h_{3-i}(u) = h_{3-i}(v)$. Therefore, u has outgoing $(3-i)$-edge (u,u'). By axiom (A3) applied to the edges $(u,v),(u,u')$, the vertices u, v, u' and $v' := F_2(v)$ form a square, and $\ell(u,u') = 1$. Then $\ell(u',v') = \ell(u,v) = 0$ and $\ell(v,v') = \ell(u,u') = 1$. The existence of i-edge (v',v'') labeled 0 follows from part (ii) in Claim 1 applied to the edge (v,v').

The second part of the claim follows from the first one applied to K^*. □

The picture below illustrates Claims 1 and 2 for the cases when (u,v) is a 1-edge labeled 0 or a 2-edge labeled 1.



For a path $(v_0, e_0, v_1, \dots, e_k, v_k)$, we may use the abbreviate notation $v_0 v_1 \dots v_k$.

Claim 3. *Let v be a critical vertex in K and let L be a left sail of maximum size that contains v. Then L has size $d := t_1(v) + h_2(v)$ and contains the paths $P_1^{\mathrm{in}}(w)$ and $P_2^{\mathrm{out}}(w)$ for all vertices*

w in L (which is equivalent to saying that $h_2(w') = 0$ for each vertex w' on the 1-side of L, and $t_1(w'') = 0$ for each vertex w'' on the 2-side of L).

(L exists since the vertex v itself forms the trivial sail $K(0,0)$.)

Proof. The claim is obvious if $d = 0$. Let $d > 0$. If v has incoming 1-edge (u, v), then $\ell(u, v) = 0$ (since v is critical). By Claim 1, v belongs to the left sail of size 1 formed by the edge (u, v) and the 2-edge incoming u. Similarly, if v has outgoing 2-edge, then it belongs to a left sail of size 1.

Thus, one may assume that the maximum-size left sail L has size $k \geqslant 1$. Consider the 1-side $P = v_0 v_1 \ldots v_k$ of L. All 1-edges of L are labeled 0, therefore, $h_2(v_0) = h_2(v_1) = \cdots = h_2(v_k)$. Suppose $h_2(v_i) > 0$. Applying Claim 2 to the edges of P, one can conclude that there exists a path $u_0 u_1 \ldots u_{k+1}$ whose edges have color 1 and label 0 and whose vertices are connected with the vertices of P by the 2-edges (v_i, u_i) labeled 1, $i = 0, \ldots, k$. But this implies that the sail L is not maximal. Hence $h_2(v_i) = 0$ for all vertices v_i in P. Considering the 2-side of L and arguing in a similar fashion, we obtain $t_1(w') = 0$ for all vertices on this side, and the claim follows. $\square$

Note that, in Claim 3, the vertex v lies on the diagonal of the sail L (since the edges in $P_1^{\mathrm{out}}(v)$ are labeled 1). One can see that v determines L uniquely, and therefore, we may call L the *maximal left sail* containing v. Also each vertex q in the diagonal of L is critical. Indeed, suppose this is not so for some q and consider the 1-line P passing through q and the critical vertex r in P. Then q occurs in P earlier than r, i.e., $q \in P_1^{\mathrm{in}}(r)$ and $q \neq r$. Applying Claim 3 to the maximum-size left sail L' containing r, one can conclude that L' includes L, contrary to the maximality of L.

Let $\mathcal{L}$ be the set of all maximal left sails (each containing a critical vertex). By above reasonings, the members of $\mathcal{L}$ are pairwise disjoint and they cover all critical vertices, all 1-edges labeled 0 and all 2-edges labeled 1.

Since any left sail of K turns into a right sail of K^*, we have similar properties for the set $\mathcal{R}$ of all maximal right sails of K: the diagonal of each member of $\mathcal{R}$ consists of critical vertices, the members of $\mathcal{R}$ are pairwise disjoint and they cover all critical vertices, all 1-edges labeled 1 and all 2-edges labeled 0.

For a sail Q, the vertices in the diagonal D of Q are ordered in a natural way, where the minimal (maximal) element is the vertex with zero indegree (respectively zero outdegree) in Q. According to this ordering, the elements of D are numbered by $0, 1, \ldots, |D|$. So if u, v are vertices in D with numbers $i, i+1$, respectively, then $v = F_1 F_2(u)$ in case of left sail, and $v = F_2 F_1(u)$ in case of right sail.

Choose a maximal left sail $L \in \mathcal{L}$. Let it have size a and diagonal $D = (v_0, v_1, \ldots, v_a)$, where v_i is the vertex with number i in D. For $i = 0, \ldots, a$, denote by R_i the maximal right sail containing v_i. It has size $h_1(v_i) + t_2(v_i)$. Since $v_{i+1} = F_1 F_2(v_i)$ and since the 2-edge $(v_i, F_2(v_i))$ of L is labeled 1, we have $h_1(v_{i+1}) = h_1(F_2(v_i)) - 1 = h_1(v_i)$. In a similar way, the fact that the 1-edge $(F_2(v_i), v_{i+1})$ of L is labeled 0 implies $t_2(v_{i+1}) = t_2(v_i)$. Therefore, the values $h_1(v_i)$ $(i = 0, \ldots, a)$ are equal to one and the same number p, and the values $t_2(v_i)$ are equal to the same number q. This gives the following important property:

(2) the maximal right sails $R_0, \ldots, R_a$ have the same size $b := p + q$, and for $i = 0, \ldots, a$, the vertex v_i has number q in the diagonal of R_i (and therefore, this number does not depend on i).

