Hindawi Publishing Corporation
Algebra
Volume 2013, Article ID 483949, 14 pages
http://dx.doi.org/10.1155/2013/483949

Research Article

Assembling Crystals of Type A

Vladimir I. Danilov,[1] Alexander V. Karzanov,[2] and Gleb A. Koshevoy[1,3]

[1] *Central Institute of Economics and Mathematics, RAS, 47 Nakhimovskii Prospect, Moscow 117418, Russia*
[2] *Institute for System Analysis, RAS, 9 Prospect 60 Let Oktyabrya, Moscow 117312, Russia*
[3] *Laboratoire J.-V. Poncelet, 11 Bolshoy Vlasyevskiy Pereulok, Moscow 119002, Russia*

Correspondence should be addressed to Alexander V. Karzanov; sasha@cs.isa.ru

Received 20 December 2012; Accepted 16 January 2013

Academic Editor: Yao-Zhong Zhang

Regular A_n-*crystals* are certain edge-colored directed graphs, which are related to representations of the quantized universal enveloping algebra $U_q(\mathfrak{sl}_{n+1})$. For such a crystal K with colors $1, 2, \dots, n$, we consider its maximal connected subcrystals with colors $1, \dots, n-1$ and with colors $2, \dots, n$ and characterize the interlacing structure for all pairs of these subcrystals. This enables us to give a recursive description of the combinatorial structure of K via subcrystals and develop an efficient procedure of assembling K.

1. Introduction

Crystals are certain "exotic" edge-colored graphs. This graph-theoretic abstraction, introduced by Kashiwara [1, 2], has proved its usefulness in the theory of representations of Lie algebras and their quantum analogues. In general, a finite *crystal* is a finite directed graph K such that the edges are partitioned into n subsets, or *color* classes, labeled $1, \dots, n$, each connected monochromatic subgraph of K is a simple directed path, and there are certain interrelations between the lengths of such paths, which depend on the $n \times n$ Cartan matrix $M = (m_{ij})$ related to a given Lie algebra $\mathfrak{g}$. Of most interest are crystals of representations, or *regular* crystals. They are associated to elements of a certain basis of the highest weight integrable modules (representations) over the quantized universal enveloping algebra $U_q(\mathfrak{g})$.

This paper continues our combinatorial study of crystals begun in [3, 4] and considers n-colored regular crystals of type A, where the number n of colors is arbitrary. Recall that *type A* concerns $\mathfrak{g} = \mathfrak{sl}_{n+1}$; in this case the Cartan matrix M is viewed as $m_{ij} = -1$ if $|i - j| = 1$, $m_{ij} = 0$ if $|i - j| > 1$, and $m_{ii} = 2$. We will refer to a regular n-colored crystal of type A as an A_n-*crystal* and omit the term n when the number of colors is not specified. Since we are going to deal with finite regular crystals only, the adjectives "finite" and "regular" will usually be omitted. Also we assume that any crystal in question is (weakly) *connected*; that is, it is not the disjoint union of two nonempty graphs (which does not lead to loss of generality).

It is known that any A-crystal K possesses the following properties. (i) K is acyclic (i.e., has no directed cycles) and has exactly one zero-indegree vertex, called the *source*, and exactly one zero-outdegree vertex, called the *sink* of K. (ii) For any $I \subseteq \{1, \dots, n\}$, each (inclusion-wise) maximal connected subgraph of K whose edges have colors from I is a crystal related to the corresponding $I \times I$ submatrix of the Cartan matrix for K. Throughout, speaking of a *subcrystal* of K, we will mean a subgraph of this sort.

Two-colored subcrystals are of especial importance, due to the result in [5] that for a crystal (of any type) with exactly one zero-indegree vertex, the regularity of all two-colored subcrystals implies the regularity of the whole crystal. Let K' be a two-colored subcrystal with colors i, j in an A-crystal K. Then K' is the Cartesian product of a path with color i and a path with color j (forming an $A_1 \times A_1$-crystal) when $|i - j| > 1$, and an A_2-crystal when $|i - j| = 1$. (The A-crystals belong to the group of *simply-laced* crystals, which are characterized by the property that each two-colored subcrystal is of type $A_1 \times A_1$ or A_2; for more details, see [6].)

Another important fact is that for any n-tuple $c = (c_1, \ldots, c_n)$ of nonnegative integers, there exists exactly one A_n-crystal K such that each c_i is equal to the length of the maximal path with color i beginning at the source (for short combinatorial explanations, see [4, Section 2]. We denote the crystal K determined by c in this way by $K(c)$ and refer to c as the *parameter* of this crystal.

There have been known several ways to define A-crystals; in particular, via Gelfand-Tsetlin pattern model, semistandard Young tableaux, and Littelmann's path model; see [7–10]. In the last decade there appeared additional, more enlightening, descriptions. A short list of "local" defining axioms for A-crystals is pointed out in [6], and an explicit construction for A_2-crystals is given in [3]. According to that construction, any A_2-crystal can be obtained from an $A_1 \times A_1$-crystal by replacing each monochromatic path of the latter by a graph viewed as a triangle-shaped half of a directed square grid.

When $n > 2$, the combinatorial structure of A_n-crystals becomes much more complicated, even for $n = 3$. Attempting to learn more about this structure, we elaborated in [4] a new combinatorial construction, the so-called *crossing model* (which is a refinement of the Gelfand-Tsetlin pattern model). This powerful tool has helped us to reveal more structural features of an A_n-crystal $K = K(c)$. In particular, K has the so-called *principal lattice*, a vertex subset Π with the following nice properties:

(P1) Π contains the source and sink of K, and the vertices $v \in \Pi$ are bijective to the elements of the integer box $\mathscr{B}(c) := \{a \in \mathbb{Z}^n : 0 \le a \le c\}$; we write $v = \check{v}[a]$;

(P2) for any $a, a' \in \mathscr{B}(c)$ with $a \le a'$, the interval of K from $\check{v}[a]$ to $\check{v}[a']$ (i.e., the subgraph of K formed by the vertices and edges contained in directed paths from $\check{v}[a]$ to $\check{v}[a']$) is isomorphic to the A_n-crystal $K(a' - a)$, and its principal lattice consists of the principal vertices $\check{v}[a'']$ of K with $a \le a'' \le a'$;

(P3) the set $\mathscr{K}^{(-n)}$ of $(n-1)$-colored subcrystals K' of K having colors $1, \ldots, n-1$ is bijective to Π; more precisely, $K' \cap \Pi$ consists of exactly one vertex, called the heart of K' w.r.t. K, and similarly for the set $\mathscr{K}^{(-1)}$ of subcrystals of K with colors $2, \ldots, n$.

Note that a sort of "principal lattice" satisfying (P1) and (P2) can be introduced for crystals of types B and C as well, and probably for the other classical types (see [11, Section 8]); for more about B_2-crystals, see also [12]). However, (P3) does not remain true in general for those types. Property (P3) is crucial in our study of A-crystals in this paper.

For $a \in \mathscr{B}(c)$, let $K^{\uparrow}[a]$ (resp., $K^{\downarrow}[a]$) denote the subcrystal in $\mathscr{K}^{(-n)}$ (resp., in $\mathscr{K}^{(-1)}$) that contains the principal vertex $\check{v}[a]$; we call it the *upper* (resp., *lower*) subcrystal at a. It is shown in [4] that the parameter of this subcrystal is expressed by a linear function of c and a, and that the total amount of upper (lower) subcrystals with a fixed parameter c' is expressed by a piecewise linear function of c and c'.