Then the sails $R_0, \dots, R_a$ are different, or, equivalently, L and R_i have a unique vertex in common, namely, v_i (since $R_i = R_j$ for $i \neq j$ would imply $t_2(v_i) \neq t_2(v_j)$).

Considering an arbitrary $R \in \mathcal{R}$ and arguing similarly, we have:

(3) the maximal left sails $L' \in \mathcal{L}$ intersecting R have the same size, and in the diagonals of these sails L', the vertices common with R have equal numbers.

Since K is connected, one can conclude from (2) and (3) that all members of $\mathcal{L}$ have size a and all members of $\mathcal{R}$ have size b.

Now let $u_0, \dots, u_b$ be the vertices in the diagonal of the maximal right sail R_0 (where the vertices are indexed according to the ordering in the diagonal), and for $j = 0, \dots, b$, define L_j to be the maximal left sail containing u_j. One may assume that L_0 is just the left sail L chosen above, i.e., $u_0 = v_0$.

So far, we have applied only axioms (A1)–(A3). The final claim essentially uses axiom (A4).

Claim 4. *For each $i = 0, \dots, a$ and each $j = 0, \dots, b$, the sails R_i and L_j are intersecting. Moreover, their (unique) common vertex w has number i in the diagonal of L_j and has number j in the diagonal of R_i.*

Proof. We use induction on $i + j$. In fact, we have seen above that the claim is valid for $j = 0$ and any i, as well as for $i = 0$ and any j. So let $0 < i \leqslant a$ and $0 < j \leqslant b$. By induction there exist a common vertex u for L_{j-1} and R_{i-1}, a common vertex v for L_{j-1} and R_i, and a common vertex v' for R_{i-1} and L_j. Furthermore, u and v have numbers $i - 1$ and i (respectively) in the diagonal of L_{j-1}, and therefore, $v = F_1F_2(u)$. In their turn, u and v' have numbers $j - 1$ and j (respectively) in the diagonal of R_{i-1}, and therefore, $v' = F_2F_1(u)$. Also v has number $j - 1 < b$ in the diagonal of R_i, while v' has number $i - 1 < a$ in the diagonal of L_j. Hence the diagonal D of R_i contains vertex w next to v (i.e., $w = F_2F_1(v)$ and w has number j in D), and the diagonal D' of L_j contains vertex w' next to v' (i.e., $w' = F_1F_2(v')$ and w' has number i in D').

Since $(u, F_2(u))$ is a 2-edge of a left sail and $(u, F_1(u))$ is a 1-edge of a right sail, both edges are labeled 1. Applying axiom (A4)(i) to them, we obtain $w = w'$, and the result follows. □

Thus, the union K' of sails $L_0, \dots, L_b, R_0, \dots, R_a$ is isomorphic to $K(a, 0) \bowtie K(0, b)$. Also each of these sails meets any other member of $\mathcal{L} \cup \mathcal{R}$ only within the set of critical vertices in the former (by Claim 3), and these critical vertices belong to K'. Therefore, the connectedness of K implies $K = K'$.

This completes the proof of the theorem.

5. Additional properties and weakened RA2-graphs

In what follows we denote the RA2-graph isomorphic to $K(a, 0) \bowtie K(0, b)$ by $K(a, b)$. The following properties of RA2-graphs are immediate from the definition of left and right sails and the construction of their diagonal-product.

Corollary 5.1. *$K(a, b)$ contains $(a + 1)(b + 1)$ critical vertices and has exactly one vertex s with zero indegree (the minimal vertex, or the* source *of the RA2-graph) and exactly one vertex t with zero outdegree (the maximal vertex, or the* sink*). This s is the common vertex of the sails L_0 and R_0 (defined in the above proof), while t is the common vertex of the sails L_b and R_a, and* *one holds $h_1(s) = t_2(t) = b$ and $h_2(s) = t_1(t) = a$. In particular, an RA2-graph with source s is determined by the parameters $h_1(s)$, $h_2(s)$. Also $K(a, b)$ has equal numbers of edges of each color, namely, $(a + 1)(b + 1)(a + b)/2$.*

(Note that as is shown in [12], for any simply-laced $n \times n$ matrix A and any numbers $c_i \in \mathbb{Z}_+$, $i = 1, \dots, n$, there exists a unique regular A-crystal having exactly one source s and satisfying $h_i(s) = c_i$ for all i.)

Next, analysing the above proof of Theorem 3.1, one can obtain additional properties. They are exhibited in Remarks 2 and 3.

Remark 2. In the proof of Theorem 3.1 we nowhere use part (ii) of axiom (A4). Therefore, this part is redundant. In fact, one can show directly that (A4)(ii) is implied by (A4)(i) and (A1)–(A3).

Remark 3. The set of connected 2-colored digraphs satisfying axioms (A1)–(A3) (but not necessarily (A4)) forms an interesting class $\mathcal{WA}2$ lying between the classes of SA2-graphs and RA2-graphs. The structure of a member K of $\mathcal{WA}2$, which may be named a *weakened RA2-graph*, or a *WA2-graph* for short, can be described relying on the fact (seen from the proof of Theorem 3.1) that K is the union of pairwise disjoint left sails of the same size a and pairwise disjoint right sails of the same size b. More precisely, K can be encoded by parameters $a, b \in \mathbb{Z}_+$, a graph $\Gamma = (V(\Gamma), E(\Gamma))$ and a map $\omega: V(\Gamma) \to \mathbb{Z}_+$ such that:

(∗) (a) Γ is a connected finite or infinite bipartite graph with vertex parts V_1, V_2;
(b) each vertex in V_1 has degree $a + 1$ and each vertex in V_2 has degree $b + 1$;
(c) $\omega(v)$ is at most b for each vertex v in V_1, and a for each vertex in V_2;
(d) for each $v \in V(\Gamma)$, the vertices u adjacent to v have different values $\omega(u)$.

The vertices in V_1 are associated with left sails of size a, and the vertices in V_2 with right sails of size b. The edges of Γ indicate how these sails are glued together, namely: if $u \in V_1$ and $v \in V_2$ are connected by an edge, then the vertex with number $\omega(v)$ in the diagonal of the sail corresponding to u is identified with the vertex with number $\omega(u)$ in the diagonal of the sail corresponding to v. The resulting K is finite if and only if Γ is finite. If Γ is a complete bipartite graph, we just obtain the RA2-graph $K(a, b)$. Another way to characterize $\mathcal{WA}2$ is as follows.

For $a, b \in \mathbb{Z}_+$, let $\Gamma_{a,b}$ denote the Cartesian product $P_a \times P_b$ of directed paths with length a and b, respectively. A *covering* over $\Gamma_{a,b}$ is a nonempty (finite or infinite) connected digraph G along with a graph homomorphism $\gamma : G \to \Gamma_{a,b}$ under which the 1-neighborhood of each vertex v of G (i.e., the subgraph induced by the edges incident with v) is isomorphically mapped to the 1-neighborhood of $\gamma(v)$. For such a (G, γ), the preimage in G of each path $(P_a, \cdot)$ of $\Gamma_{a,b}$ is a collection of pairwise disjoint paths Q of length a, and we can replace each of these Q by a copy of the left sail L of size a in a natural way (the vertices of Q are identified with the elements of the diagonal of L in the natural order). The preimages of each path $(\cdot, P_b)$ are replaced by copies of the right sail R of size b in a similar fashion. One can check that the resulting digraph K, with a due assignment of edge colors, satisfy (A1)–(A3), and conversely, each member of $\mathcal{WA}2$ can be obtained by this construction.

Proposition 5.2. *There is a bijection between the set $\mathcal{WA}2$ of 2-colored digraphs satisfying axioms* (A1)–(A3) *and the set of coverings over grids $\Gamma_{a,b}$ for all $a, b \in \mathbb{Z}_+$.*

Adding axiom (A4) removes all nontrivial coverings (i.e., different from the grid $\Gamma_{a,b}$ itself). On the other hand, the fact that the grid $\Gamma_{a,b}$ has exactly one source and one sink implies that the quantity of sources (sinks) in a covering G is equal to the quantity of preimages in G of any vertex of $\Gamma_{a,b}$. This gives the following important property.

Proposition 5.3. *Under validity of* (A1)–(A3), *axiom* (A4) *is equivalent to the requirement that the digraph has only one source or only one sink. In other words, the RA*2*-graphs are precisely the WA*2*-graphs with one source* (*one sink*).

Another interesting case arises when we take as G the universal covering over $\Gamma_{a,b}$ (i.e., the corresponding infinite tree). This produces a crystal of type A_2 without the Verma relation.

The final remark demonstrates one more property of an RA2-graph $K = K(a,b)$.

Remark 4. The fact that K is graded for each color (cf. Corollary 2.1) implies that K is the Hasse diagram of a poset $(V, \preccurlyeq)$ on the vertex set (it is generated by the relations $u \prec v$ for edges (u,v)). Considering the sail structure of K, it is not difficult to obtain the following sharper property. Here v_{ij} stands for the common critical vertex of sails R_i and L_j, and for a vertex v, we denote by $p(v)$ ($q(v)$) the minimal (respectively maximal) critical vertex greater (respectively smaller) than or equal to v in a maximal sail containing v.