In this paper, we further essentially use the crossing model, aiming to obtain a refined description of the structure of an A_n-crystal K. We study the intersections of subcrystals $K^{\uparrow}[a]$ and $K^{\downarrow}[b]$ for any $a, b \in \mathscr{B}(c)$. This intersection may be empty or consist of one or more subcrystals with colors $2, \ldots, n-1$, called *middle* subcrystals of K. Each of these middle subcrystals $\widetilde{K}$ is therefore a lower subcrystal of $K^{\uparrow}[a]$ and an upper subcrystal of $K^{\downarrow}[b]$; so $\widetilde{K}$ has a unique vertex z in the principal lattice $\Pi^{\uparrow}$ of the former, and a unique vertex z' in the principal lattice $\Pi^{\downarrow}$ of the latter. Our main structural results—Theorems 7 and 8—give explicit expressions showing how the "loci" a and b in Π, the "deviation" of (the heart of) $\widetilde{K}$ from z in $\Pi^{\uparrow}$, and the "deviation" of $\widetilde{K}$ from z' in $\Pi^{\downarrow}$ are interrelated.

This gives rise to a recursive procedure of assembling of the A_n-crystal $K(c)$. More precisely, suppose that the $(n-1)$-colored crystals $K^{\uparrow}[a]$ and $K^{\downarrow}[b]$ for all $a, b \in \mathscr{B}(c)$ are already constructed. Then we can combine these subcrystals to obtain the desired crystal $K(c)$, by properly identifying the corresponding middle subcrystals (if any) for each pair $K^{\uparrow}[a], K^{\downarrow}[b]$. This recursive method is implemented as an efficient algorithm which, given a parameter $c \in \mathbb{Z}_+^n$, outputs the crystal $K(c)$. The running time of the algorithm and the needed space are bounded by $Cn^2|K(c)|$, where C is a constant and $|K(c)|$ is the size of $K(c)$. (It may be of practical use for small n and c; in general, an A_n-crystal has "dimension" $n(n+1)/2$ and its size grows sharply by increasing c.)

This paper is organized as follows. Section 2 contains basic definitions and backgrounds. Here we recall "local" axioms and the crossing model for A-crystals and review the needed results on the principal lattice Π of an A_n-crystal and relations between Π and the $(n-1)$-colored subcrystals from [4]. Section 3 states Theorems 7 and 8 and gives a recursive description of the structure of an A_n-crystal K and the algorithm of assembling K. These theorems are proved in Section 4. Section 5 illustrates our assembling method for two special cases of A-crystals: for an arbitrary A_2-crystal (in which case the method can be compared with the explicit combinatorial construction in [3]) and for the particular A_3-crystal $K(1, 1, 1)$.

It should be noted that the obtained structural results on A-crystals can also be applied to give a direct combinatorial proof of the known fact that any regular B_n-crystal (C_n-crystal) can be extracted, in a certain way, from a symmetric A_{2n-1}-crystal (resp., A_{2n}-crystal); this is discussed in detail in ([11], Sections 5–8). Here an A_k-crystal with parameter $(c_1, \ldots, c_k)$ is called symmetric if $c_i = c_{k+1-i}$.

2. Preliminaries

In this section we recall "local" axioms defining A-crystals, explain the construction of crossing model, and review facts about the principal lattice and subcrystals established in [4] that will be needed later.

2.1. A-Crystals. Stembridge [6] pointed out a list of "local" graph-theoretic axioms for the regular simply laced crystals. The (regular) A-crystals form a subclass of those and are defined by axioms (A1)–(A5) below; these axioms are given in a slightly different, but equivalent, form compared with [6].

Let $K = (V(K), E(K))$ be a directed graph whose edge set is partitioned into n subsets $E_1, \dots, E_n$, denoted as $K = (V(K), E_1 \sqcup \cdots \sqcup E_n)$. We assume K to be (weakly) connected. We say that an edge $e \in E_i$ has *color* i or is an i-*edge*.

Unless explicitly stated otherwise, by a *path* we mean a simple finite directed path, that is, a sequence of the form $(v_0, e_1, v_1, \dots, e_k, v_k)$, where $v_0, v_1, \dots, v_k$ are distinct vertices and each e_i is an edge from v_{i-1} to v_i (admitting $k = 0$).

The first axiom concerns the structure of monochromatic subgraphs of K.

(A1) For $i = 1, \dots, n$, each connected subgraph of $(V(K), E_i)$ is a path.

So each vertex of K has at most one incoming i-edge and at most one outgoing i-edge, and therefore one can associate to the set E_i a *partial invertible operator* F_i acting on vertices: (u, v) is an i-edge if and only if F_i *acts* at u and $F_i(u) = v$ (or $u = F_i^{-1}(v)$, where F_i^{-1} is the partial operator inverse to F_i). Since K is connected, one can use the operator notation to express any vertex via another one. For example, the expression $F_1^{-1}F_3^2F_2(v)$ determines the vertex w obtained from a vertex v by traversing 2-edge (v, v'), followed by traversing 3 edges (v', u) and (u, u'), followed by traversing 1-edge (w, u') in backward direction. Emphasize that every time we use such an operator expression in what follows, this automatically says that all corresponding edges do exist in K.

We refer to a monochromatic path with color i on the edges as an i-*path*. So each maximal i-path is an A_1-subcrystal with color i in K. The maximal i-path passing a given vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\mathrm{in}}(v)$, and its part from v to the last vertex by $P_i^{\mathrm{out}}(v)$ (the *tail* and *head* parts of P w.r.t. v). The lengths (i.e., the numbers of edges) of $P_i^{\mathrm{in}}(v)$ and $P_i^{\mathrm{out}}(v)$ are denoted by $t_i(v)$ and $h_i(v)$, respectively.

Axioms (A2)–(A5) concern interrelations of different colors i, j. They say that each component of the two-colored graph $(V(K), E_i \sqcup E_j)$ forms an A_2-crystal when colors i, j are *neighboring*, which means that $|i - j| = 1$, and forms an $A_1 \times A_1$-crystal otherwise.

When we traverse an edge of color i, the head and tail part lengths of maximal paths of another color j behave as follows.

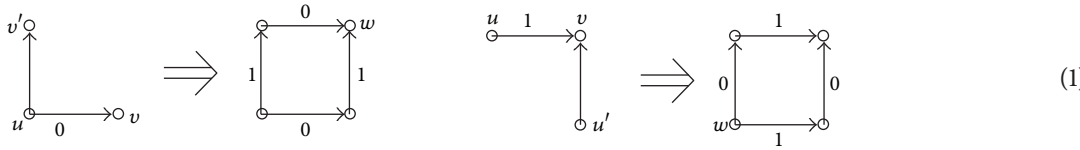
(A2) For different colors i, j and for an edge (u, v) with color i, one holds $t_j(v) \le t_j(u)$ and $h_j(v) \ge h_j(u)$. The value $(h_j(u) - t_j(u)) - (h_j(v) - t_j(v))$ is the constant m_{ij} equal to -1 if $|i - j| = 1$, and 0 otherwise. Furthermore, h_j is convex on each i-path, in the sense that if (u, v), (v, w) are consecutive i-edges, then $h_j(u) + h_j(w) \ge 2h_j(v)$.

These constants m_{ij} are just the off-diagonal entries of the Cartan $n \times n$ matrix M related to the crystal type A and the number n of colors.

It follows that for neighboring colors i, j, each maximal i-path P contains a unique vertex r such that when traversing any edge e of P before r (i.e., $e \in P_i^{\mathrm{in}}(r)$), the tail length t_j decreases by 1 while the head length h_j does not change, and when traversing any edge of P after r, t_j does not change while h_j increases by 1. This r is called the *critical* vertex for P, i, j. To each i-edge $e = (u, v)$, we associate *label* $\ell_j(e) := h_j(v) - h_j(u)$; then $\ell_j(e) \in \{0, 1\}$ and $t_j(v) = t_j(u) - 1 + \ell_j(e)$. We emphasize that the critical vertices on a maximal i-path P w.r.t. its neighboring colors $j = i - 1$ and $j = i + 1$ may be different (and so are the edge labels on P).

Two operators $F = F_i^{\alpha}$ and $F' = F_j^{\beta}$, where $\alpha, \beta \in \{1, -1\}$, are said to *commute* at a vertex v if each of F, F' acts at v (i.e., corresponding i-edge and j-edge incident with v exist) and $FF'(v) = F'F(v)$. The third axiom indicates situations when such operators commute for neighboring i, j.

(A3) Let $|i - j| = 1$. (a) If a vertex u has outgoing i-edge (u, v) and outgoing j-edge (u, v') and if $\ell_j(u, v) = 0$, then $\ell_i(u, v') = 1$ and F_i, F_j commute at v. Symmetrically: (b) if a vertex v has incoming i-edge (u, v) and incoming j-edge (u', v) and if $\ell_j(u, v) = 1$, then $\ell_i(u', v) = 0$ and F_i^{-1}, F_j^{-1} commute at v. (See the following picture.)

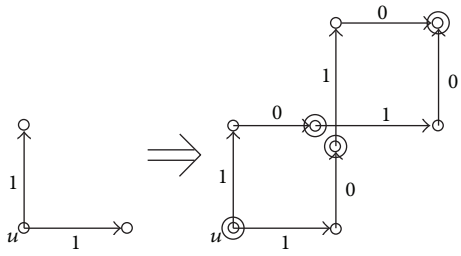


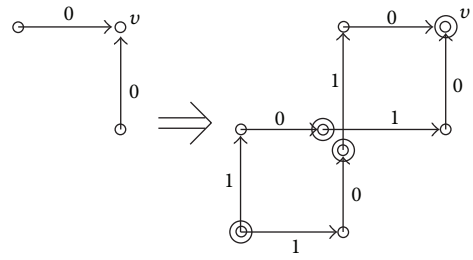
(1)

One easily shows that if four vertices are connected by two i-edges e, e' and two j-edges $\tilde{e}, \tilde{e}'$ (forming a "square"), then $\ell_j(e) = \ell_j(e') \neq \ell_i(\tilde{e}) = \ell_i(\tilde{e}')$ (as illustrated in the picture). Another important consequence of (A3) is that for neighboring colors i, j, if v is the critical vertex on a maximal i-path w.r.t. color j, then v is also the critical vertex on the maximal j-path passing v w.r.t. color i; that is, we can speak of common critical vertices for the pair $\{i, j\}$.

The fourth axiom points out situations when, for neighboring i, j, the operators F_i, F_j and their inverse ones "remotely commute" (forming the "Verma relation of degree 4").

(A4) Let $|i - j| = 1$. (i) If a vertex u has outgoing edges with color i and color j and if each edge is labeled 1 w.r.t. the other color, then $F_iF_j^2F_i(u) = F_jF_i^2F_j(u)$. Symmetrically, (ii) if v has incoming edges with color i and color j and if both are labeled 0, then $F_i^{-1}(F_j^{-1})^2F_i^{-1}(v) = F_j^{-1}(F_i^{-1})^2F_j^{-1}(v)$. (See the following picture.)





(2)

Again, one shows that the label w.r.t. i, j of each of the eight involved edges is determined uniquely, just as indicated in the above picture (where the bigger circles indicate critical vertices).

The final axiom concerns nonneighboring colors.

(A5) Let $|i-j| \geq 2$. Then for any $F \in \{F_i, F_i^{-1}\}$ and $F' \in \{F_j, F_j^{-1}\}$, the operators F, F' commute at each vertex where both act.

This is equivalent to saying that each component of the two-colored subgraph $(V(K), E_i \sqcup E_j)$ is the Cartesian product of an i-path P and a j-path P', or that each subcrystal of K with nonneighboring colors i, j is an $A_1 \times A_1$-*crystal*.

One shows that any A_n-crystal K is finite and has exactly one zero-indegree vertex s_K and one zero-outdegree vertex t_K, called the *source* and *sink* of K, respectively. Furthermore, the A_n-crystals K admit a nice parameterization: the lengths $h_1(s_K), \ldots, h_n(s_K)$ of monochromatic paths starting at the source determine K, and for each tuple $c = (c_1, \ldots, c_n)$ of nonnegative integers, there exists a (unique) A_n-crystal K such that $c_i = h_i(s_K)$ for $i = 1, \ldots, n$. (See [4, 6].) We call c the *parameter* of K and denote K by $K(c)$.

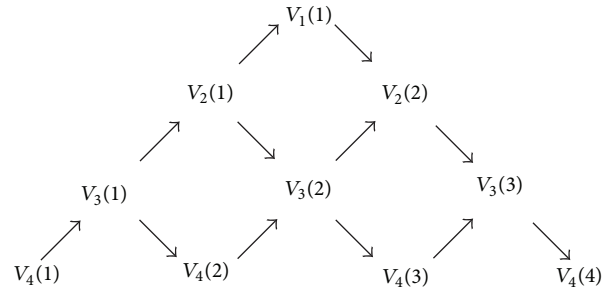
2.2. The Crossing Model for A_n-Crystals. Following [4], the *crossing model* $\mathscr{M}_n(c)$ generating the A_n-crystal $K = K(c)$ with a parameter $c = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n$ consists of three ingredients:

(i) a directed graph $G_n = G = (V(G), E(G))$ depending on n, called the *supporting graph* of the model;

(ii) a set $\mathscr{F} = \mathscr{F}(c)$ of *feasible* functions on $V(G)$;

(iii) a set $\mathscr{E} = \mathscr{E}(c)$ of transformations $f \mapsto f'$ of feasible functions, called *moves*.

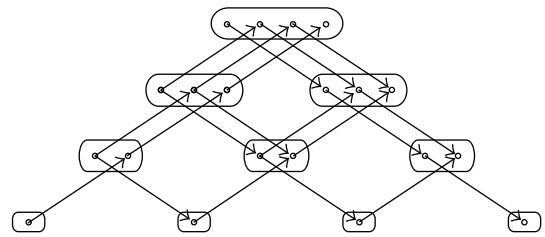
To explain the construction of the supporting graph G, we first introduce another directed graph $\mathscr{G} = \mathscr{G}_n$ that we call the *protograph* of G. Its node set consists of elements $V_i(j)$ for all $i, j \in \{1, \ldots, n\}$ such that $j \leq i$. (To avoid a possible mess, we prefer to use the term "node" for vertices in the crossing model, and the term "vertex" for vertices of crystals.) Its edges are all possible pairs of the form $(V_i(j), V_{i-1}(j))$ (*ascending* edges) or $(V_i(j), V_{i+1}(j+1))$ (*descending* edges). We say that the nodes $V_i(1), \ldots, V_i(i)$ form ith *level* of $\mathscr{G}$ and order them as indicated (by increasing j). We visualize $\mathscr{G}$ by drawing it on the plane so that the nodes of the same level lie in a horizontal line, the ascending edges point North-East, and the descending edges point South-East. See the picture where $n = 4$.