Proposition 5.4. *The poset $(V, \preccurlyeq)$ is a lattice, that is, any two vertices u, v have a unique minimal upper bound $u \vee v$ and a unique maximal lower bound $u \wedge v$. More precisely*:

(i) *if u, v are critical vertices v_{ij}, $v_{i'j'}$, then $u \vee v = v_{\max\{i,i'\},\max\{j,j'\}}$ and $u \wedge v = v_{\min\{i,i'\},\min\{j,j'\}}$*;
(ii) *if u, v occur in the same maximal sail, then both $u \vee v$ and $u \wedge v$ belong to this sail* (*they are computed in a straightforward way*; *in particular, $p(u \vee v) = p(u) \vee p(v)$ and $q(u \vee v) = q(u) \vee q(v)$*);
(iii) *for vertices u, v occurring in different maximal sails*: (a) *if $p(u) \preccurlyeq p(v)$, then v and $w := u \vee v$ belong to the same maximal sail* (*sails*) *Q and one holds $p(w) = p(v)$ and $q(w) = p(u) \vee q(v)$* (*the latter vertex belongs to Q as well*); *and* (b) *if $p(u)$ and $p(v)$ are incomparable, then $u \vee v = p(u) \vee p(v)$.*

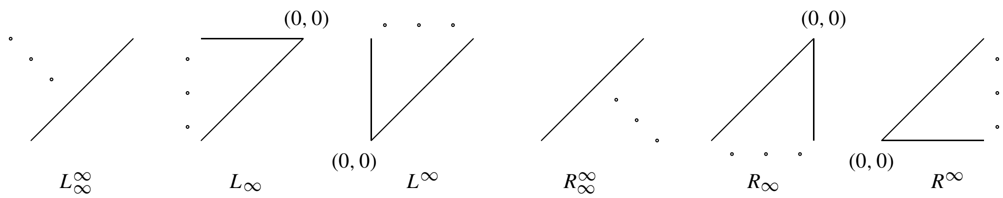
(For vertices u, v in different maximal sails, computing $u \wedge v$ is symmetric to (iii).) Note that this lattice is not distributive already for $a, b = 1$.

6. Infinite RA2-graphs

It is tempting to attempt to extend the notion of RA2-graphs to infinite graphs (then we should allow infinite monochromatic paths since the finiteness of each monochromatic path leads to the finiteness of an RA2-graph, by Theorem 3.1). Infinite analogs of RA2-graphs can be introduced formally via the operation $\bowtie$ (defined in Section 3) applied to two sails of which at least one is infinite or semiinfinite. The largest graph obtained in this way possesses an important property, as we shall see later.

More precisely, we distinguish between three types of infinite paths. A *fully infinite path* is a sequence of the form $\dots, v_i, e_i, v_{i+1}, e_{i+1}, \dots$, where the index set I of vertices v_i ranges $\mathbb{Z}$ (and, as before, e_i is the edge from v_i to v_{i+1}). If $I = \mathbb{Z}_+$ (respectively $I = \mathbb{Z}_-$), we deal with a *semiinfinite path* in forward (respectively backward) direction.

Next we produce, in a natural way, three possible sorts of infinite left sails. The left sail denoted by L_∞^∞ ("infinite up and to the left") has vertex set $\{(i,j)\colon i,j\in\mathbb{Z},\ i\leqslant j\}$, the sail L^∞ ("infinite up") has the vertices (i,j) for $0\leqslant i\leqslant j$, and the sail L_∞ ("infinite to the left") has the vertices (i,j) for $i\leqslant j\leqslant 0$. Doing similarly, we also produce three sorts of infinite right sails: R_∞^∞ ("infinite down and to the right") with the vertices (i,j) for $j\leqslant i$, R^∞ ("infinite to the right") with the vertices (i,j) for $0\leqslant j\leqslant i$, and R_∞ ("infinite down") with the vertices (i,j) for $j\leqslant i\leqslant 0$. In all cases the 1-edges correspond to the pairs $((i,j),(i+1,j))$, and 2-edges to the pairs $((i,j),(i,j+1))$. In a left sail all 1-edges are labeled 0, and all 2-edges are labeled 1, while in a right sail the labels are interchanged. Infinite sails are illustrated in the picture.



As before, the distinguished subsets in the above sails are the corresponding "diagonals" consisting of the vertices (i,i).

Now we use the construction $L\bowtie R$, where L is either a finite left sail or any of the above three infinite left sails, and similarly, R is either a finite right sail or any of three infinite right sails. Combining any of the four sorts of left sails with any of the four sorts of right sails, we obtain 16 types of RA2-graphs $L\bowtie R$, of which one is finite, while the other 15 contain a fully infinite or semiinfinite monochromatic path. For example, $L_\infty\bowtie R^\infty$ has fully infinite 1-lines and finite 2-lines, and $K(a,0)\bowtie R_\infty^\infty$ has semiinfinite in forward direction 1-lines and semiinfinite in backward direction 2-lines.

The largest infinite RA2-graph $L_\infty^\infty\bowtie R_\infty^\infty$, denoted by K_∞^∞, is of most interest for us. It has the set $C=\{v_{ij}\colon i,j\in\mathbb{Z}\}$ of "critical" vertices, where v_{ij} is the common vertex of jth left sail and ith right sail (v_{ij} has the coordinates (i,i) in the former, and (j,j) in the latter). So C is isomorphic to the lattice $\mathbb{Z}^2$ by $v_{ij}\mapsto(i,j)$.

For a subset $C'\subseteq C$, let $\mathrm{Int}(C')$ denote the subgraph of K_∞^∞ formed as the union of all paths connecting pairs in C'. Let us call such a subgraph *principal* if C' is of the form $\{v_{ij}\colon i\in A,\ j\in B\}$ for some (finite or infinite) intervals $A,B\subseteq\mathbb{Z}$, i.e., $i,i'\in A$, $i<i'\Rightarrow i+1\in A$, and similarly for B. Examining the sails structure of the above graphs, it is not difficult to conclude with the following.

Proposition 6.1. *Any principal subgraph of K_∞^∞ is a (finite or infinite) RA2-graph, and conversely, any RA2-graph is isomorphic (respecting the colors and labels of edges) to a principal subgraph of K_∞^∞. A finite RA2-graph is also representable as* $\mathrm{Int}(\{v_{ij},v_{i'j'}\})$ *for some $i<i'$ and $j<j'$.*

In light of this property, K_∞^∞ can be considered as the universal graph in the class of RA2-graphs. (Note also that infinite RA2-graphs can be of use for the study of so-called free combinatorial crystals [1]. The graph $L^\infty\bowtie R^\infty$ from the above list is, in fact, the crystal $B(\infty)$ for sl_3.)

Remark 5. One can characterize the extended class of RA2-graphs (including the above infinite graphs) axiomatically by modifying axioms (A1)–(A3) and preserving (A4). The modified axiom (A1) allows infinite and semiinfinite monochromatic paths. The axioms (A2) and (A3) are replaced by one axiom (A′) that postulates properties exposed in Claims 1 and 2 from the proof of Theorem 3.1. As before, each edge e is endowed with label $\ell(e) \in \{0, 1\}$, the labels are monotonically nondecreasing along each monochromatic path, and $t_i(v)$, $h_i(v)$ denote the lengths of corresponding paths (which may be infinite). The new axiom is stated as follows:

(A′) K is graded for each color. Also: (a) for each i-edge (u, v) labeled 0, there exists $(3-i)$-edge (w, u) labeled 1; moreover, u has outgoing $(3-i)$-edge (u, u') if and only if v has outgoing $(3-i)$-edge (v, v'), and in this case both (u, u'), (v, v') are labeled 1 and there exist i-edges (u', v'), (v', v'') labeled 0. Symmetrically: (b) for each i-edge (u, v) labeled 1, there exists $(3-i)$-edge (v, w) labeled 0; moreover, u has incoming $(3-i)$-edge (u', u) if and only if v has incoming $(3-i)$-edge (v', v), and in this case both (u', u), (v', v) are labeled 0 and there exist i-edges (u', v'), (u'', u') labeled 1.

A verification that the above finite and infinite graphs satisfy (A′) and (A4) is straightforward. Arguing as in the proof of Theorem 3.1, with some refinements, one can show a converse property (we omit details here), namely: these axioms produce only the above graphs, except for five "poor" infinite graphs (defined up to swapping the edge colors) without critical vertices at all. In these graphs the vertices are the pairs (i, j) such that: either (i) $i, j \in \mathbb{Z}$, or (ii) $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, or (iii) $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_-$, or (iv) $i \in \mathbb{Z}_-$ and $j \in \mathbb{Z}_+$, or (v) $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$. Formally, in cases (ii), (iv), the 1-edges $((i, j), (i+1, j))$ are labeled 1, and 2-edges $((i, j), (i, j+1))$ are labeled 0, while in cases (iii), (v), the labels are interchanged.

Remark 6. The construction of diagonal-product can be used for extending the notion of RA2-graphs to more abstract structures. More precisely, let I, J be two linearly ordered sets. (For example, we can take as I, J intervals in $\mathbb{R}$ or $\mathbb{Q}$. In essence, so far we have dealt with intervals in $\mathbb{Z}$.) We define the left sail over I in a natural way, to be the set $L := \{(x, y) \in I^2\colon x \preccurlyeq y\}$, and define the right sail over J to be $R := \{(x, y) \in J^2\colon x \succcurlyeq y\}$. The distinguished subsets D, D', or the diagonals, in L, R, respectively, consist of the identical pairs (x, x). Then we can form the corresponding "diagonal-product" $K := L \bowtie R$. Fixing the second coordinate y (respectively the first coordinate x) in the left sail L or in the right sail R, we obtain a "line" of color 1 (respectively 2) in this sail. When I, J are intervals in $\mathbb{R}$ (in which case K may be conditionally named a *continuous A_2-crystal*), one can introduce a reasonable metric on K, which determines its intrinsic topological structure, as follows. The distance between points (x, y) and (x', y') in each sail of K is assigned to be the ℓ_1-distance $|x - x'| + |y - y'|$, and the distance within the critical set $W := D \times D'$ is also assigned to be the corresponding distance of ℓ_1-type. This induces a metric d on the entire K: for different sails Q, Q' of K and points $u \in Q$ and $v \in Q'$, $d(u, v)$ is equal to $\inf\{d(u, u') + d(u', v') + d(v', v)\colon u' \in Q \cap W,\ v' \in Q' \cap W\}$. The resulting metric space need not be compact even if the intervals I, J are bounded and closed.

7. Arithmetization of regular A_2-crystals and polyhedral aspects

We return to a finite RA2-graph $K = K(a, b)$, with vertex set V, and discuss some natural embeddings of K and other properties, using definitions, notation and results from Sections 3–5. (The description can be extended, with a due care, to infinite RA2-graphs as well.)