(3)

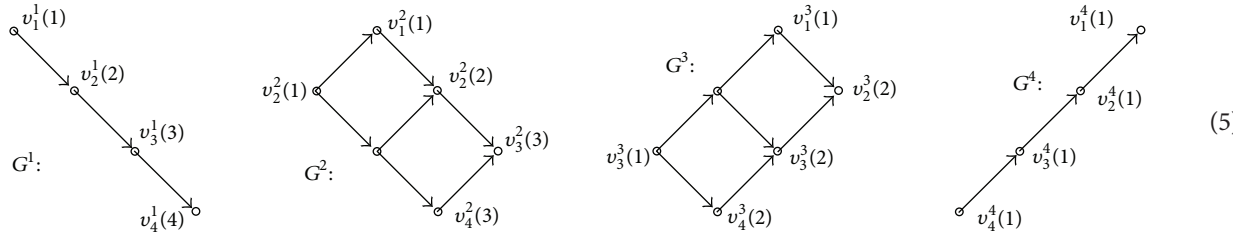
The supporting graph G is produced by replicating elements of $\mathscr{G}$ as follows. Each node $V_i(j)$ generates $n-i+1$ nodes of G, denoted as $v_i^k(j)$ for $k = i-j+1, \ldots n-j+1$, which are ordered by increasing k (and accordingly follow from left to right in the visualization). We identify $V_i(j)$ with the set of these nodes and call it a *multinode* of G. Each edge of $\mathscr{G}$ generates a set of edges of G (a *multiedge*) connecting elements with equal upper indices. More precisely, $(V_i(j), V_{i-1}(j))$ produces $n-i+1$ ascending edges $(v_i^k(j), v_{i-1}^k(j))$ for $k = i-j+1, \ldots, n-j+1$, and $(V_i(j), V_{i+1}(j+1))$ produces $n-i$ descending edges $(v_i^k(j), v_{i+1}^k(j+1))$ for $k = i-j+1, \ldots, n-j$.

The resulting G is the disjoint union of n directed graphs $G^1, \ldots, G^n$, where each G^k contains all vertices of the form $v_i^k(j)$. Also G^k is isomorphic to the Cartesian product of two paths, with the lengths $k-1$ and $n-k$. For example, for $n = 4$, the graph G is viewed as



(4)

(where the multinodes are surrounded by ovals) and its components G^1, G^2, G^3, G^4 are viewed as



(5)

So each node $v = v_i^k(j)$ of G has at most four incident edges, namely, $(v_{i-1}^k(j-1), v)$, $(v_{i+1}^k(j), v)$, $(v, v_{i-1}^k(j))$, and $(v, v_{i+1}^k(j+1))$; we refer to them, when exist, as the NW-, SW-, NE-, and SE-*edges* and denote them by $e^{\mathrm{NW}}(v)$, $e^{\mathrm{SW}}(v)$, $e^{\mathrm{NE}}(v)$, and $e^{\mathrm{SE}}(v)$, respectively.

By a *feasible* function in the model (with a given c), we mean a function $f : V(G) \to \mathbb{Z}_+$ satisfying the following three conditions, where for an edge $e = (u, v)$, $\partial f(e)$ denotes the increment $f(u) - f(v)$ of f on e, and e is called *tight* for f, or f- *tight*, if $\partial f(e) = 0$:

(i) f is *monotone* on the edges, in the sense that $\partial f(e) \geq 0$ for all $e \in E(G)$;

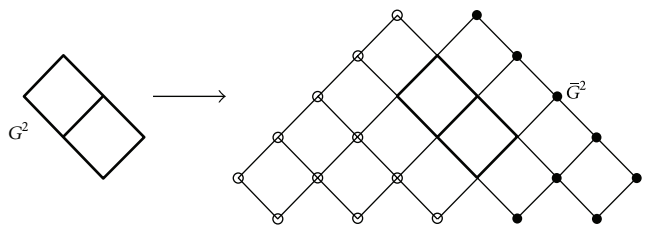
(ii) $0 \leq f(v) \leq c_k$ for each $v \in V\left(G^k\right)$, $k = 1, \ldots, n$;

(iii) each multinode $V_i(j)$ contains a node v with the following property: the edge $e^{\mathrm{SE}}(u)$ is tight for each node $u \in V_i(j)$ preceding v, and $e^{\mathrm{SW}}\left(u'\right)$ is tight for each node $u' \in V_i(j)$ succeeding v.

(6)

The *first* node $v = v_i^k(j)$ (i.e., with k minimum) satisfying the property in (iii) is called the *switch-node* of the multinode $V_i(j)$. These nodes play an important role in our transformations of feasible functions in the model.

To describe the rule of transforming $f \in \mathcal{F}(c)$, we first extend each G^k by adding extra nodes and extra edges (following [4] and aiming to slightly simplify the description). In the extended directed graph $\overline{G}^k$, the node set consists of elements $v_i^k(j)$ for all $i = 0, \ldots, n+1$ and $j = 0, \ldots, n$ such that $j \leq i$. The edge set of $\overline{G}^k$ consists of all possible pairs of the form $(v_i^k(j), v_{i-1}^k(j))$ or $(v_i^k(j), v_{i+1}^k(j+1))$. Then all $\overline{G}^k$ are isomorphic. The disjoint union of these $\overline{G}^k$ gives the *extended supporting graph* $\overline{G}$. The creation of $\overline{G}^2$ from G^2 for $n = 4$ is illustrated in the picture:



(7)

Each feasible function on $V(G)$ is extended to the extra nodes $v = v_i^k(j)$ as follows: $f(v) := c_k$ if there is a path from v to a node of G^k, and $f(v) := 0$ otherwise (one may say that v lies on the left of G^k in the former case and on the right of G^k in the latter case; in the above picture, such nodes v are marked by white and black circles, resp.). ∂f is extended to the extra edges accordingly. In particular, each edge e of $\overline{G}$ not incident with a node of G is tight; that is, $\partial f(e) = 0$. For a node $v = v_i^k(j)$ with $1 \leq j \leq i \leq n$, define the value $\varepsilon(v) = \varepsilon_f(v)$ by

$$\begin{aligned} \varepsilon(v) &:= \partial f\left(e^{\mathrm{NW}}(v)\right) - \partial f\left(e^{\mathrm{SE}}(u)\right) \\ &= \partial f\left(e^{\mathrm{SW}}(v)\right) - \partial f\left(e^{\mathrm{NE}}(u)\right), \end{aligned} \tag{8}$$

where $u := v_i^k(j-1)$. For a multinode $V_i(j)$, define the numbers

$$\varepsilon_i(j) := \sum\left(\varepsilon(v) : v \in V_i(j)\right), \tag{9}$$

$$\begin{aligned} \varepsilon_i(p, j) &:= \varepsilon_i(p) + \varepsilon_i(p+1) + \cdots + \varepsilon_i(j) \quad \text{for } 1 \leq p \leq j, \\ \widetilde{\varepsilon}_i(j) &:= \max\left\{0, \min\left\{\varepsilon_i(p, j) : p = 1, \ldots, j\right\}\right\}. \end{aligned} \tag{10}$$

We call $\varepsilon(v)$, $\varepsilon_i(j)$, and $\widetilde{\varepsilon}_i(j)$ the *slack* at a node v, the *total slack* at a multinode $V_i(j)$, and the *reduced slack* at $V_i(j)$, respectively. (Note that ε, $\widetilde{\varepsilon}$ are defined in (8), (9), and (10) in a slightly different way than in [4], which, however, does not affect the choice of active multinodes and switch-nodes below.)

Now we are ready to define the transformations of f (or the moves from f). At most n transformations $\phi_1, \ldots, \phi_n$ are possible. Each ϕ_i changes f within level i and is applicable when this level contains a multinode $V_i(j')$ with $\widetilde{\varepsilon}_i(j') > 0$. In this case we take the multinode $V_i(j)$ such that

$$\widetilde{\varepsilon}_i(j) > 0, \qquad \widetilde{\varepsilon}_i(q) = 0 \quad \text{for } q > j, \tag{11}$$

referring to it as the *active* multinode for the given f and i. We increase f by 1 at the *switch-node* in $V_i(j)$, preserving f on the other nodes of G. It is shown [4] that the resulting function $\phi_i(f)$ is again feasible.

As a result, the model generates n-colored directed graph $\mathcal{K}(c) = (\mathcal{F}, \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_n)$, where each color class $\mathcal{E}_i$ is formed by the edges $(f, \phi_i(f))$ for all feasible functions f to which the operator ϕ_i is applicable. This graph is just an A_n-crystal.