Recall that K has set $\mathcal{L} = \{L_0, \dots, L_b\}$ of maximal left sails of size a and set $\mathcal{R} = \{R_0, \dots, R_a\}$ of maximal right sails of size b. These sails and the vertices in their diagonals are numbered as in the proof of Theorem 3.1. Under these numerations, sails L_j and R_i intersect at the critical vertex that has number i in the diagonal D_j of L_j and number j in the diagonal D'_i of R_i. We denote the vertex with number 0 in D_j (the *source* of L_j) by s_j, and denote the vertex with number 0 in D'_i (the source of R_i) by s'_i. Each vertex v of L_j is determined by (*local*) *coordinates* (p, q), where p (respectively q) is the number of 1-edges (respectively 2-edges) in a path from s_j to v, i.e., $v = F_1^p F_2^q(s_j)$. Analogous coordinates are assigned in R_i with respect to s'_i. The vertex $s_0 = s'_0$ is the source of the whole K, denoted by s_K.

7.1. One way to embed K in $\mathbb{Z}^4$ relies on the observation that the vertices $v \in V$ have different length-tuples $\tau(v) = (t_1(v), h_1(v), t_2(v), h_2(v))$. Moreover, the vertices differ from each other even if three parameters involved in τ are considered, e.g., $t_1(v)$, $h_1(v)$, $t_2(v)$. This is seen from the following lemma.

Lemma 7.1. *For $v \in V$, define $\varepsilon := a - t_1(v) - h_2(v)$ and $\delta := b - t_2(v) - h_1(v)$.*

(i) *If v occurs in a left sail L_j, then $-2\delta = \varepsilon \geqslant 0$, v has coordinates $(t_1(v), a - h_2(v))$ in L_j, and j is equal to $t_2(v) - \varepsilon = b - h_1(v) + \varepsilon$.*
(ii) *If v occurs in a right sail R_i, then $-2\varepsilon = \delta \geqslant 0$, v has coordinates $(b - h_1(v), t_2(v))$ in R_i, and i is equal to $t_1(v) - \delta = a - h_2(v) + \delta$.*
(iii) *The vertex v is critical if and only if $\varepsilon = \delta = 0$.*

Proof. The assertions are obvious when v is critical. If v lies in a left sail L_j, then the assertions in (i) can be obtained by comparing $\tau(v)$ with the length-tuples of the critical vertices in the lines $P_1(v)$ and $P_2(v)$ and by using the fact that both critical vertices have number j in the diagonals of the corresponding maximal right sails. If v lies in a right sail R_i, the proof is analogous. □

Note that the edges (u, v) with the same color and the same label have the same difference $\tau(v) - \tau(u)$ (e.g., for color 1 and label 0, the difference is $(1, -1, -1, 0)$). So τ induces an embedding of K in the corresponding subgroup of $\mathbb{Z}^4$ shifted by the vector $-\tau(s_K) = (0, -b, 0, -a)$.

7.2. Next we are interested in embeddings with the property that the edges of K correspond to parallel translations of unit base vectors. To obtain an embedding of this sort in $\mathbb{Z}^4$, we associate to each vertex v quadruple $\bar{\rho}(v) = (\alpha_0(v), \alpha_1(v), \beta_0(v), \beta_1(v))$ defined as follows: (i) if v occurs in a left sail L_j and has coordinates (p, q) in it, then $\alpha_1(v) := \beta_0(v) := j$ and $(\alpha_0(v), \beta_1(v)) := (p, q)$; and (ii) if v occurs in a right sail R_i and has coordinates (p, q) in it, then $\alpha_0(v) := \beta_1(v) := i$ and $(\alpha_1(v), \beta_0(v)) := (p, q)$. Observe that if v is the common (critical) vertex of L_j and R_i, then in both (i), (ii) we have the same quadruple (i, j, j, i); so $\bar{\rho}$ is well defined. The next lemma strengthens Corollary 2.1, by showing that K is graded for each combination of color and label.

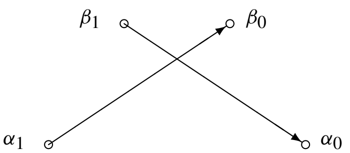
Lemma 7.2. *For any path P in K beginning at the source s_K and ending at a vertex $v \in V$ and for each $\ell = 0, 1$, the number of 1-edges (2-edges) of P labeled ℓ is equal to $\alpha_\ell(v)$ (respectively $\beta_\ell(v)$).*

Proof. Apply induction on the length $|P|$ of P. Let $e=(u,v)$ be the last edge in P. Suppose e occurs in a left sail. One can see that $\bar\rho(v)-\bar\rho(u)$ is equal to $(1,0,0,0)$ if e is a 1-edge, and $(0,0,0,1)$ if e is a 2-edge. Also $\ell(e)=0$ in the former case, and $\ell(e)=1$ in the latter case. Now the required assertion for P follows by induction from that for the part of P from s_K to u. And if e occurs in a right sail then: either e has color 1 and label 1 and $\bar\rho(v)-\bar\rho(u)=(0,1,0,0)$, or e has color 2 and label 0 and $\bar\rho(v)-\bar\rho(u)=(0,0,1,0)$, and the result again follows by induction. $\square$

Clearly the quadruples $\bar\rho(v)$ are different for all vertices v of K, i.e., the map $\bar\rho$: $V\to\mathbb{Z}^4$ is injective. Under this map, traversing an edge of K corresponds to adding a unit base vector associated with the color and label of the edge. Since the local coordinates (p,q) satisfy the relation $0\leqslant p\leqslant q\leqslant a$ for the left sails, and $0\leqslant q\leqslant p\leqslant b$ for the right sails, we obtain the following.