Theorem 1 (see [4, Th. 5.1]). *For each $c \in \mathbb{Z}_+^n$, the n-colored graph $\mathcal{K}(c)$ is exactly the A_n-crystal $K(c)$.*

2.3. Principal Lattice and $(n-1)$-Colored Subcrystals of an A_n-Crystal. Based on the crossing model, [4] reveals some important ingredients and relations for an A_n-crystal $K = K(c)$. One of them is the so-called principal lattice, which is defined as follows.

Let $a \in \mathbb{Z}_+^n$ and $a \le c$. One easily checks that the function on the vertices of the supporting graph G that takes the constant value a_k within each subgraph G^k of G, $k = 1, \dots, n$, is feasible. We denote this function and the vertex of K corresponding to it by $f[a]$ and $\check{v}[a]$, respectively, and call them *principal* ones. So the set of principal vertices is bijective to the integer box $\mathscr{B}(c) := \{a \in \mathbb{Z}^n : 0 \le a \le c\}$; this set is called the *principal lattice* of K and denoted by $\Pi = \Pi(c)$. When it is not confusing, the term "principal lattice" may also be applied to $\mathscr{B}(c)$.

The following properties of the principal lattice will be essentially used later.

Proposition 2 (see [4, Statement (6.4)]). *Let $a \in \mathscr{B}(c)$, $k \in \{1, \dots, n\}$, and $a' := a + 1_k$ (where 1_k is ith unit base vector in $\mathbb{R}^n$). The principal vertex $\check{v}[a']$ is obtained from $\check{v}[a]$ by applying the operator string*

$$S_{n,k} := w_{n,k,n-k+1} \cdots w_{n,k,2} w_{n,k,1}, \tag{12}$$

where for $j = 1, \dots, n-k+1$, the substring $w_{n,k,j}$ is defined as

$$w_{n,k,j} := F_j F_{j+1} \cdots F_{j+k-1}. \tag{13}$$

When acting on Π, any two (applicable) strings $S_{n,k}, S_{n,k'}$ commute. In particular, any principal vertex $\check{v}[a]$ is expressed via the source $s_K = \check{v}[0]$ as

$$\check{v}[a] = S_{n,n}^{a_n} S_{n,n-1}^{a_{n-1}} \cdots S_{n,1}^{a_1} (s_K). \tag{14}$$

Proposition 3 (see [4, Prop. 6.1]). *For $c', c'' \in Z_+^n$ with $c' \le c'' \le c$, let $K(c' : c'')$ be the subgraph of $K(c)$ formed by the vertices and edges contained in (directed) paths from $\check{v}[c']$ to $\check{v}[c'']$ (the interval of $K(c)$ from $\check{v}[c']$ to $\check{v}[c'']$). Then $K(c' : c'')$ is isomorphic to the A_n-crystal $K(c'' - c')$, and the principal lattice of K' consists of the principal vertices $\check{v}[a]$ of $K(c)$ with $c' \le a \le c''$.*

Let $\mathscr{K}^{(-n)}(c)$ denote the set of subcrystals with colors $1, \dots, n-1$, and $\mathscr{K}^{(-1)}$ the set of subcrystals with colors $2, \dots, n$ in K (recall that a subcrystal is assumed to be connected and maximal for the corresponding subset of colors).

Proposition 4 (see [4, Prop. 7.1]). *Each subcrystal in $\mathscr{K}^{(-n)}$ (in $\mathscr{K}^{(-1)}$) contains precisely one principal vertex. This gives a bijection between $\mathscr{K}^{(-n)}$ and Π (resp., between $\mathscr{K}^{(-1)}$ and Π).*

We refer to the members of $\mathscr{K}^{(-n)}$ and $\mathscr{K}^{(-1)}$ as *upper* and *lower* ($(n-1)$-colored) subcrystals of K, respectively. For $a \in \mathscr{B}(c)$, the upper subcrystal containing the vertex $\check{v}[a]$ is denoted by $K^{\uparrow}[a]$. This subcrystal has its own principal lattice of dimension $n-1$, which is denoted by $\Pi^{\uparrow}[a]$. We say that the coordinate tuple a is the *locus* of $K^{\uparrow}[a]$ (and of $\Pi^{\uparrow}[a]$) in Π. Analogously, for $b \in \mathscr{B}(c)$, the lower subcrystal containing $\check{v}[b]$ is denoted by $K^{\downarrow}[b]$ and its principal lattice by $\Pi^{\downarrow}[b]$; we say that b is the locus of $K^{\downarrow}[b]$ (and of $\Pi^{\downarrow}[b]$) in Π. It turns out that the parameters of upper and lower subcrystals can be expressed explicitly as follows.

Proposition 5 (see [4, Props. 7.2, 7.3]). *For $a \in \mathscr{B}(c)$, the upper subcrystal $K^{\uparrow}[a]$ is isomorphic to the A_{n-1}-crystal $K(c^{\uparrow})$, where $c^{\uparrow}$ is the tuple $(c_1^{\uparrow}, \dots, c_{n-1}^{\uparrow})$ defined by*

$$c_i^{\uparrow} := c_i - a_i + a_{i+1}, \quad i = 1, \dots, n-1. \tag{15}$$

The principal vertex $\check{v}[a]$ is contained in the upper lattice $\Pi^{\uparrow}[a]$ and its coordinate $\hbar^{\uparrow} = (\hbar_1^{\uparrow}, \dots, \hbar_{n-1}^{\uparrow})$ in $\Pi^{\uparrow}[a]$ satisfies

$$\hbar_i^{\uparrow} = a_{i+1}, \quad i = 1, \dots, n-1. \tag{16}$$

Symmetrically, for $b \in \mathscr{B}(c)$, the lower subcrystal $K^{\downarrow}[b]$ is isomorphic to the A_{n-1}-crystal $K(c^{\downarrow})$ with colors $2, \dots, n$, where $c^{\downarrow}$ is defined by

$$c_i^{\downarrow} := c_i - b_i + b_{i-1}, \quad i = 2, \dots, n. \tag{17}$$

The principal vertex $\check{v}[b]$ is contained in the lower lattice $\Pi^{\downarrow}[b]$ and its coordinate $\hbar^{\downarrow} = (\hbar_2^{\downarrow}, \dots, \hbar_n^{\downarrow})$ in $\Pi^{\downarrow}[b]$ satisfies

$$\hbar_i^{\downarrow} = b_{i-1}, \quad i = 2, \dots, n. \tag{18}$$

We call $\check{v}[a]$ the *heart* of $K^{\uparrow}[a]$ w.r.t. K, and similarly for lower subcrystals.

(One more result given in [4, Remark 5] is a piecewise linear formula to compute, for an $(n-1)$-tuple q, the number of upper subcrystals of $K(c)$ with the parameter equal to q, but we do not need this in what follows.)

Remark 6. As is mentioned in the Introduction, the crossing model is, in fact, a refinement of the Gelfand-Tsetlin pattern (or *GT-pattern*) model [7]. More precisely, for $c \in \mathbb{Z}_+^n$, form the partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge \lambda_{n+1} = 0)$ by setting $\lambda_i := c[1 : n-i+1]$, where $c[p : q]$ denotes $c_p + c_{p+1} + \cdots + c_q$. A GT-pattern for λ is a triangular array $X = (x_{ij})_{1 \le j \le i \le n}$ of integers satisfying (a) $x_{ij} \ge x_{i-1,j}, x_{i+1,j+1}$ and (b) $\lambda_i \ge x_{n,i} \ge \lambda_{i+1}$, for all possible i, j. It is shown in [4] that the set of feasible functions f in the crossing model $\mathscr{M}_n(c)$ is bijective to the set of GT-patterns X for λ; such a correspondence is given by $x_{i,j} := \widehat{f}_i(j) + c[1 : i-j]$, where $\widehat{f}_i(j)$ denotes the sum of values of f over the multinode $V_i(j)$. However, it is not clear how to visualize, and work with, principal vertices directly in terms of GT-patterns, whereas such vertices are well visualized and fit to handle in the crossing model.

3. Assembling an A_n-Crystal

As mentioned in the Introduction, the structure of an A_n-crystal $K = K(c)$ can be described in a recursive manner. The idea is as follows. We know that K contains $|\Pi| = (c_1 + 1) \times \cdots \times (c_n + 1)$ upper subcrystals (with colors $1, \dots, n-1$) and $|\Pi|$ lower subcrystals (with colors $2, \dots, n$). Moreover, the parameters of these subcrystals are expressed explicitly by

(15) and (17). Assume that the set $\mathscr{K}^{(-n)}$ of upper subcrystals and the set $\mathscr{K}^{(-1)}$ of lower subcrystals are already available. Then in order to assemble K, it suffices to point out, in appropriate terms, the intersection $K^{\uparrow}[a] \cap K^{\downarrow}[b]$ for all pairs $a, b \in \mathscr{B}(c)$ (the intersection may either be empty or consist of one or more $(n-2)$-colored subcrystals with colors $2, \ldots, n-1$ in K). We give an appropriate characterization in Theorems 7 and 8.

To state them, we need additional terminology and notation. Consider a subcrystal $K^{\uparrow}[a]$, and let $c^{\uparrow}, \hbar^{\uparrow}$ be defined as in (15), (16). For $p = (p_1, \ldots, p_{n-1}) \in \mathscr{B}(c^{\uparrow})$, the vertex in the upper lattice $\Pi^{\uparrow}[a]$ having the coordinate p is denoted by $v^{\uparrow}[a, p]$. We call the vector $\Delta := p - \hbar^{\uparrow}$ the *deviation* of $v^{\uparrow}[a, p]$ from the heart $\check{v}[a]$ in $\Pi^{\uparrow}[a]$ and we will use the alternative notation $v^{\uparrow}[a \mid \Delta]$ for this vertex. In particular, $\check{v}[a] = v^{\uparrow}[a, \hbar^{\uparrow}] = v^{\uparrow}[a \mid 0]$.

Similarly, for a lower subcrystal $K^{\downarrow}[b]$, let $c^{\downarrow}, \hbar^{\downarrow}$ be as in (17), (18). For $q = (q_2, \ldots, q_n) \in \mathscr{B}(c^{\downarrow})$, the vertex with the coordinate q in $\Pi^{\downarrow}[b]$ is denoted by $v^{\downarrow}[a, q]$. Its deviation is $\nabla := q - \hbar^{\downarrow}$, and we may alternatively denote this vertex by $v^{\downarrow}[b \mid \nabla]$.

We call an $(n-2)$-colored subcrystal with colors $2, \ldots, n-1$ in K a *middle subcrystal* and denote the set of these by $\mathscr{K}^{(-1,-n)}$. Each middle subcrystal $K^{\uparrow\downarrow}$ is a *lower* subcrystal of some upper subcrystal $K' = K^{\uparrow}[a]$ of K. By Proposition 4 applied to K', $K^{\uparrow\downarrow}$ has a unique vertex $v^{\uparrow}[a \mid \Delta]$ in the lattice $\Pi^{\uparrow}[a]$. So each $K^{\uparrow\downarrow}$ can be encoded by a pair (a, Δ) formed by a point $a \in \mathscr{B}(c)$ and a deviation Δ in $\Pi^{\uparrow}[a]$. At the same time, $K^{\uparrow\downarrow}$ is an *upper* subcrystal of some lower subcrystal $K^{\downarrow}[b]$ of K and has a unique vertex $v^{\downarrow}[b \mid \nabla]$ in $\Pi^{\downarrow}[b]$. Therefore, the members of $\mathscr{K}^{(-1,-n)}$ determine a bijection

$$\zeta : (a, \Delta) \longmapsto (b, \nabla) \tag{19}$$

between all pairs (a, Δ) concerning upper subcrystals and all pairs (b, ∇) concerning lower subcrystals.

The map ζ is expressed explicitly in the following two theorems. Here for a tuple $\rho = (\rho_i : i \in I)$ of reals, we denote by ρ^{+} (ρ^{-}) the tuple formed by $\rho_i^{+} := \max\{0, \rho_i\}$ (resp., $\rho_i^{-} := \min\{0, \rho_i\}$), $i \in I$.

Theorem 7 (on two deviations). *Let $a \in \mathscr{B}(c)$ and let $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$ be a deviation in $\Pi^{\uparrow}[a]$ (from the heart of $K^{\uparrow}[a]$). Let $(b, \nabla) = \zeta(a, \Delta)$. Then*

$$\nabla_i = -\Delta_{i-1}, \quad i = 2, \ldots, n. \tag{20}$$

Theorem 8 (on two loci). *Let a, b, Δ, ∇ be as in the previous theorem. Then*

$$b_i = a_i + \Delta_i^{+} + \Delta_{i-1}^{-}, \quad i = 1, \ldots, n, \tag{21}$$

letting $\Delta_0 = \Delta_n := 0$.

Proofs of these theorems will be given in the next section.

Based on Theorems 7 and 8, the crystal $K(c)$ is assembled as follows. By recursion we assume that all upper and lower subcrystals are already constructed. We also assume that for each upper subcrystal $K^{\uparrow}[a] \simeq K(c^{\uparrow})$, its principal lattice is distinguished by the use of the corresponding injective map $\sigma : \mathscr{B}(c^{\uparrow}) \to V(K(c^{\uparrow}))$, and similarly for the lower subcrystals. We delete the edges with color 1 in each $K^{\uparrow}[a]$ and extract the components of the resulting graphs, forming the set $\mathscr{K}^{(-1,-n)}$ (arranged as a list) of all middle subcrystals of $K(c)$. Each $K^{\uparrow\downarrow} \in \mathscr{K}^{(-1,-n)}$ is encoded by a corresponding pair (a, Δ), where $a \in \mathscr{B}(c)$ and the deviation Δ in $\Pi^{\uparrow}[a]$ is determined by the use of σ as above. Acting similarly for the lower subcrystals $K^{\downarrow}[b]$ (by deleting the edges with color n), we obtain the same set of middle subcrystals (arranged as another list), each of which being encoded by a corresponding pair (b, ∇), where $b \in \mathscr{B}(c)$ and ∇ is a deviation in $\Pi^{\downarrow}(b)$. Relations (21) and (20) indicate how to identify each member of the first list with its counterpart in the second one. Now restoring the deleted edges with colors 1 and n, we obtain the desired crystal $K(c)$. The corresponding map $\mathscr{B}(c) \to V(K(c))$ is constructed easily (e.g., by the use of operator strings as in Proposition 2).

We conclude this section with several remarks.