Proposition 7.3. *For each vertex v, one has $0\leqslant\alpha_0(v)\leqslant\beta_1(v)\leqslant a$ and $0\leqslant\beta_0(v)\leqslant\alpha_1(v)\leqslant b$; moreover, at least one of $\alpha_0(v)\leqslant\beta_1(v)$ and $\beta_0(v)\leqslant\alpha_1(v)$ turns into equality (and both equalities here characterize the critical vertices). Conversely, if integers $\alpha_0,\alpha_1,\beta_0,\beta_1$ satisfy $0\leqslant\alpha_0\leqslant\beta_1\leqslant a$ and $0\leqslant\beta_0\leqslant\alpha_1\leqslant b$ and if at least one of $\alpha_0=\beta_1$ and $\beta_0=\alpha_1$ holds, then there is a vertex v with $\bar\rho(v)=(\alpha_0,\alpha_1,\beta_0,\beta_1)$.*

This proposition prompts a model to characterize (finite) regular A_2-crystals; we call it the *numerical* (*or arithmetical*) *model with four variables*. It involves integer parameters a, a', b, b' with $a\geqslant a'$ and $b\geqslant b'$ and variables α_0, α_1, β_0, β_1; the latter range the integers satisfying the relations as in Proposition 7.3, with lower bounds 0 replaced by a' or b' (so that the models with the same $a-a'$ and the same $b-b'$ are equivalent). This produces the RA2-graph $K(a-a',b-b')$; its vertices are the corresponding quadruples $(\alpha_0,\alpha_1,\beta_0,\beta_1)$, and the edges are the corresponding one-element transformations of these quadruples (they are defined easily from the description above). The extended set of RA2-graphs, including the infinite ones (with critical vertices) from Section 6, is obtained by considering all $a,b\in\mathbb{Z}\cup\{+\infty\}$ and $a',b'\in\mathbb{Z}\cup\{-\infty\}$. This model is illustrated by the diagram



in which the arrows indicate the corresponding inequalities on variables. (This diagram illustrates the simplest case of the so-called *crossing model* in [3] giving a combinatorial characterization for regular A_n-crystals.)

7.3. Proposition 7.3 also enables us to transform the map $\bar\rho$ defined in part 2 into an injective map $\rho: V\to\mathbb{Z}^3$, by combining β_0 and β_1 into one coordinate. More precisely, define $\beta:=\beta_0+\beta_1$ and $\rho:=(\alpha_0,\alpha_1,\beta)$. The fact that ρ is injective follows from the possibility of uniquely restoring $\beta_0(v)$, $\beta_1(v)$ if we know $\rho(v)$, namely:

(4) $\beta_0(v)=\alpha_1(v)$ if $\beta(v)\geqslant\alpha_0(v)+\alpha_1(v)$, and $\beta_1(v)=\alpha_0(v)$ otherwise; equivalently: $\beta_0(v)=\min\{\alpha_1(v),\beta(v)-\alpha_0(v)\}$ and $\beta_1(v)=\max\{\alpha_0(v),\beta(v)-\alpha_1(v)\}$.

In a similar way, one can combine α_0 and α_1, by setting $\alpha := \alpha_0 + \alpha_1$ and $\rho' := (\alpha, \beta_0, \beta_1)$. Then the injectivity of ρ' is provided by:

(5) $\alpha_0(v) = \beta_1(v)$ if $\alpha(v) \geqslant \beta_0(v) + \beta_1(v)$, and $\alpha_1(v) = \beta_0(v)$ otherwise; equivalently: $\alpha_0(v) = \min\{\beta_1(v), \alpha(v) - \beta_0(v)\}$ and $\alpha_1(v) = \max\{\beta_0(v), \alpha(v) - \beta_1(v)\}$.

Consider the map ρ and identify V with the set $\rho(V)$ of points in the space $\mathbb{R}^3$ with coordinates $(\alpha_0, \alpha_1, \beta)$. Let $\mathcal{P} = \mathcal{P}(a, b)$ denote the convex hull of V. Using Proposition 7.3, it is not difficult to obtain the following description and properties of the polytope $\mathcal{P}$.