Remark 9. For each $a \in \mathscr{B}(c)$ and each vertex $v = v^{\uparrow}[a, p]$ in the upper lattice $\Pi^{\uparrow}[a]$, one can express the parameter $c^{\uparrow\downarrow} = (c_2^{\uparrow\downarrow}, \ldots, c_{n-1}^{\uparrow\downarrow})$ of the middle subcrystal $K^{\uparrow\downarrow}$ containing v, as well as the coordinate $\hbar^{\uparrow\downarrow} = (\hbar_2^{\uparrow\downarrow}, \ldots, \hbar_{n-1}^{\uparrow\downarrow})$ of its heart w.r.t. $K^{\uparrow}[a]$ in the principal lattice of $K^{\uparrow\downarrow}$. Indeed, since $K^{\uparrow\downarrow}$ is a lower subcrystal of $K^{\uparrow}[a]$, one can apply relations as in (17) and (18). Denoting the parameter of $K^{\uparrow}[a]$ by $c^{\uparrow}$ and the coordinate of its heart in $\Pi^{\uparrow}[a]$ by $\hbar^{\uparrow}$, letting $\Delta := p - \hbar^{\uparrow}$, and using (15), (16), we have for $i = 2, \ldots, n-1$,

$$\begin{aligned} c_i^{\uparrow\downarrow} &= c_i^{\uparrow} - p_i + p_{i-1} = (c_i - a_i + a_{i+1}) \\ &\quad - (a_{i+1} + \Delta_i) + (a_i + \Delta_{i-1}) = c_i - \Delta_i + \Delta_{i-1}, \end{aligned} \tag{22}$$

$$\hbar_i^{\uparrow\downarrow} = p_{i-1} = \hbar_{i-1}^{\uparrow} + \Delta_{i-1} = a_i + \Delta_{i-1}. \tag{23}$$

Symmetrically, if $K^{\uparrow\downarrow}$ is contained in $K^{\downarrow}[b]$ and has deviation ∇ in $\Pi^{\downarrow}[b]$, then for $i = 2, \ldots, n-1$,

$$\begin{aligned} c_i^{\uparrow\downarrow} &= c_i - \nabla_i + \nabla_{i+1}, \\ \hbar_i^{\downarrow\uparrow} &= b_i + \nabla_{i+1}, \end{aligned} \tag{24}$$

where $\hbar^{\downarrow\uparrow}$ is the coordinate of the heart of $K^{\uparrow\downarrow}$ w.r.t. $K^{\downarrow}[b]$ in the principal lattice of $K^{\uparrow\downarrow}$ (note that $\hbar^{\downarrow\uparrow}$ may differ from $\hbar^{\uparrow\downarrow}$). We will use (22) and (24) in Section 4.

Remark 10. A straightforward implementation of the above recursive method of constructing $K = K(c)$ takes $O(2^{q(n)}N)$ time and space, where $q(n)$ is a polynomial in n and N is the number of vertices of K. Here the factor $2^{q(n)}$ appears because the total number of vertices in the upper and lower subcrystals is $2N$ (implying that there appear $4N$ vertices in total on the previous step of the recursion, and so on). Therefore,

such an implementation has polynomial complexity of the size of the output for each fixed n, but not in general. However, many intermediate subcrystals arising during the recursive process are repeated, and we can use this fact to improve the implementation. More precisely, the colors occurring in each intermediate subcrystal in the process form an interval of the ordered set $(1,\dots,n)$. We call a subcrystal of this sort a *color-interval subcrystal*, or a *CI-subcrystal*, of K. In fact, *every* CI-subcrystal of K appears in the process. Since the number of intervals is $n(n+1)/2$ and the CI-subcrystals concerning equal intervals are pairwise disjoint, the total number of vertices of all CI-subcrystals of K is $O(n^2N)$. It is not difficult to implement the recursive process so that each CI-subcrystal K' be explicitly constructed only once. As a result, we obtain the following.

Proposition 11. *Let $c \in \mathbb{Z}_+^n$. The A_n-crystal $K(c)$ and all its CI-subcrystals can be constructed in $O(q'(n)|V(K(c))|)$ time and space, where $q'(n)$ is a polynomial in n.*

Remark 12. Relation (21) shows that the intersection of $K^{\uparrow}[a]$ and $K^{\downarrow}[b]$ may consist of many middle subcrystals. Indeed, if $\Delta_i > 0$ and $\Delta_{i-1} < 0$ for some i, then b does not change by simultaneously decreasing Δ_i by 1 and increasing Δ_{i-1} by 1. The number of common middle subcrystals of $K^{\uparrow}[a]$ and $K^{\downarrow}[b]$ for arbitrary $a, b \in \mathscr{B}(c)$ can be expressed by an explicit piecewise linear formula, using (21) and the box constraints $-a_{i+1} \le \Delta_i \le c_i - a_i$, $i = 1,\dots,n-1$, on the deviations Δ in $\Pi^{\uparrow}[a]$ (which follow from (15) and (16)).

4. Proofs of Theorems 7 and 8

Let a, Δ, b, ∇ be as in the hypotheses of Theorem 7. First we show that Theorem 7 follows from Theorem 8.

Proof of (20) *(in the assumption that* (21) *is valid).* The middle subcrystal $K^{\uparrow\downarrow}$ determined by (a,Δ) is the same as the one determined by (b,∇). The parameter $c^{\uparrow\downarrow}$ of $K^{\uparrow\downarrow}$ is expressed simultaneously by (22) and by (24). Then $c_i - \Delta_i + \Delta_{i-1} = c_i - \nabla_i + \nabla_{i+1}$ for $i = 2,\dots,n-1$. Therefore,

$$\Delta_1 + \nabla_2 = \Delta_2 + \nabla_3 = \cdots = \Delta_{n-1} + \nabla_n =: \alpha. \tag{25}$$

In order to obtain (20), one has to show that $\alpha = 0$. We argue as follows. Renumber the colors $1,\dots,n$ as $n,\dots,1$, respectively; this yields the crystal $\widehat{K} = K(\widehat{c})$ symmetric to $K(c)$. Then $K^{\downarrow}[b]$ turns into the upper subcrystal $\widehat{K}^{\uparrow}[\widehat{b}]$ of $\widehat{K}$, where $(\widehat{b}_1,\dots,\widehat{b}_n) = (b_n,\dots,b_1)$. Also the deviation ∇ in $\Pi^{\downarrow}[b]$ turns into the deviation $\widehat{\nabla} = (\widehat{\nabla}_1,\dots,\widehat{\nabla}_{n-1}) = (\nabla_n,\dots,\nabla_2)$ in the principal lattice of $\widehat{K}^{\uparrow}[\widehat{b}]$. Applying relations as in (21) to $(\widehat{b},\widehat{\nabla})$, we have

$$\widehat{a}_i = \widehat{b}_i + \widehat{\nabla}_i^{+} + \widehat{\nabla}_{i-1}^{-} = b_{n-i+1} + \nabla_{n-i+1}^{+} + \nabla_{n-i+2}^{-}, \quad i = 1,\dots,n, \tag{26}$$

where $\widehat{a}_i := a_{n-i+1}$ and $\widehat{\nabla}_n^{+} := \widehat{\nabla}_0^{-} := 0$. On the other hand, (21) for (a,Δ) gives

$$\widehat{b}_i = b_{n-i+1} = a_{n-i+1} + \Delta_{n-i+1}^{+} + \Delta_{n-i}^{-}, \quad i = 1,\dots,n. \tag{27}$$

Relations (26) and (27) imply

$$\begin{aligned} a_{n-i+1} &= b_{n-i+1} + \nabla_{n-i+1}^{+} + \nabla_{n-i+2}^{-} \\ &= \left(a_{n-i+1} + \Delta_{n-i+1}^{+} + \Delta_{n-i}^{-}\right) + \nabla_{n-i+1}^{+} + \nabla_{n-i+2}^{-}, \end{aligned} \tag{28}$$

whence

$$\Delta_{n-i+1}^{+} + \Delta_{n-i}^{-} + \nabla_{n-i+1}^{+} + \nabla_{n-i+2}^{-} = 0, \quad i = 1,\dots,n. \tag{29}$$

Adding up the latter equalities, we obtain

$$\left(\Delta_1 + \cdots + \Delta_{n-1}\right) + \left(\nabla_2 + \cdots + \nabla_n\right) = 0. \tag{30}$$

This and (25) imply $(n-1)\alpha = 0$. Hence $\alpha = 0$, yielding (20) and Theorem 7. □

Proof of Theorem 8. It is more intricate and essentially uses the crossing model.