Proposition 7.4. *P is formed by the vectors $(\alpha_0, \alpha_1, \beta) \in \mathbb{R}^3$ satisfying*

$$\begin{aligned} &\text{(i)} \quad 0 \leqslant \alpha_0 \leqslant a;\\ &\text{(ii)} \quad 0 \leqslant \alpha_1 \leqslant b;\\ &\text{(iii)} \quad \alpha_0 \leqslant \beta \leqslant \alpha_1 + a. \end{aligned} \tag{6}$$

Also $\mathcal{P}$ is represented as the Minkowsky sum of the convex hulls of sails L_0 and R_0 (considered as sets of points), and the set of integer points in $\mathcal{P}$ is exactly V. The vertices of $\mathcal{P}$ are $(0,0,0)$, $(0,0,a)$, $(0,b,0)$, $(a,0,a)$, (a,b,a), $(0,b,a+b)$, $(a,b,a+b)$ (*some of which coincide when* $a = 0$ *or* $b = 0$).

So, in the nondegenerate case $a, b > 0$, $\mathcal{P}$ has 6 facets and 7 vertices. All critical vertices of K are contained in the cutting plane $\beta = \alpha_0 + \alpha_1$ (cf. Proposition 7.3). It intersects $\mathcal{P}$ by the parallelogram Π whose vertices are $(0,0,0)$, $(a,0,a)$, $(a,b,a+b)$ and a point lying on the edge of $\mathcal{P}$ connecting $(0,b,0)$ and $(0,b,a+b)$. This "critical section" Π subdivides $\mathcal{P}$ into two triangular prisms being, respectively, the convex hulls of the sails in $\mathcal{L}$ and of the sails in $\mathcal{R}$. The polytope $\mathcal{P}$ is illustrated in Fig. 3.

A similar description can be obtained for the convex hull $\mathcal{P}'$ of the RA2-graph K when K is embedded by use of ρ' in the space $\mathbb{R}^3$ with coordinates $(\alpha, \beta_0, \beta_1)$. Comparing (4) and (5), one can determine the canonical bijection $\omega : \mathcal{P} \to \mathcal{P}'$ (preserving the vertices of K). This ω is piecewise-linear and maps a point $(\alpha_0, \alpha_1, \beta)$ to $(\alpha, \beta_0, \beta_1)$ such that

$$\alpha = \alpha_0 + \alpha_1, \qquad \beta_0 = \min\{\alpha_1, \beta - \alpha_0\}, \qquad \beta_1 = \max\{\alpha_0, \beta - \alpha_1\}. \tag{7}$$

Remark 7. The map ρ gives rise to the corresponding *numerical model with three variables* for (finite) regular A_2-crystals. It involves integer parameters $a, b \geqslant 0$, integer variables $\alpha_0, \alpha_1, \beta$ satisfying (6) and corresponding one-element transformations of these triples (yielding $K(a,b)$). It can easily be extended to include infinite RA2-graphs as well. In fact, this model is nothing else than a special case of the well-known Gel'fand–Tsetlin pattern model [5], which in turn is close to (a special case of) the semi-standard Young tableaux model (see, e.g., [4]). In our case, introducing new variables $x_{11} := \alpha_1 + a$, $x_{12} := \alpha_0$ and $x_{22} := \beta$, we obtain the triangular Gel'fand–Tsetlin pattern $(x_{ij})_{1 \leqslant i \leqslant j \leqslant 2}$ of size 2 bounded by $(a+b, a)$, i.e., satisfying $a + b \geqslant x_{11} \geqslant x_{22} \geqslant x_{12} \geqslant 0$ and $x_{11} \geqslant a \geqslant x_{12}$. (About relationships between crystal bases, Young tableaux and Gel'fand–Tsetlin patterns for classical groups, see, e.g., [8,9].) A related numerical model with three variables arises when we use the map ρ' instead of ρ.

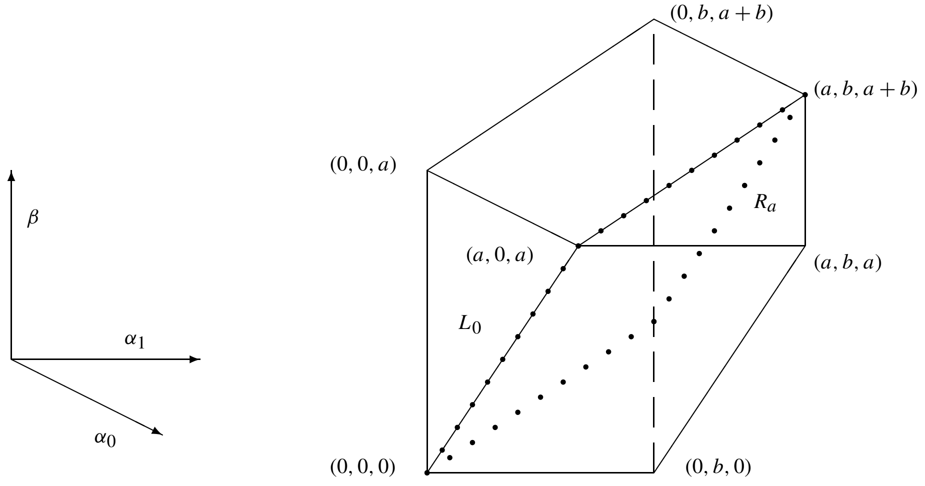


Fig. 3. The polytope $\mathcal{P}(a, b)$. The critical section Π is indicated by dots.

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