For a feasible function $f \in \mathscr{F}(c)$ and its corresponding vertex v in $K = K(c)$, we may denote v as v_f and f as f_v. From the crossing model it is seen that

$$\begin{aligned} &\text{if a vertex } v \in V(K) \text{ belongs to } K^{\uparrow}[a] \text{ and to } K^{\downarrow}[b], \\ &\text{then the tuples } a \text{ and } b \text{ are expressed via the values of} \\ &f = f_v \text{ in levels } n \text{ and } 1 \text{ as follows:} \\ &a_k = f\left(v_n^k(n-k+1)\right), \\ &\text{and } b_k = f\left(v_1^k(1)\right) \quad \text{for } k = 1,\dots,n. \end{aligned} \tag{31}$$

Indeed, the principal vertex $\check{v}[a]$ is reachable from v by applying operators F_i or F_i^{-1} with $i \ne n$. The corresponding moves in the crossing model do not change f within level n. Similarly, $\check{v}[b]$ is reachable from v by applying operators F_i or F_i^{-1} with $i \ne 1$, and the corresponding moves do not change f within level 1. Also the first (second) equality in (31) is valid for the principal function $f_{\check{v}[a]}$ (resp., $f_{\check{v}[b]}$).

Next we introduce special functions on the node set $V(G)$ of the supporting graph $G = G_n$. Consider a component $G^k = (V^k, E^k)$ of G. It is a rectangular grid of size $k \times (n-k+1)$ (rotated by 45° in the visualization of G), and its vertex set is

$$V^k = \left\{v_i^k(j) : j = 1,\dots,n-k+1,\ i = j,\dots,j+k-1\right\}. \tag{32}$$

To represent it in a more convenient form, let us rename $v_i^k(j)$ as $x_{i-j+1}^k(j)$ or $x_{i-j+1}(j)$ (as though rotating G^k by 45°). Then

$$V^k = \left\{x_m(j) : m = 1,\dots,k,\ j = 1,\dots,n-k+1\right\}, \tag{33}$$

the SE-edges of G^k become of the form $(x_m(j), x_m(j+1))$, and the NE-edges become of the form $(x_m(j), x_{m-1}(j))$. We distinguish the following subsets of V^k:

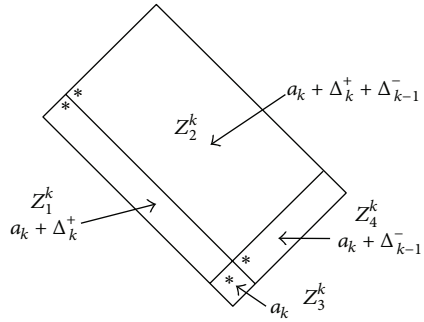


FIGURE 1: The partition of V^k (see Section 4).

(i) the SW-*side* $P^k := \{x_k(1), \dots, x_k(n-k+1)\}$;

(ii) the *right rectangle* $R^k := \{x_m(j) : 1 \le m \le k-1,\ 1 \le j \le n-k+1\}$;

(iii) the *left rectangle* $L^k := \{x_m(j) : 1 \le m \le k,\ 1 \le j \le n-k\}$.

Denote the characteristic functions (in $\mathbb{R}^{V^k}$) of P^k, R^k, and L^k as π^k, ρ^k, and λ^k, respectively.

Return to a and Δ as above. We associate to (a, Δ) the functions

$$f_{a,\Delta}^k := a_k \pi^k + \left(a_k + \Delta_{k-1}^-\right) \rho^k + \Delta_k^+ \lambda^k \tag{34}$$

on V^k for $k = 1, \dots, n$ (see Figure 1) and define $f_{a,\Delta}$ to be the function on $V(G)$ whose restriction to each V^k is $f_{a,\Delta}^k$.

In view of (31), $f = f_{a,\Delta}$ takes the values in levels n and 1 as required in (21) (with k in place of i); namely, $f(v_n^k(n-k+1)) = a_k$ and $f(v_1^k(1)) = a_k + \Delta_k^+ + \Delta_{k-1}^-$ for $k = 1, \dots, n$. Therefore, to obtain (21) it suffices to show the following.

Lemma 13. *(i) The function $f = f_{a,\Delta}$ is feasible. (ii) The vertex v_f belongs to $\Pi^\uparrow[a]$ and has the deviation Δ in it; in other words, $f = f_{\check{v}[a|\Delta]}$.*

Proof. First we prove assertion (i). Let $k \in \{1, \dots, n\}$. We partition V^k into four subsets (rectangular *pieces*):

$$Z_1^k := P^k - \left\{x_k^k(n-k+1)\right\},$$
$$Z_2^k := L^k \backslash P^k, \qquad Z_3^k := \left\{x_k^k(n-k+1)\right\}, \qquad Z_4^k := R^k \backslash L^k, \tag{35}$$

where $Z_2^k = Z_4^k = \emptyset$ when $k = 1$, and $Z_1^k = Z_2^k = \emptyset$ when $k = n$. By (34),

$$f \text{ takes a constant value within each piece } Z_q^k; \text{ namely,}$$
$$\begin{matrix} a_k + \Delta_k^+ \text{ on } Z_1^k, \\ a_k + \Delta_k^+ + \Delta_{k-1}^- \text{ on } Z_2^k; \qquad a_k \text{ on } Z_3^k, \\ a_k + \Delta_{k-1}^- \text{ on } Z_4^k \end{matrix} \tag{36}$$

(as illustrated in Figure 1). Also each edge of G^k connecting different pieces goes either from Z_1^k to $Z_2^k \cup Z_3^k$ or from $Z_2^k \cup Z_3^k$ to Z_4^k. This and (36) imply that $\partial f(e) \ge 0$ for each edge $e \in E^k$, whence f satisfies (6)(i).

The deviation Δ is restricted as $-\hbar^\uparrow \le \Delta \le c^\uparrow - \hbar^\uparrow$, where $c^\uparrow$ is the parameter of the subcrystal $K^\uparrow[a]$ and $\hbar^\uparrow$ is the coordinate of its heart $\check{v}[a]$ in $\Pi^\uparrow[a]$. Formulas (15) and (16) for $c^\uparrow$ and $\hbar^\uparrow$ give

$$-a_{k+1} \le \Delta_k \le c_k - a_k, \qquad -a_k \le \Delta_{k-1} \le c_{k-1} - a_{k-1}. \tag{37}$$

The inequalities $\Delta_k \le c_k - a_k$ and $a_k \le c_k$ imply $a_k + \Delta_k^+ \le c_k$. The inequalities $-a_k \le \Delta_{k-1}$ and $a_k \ge 0$ imply $a_k + \Delta_{k-1}^- \ge 0$. Then, in view of (36), we obtain $0 \le f(v) \le c_k$ for each node v of G^k, yielding (6)(ii).

To verify the switch condition (6)(iii), consider a multinode $V_i(j)$ with $i < n$. It consists of $n-i+1$ nodes $v_i^k(j)$, where $i-j+1 \le k \le n-j+1$.

Let $i \le n-2$. Suppose that $v = v_i^k(j)$ is a node whose SW-edge $e = (u, v)$ exists and is not f-tight. This is possible only if $u \in Z_1^k$ and $v \in Z_2^k$. In this case, k is determined as $k = i - j + 2$; that is, v is the second node in $V_i(j)$. We observe that (a) for the first node $v_i^{k-1}(j)$ of $V_i(j)$, both ends of its SE-edge e' belong to the piece Z_1^{k-1}; and (b) for any node $v_i^{k'}(j)$ with $k' > k$ in $V_i(j)$, both ends of its SW-edge e'' belong either to $Z_2^{k'}$ or to $Z_4^{k'}$. So, such e' and e'' are f-tight. Therefore, the node v satisfies the condition in (6)(iii) for $V_i(j)$.

Now let $i = n-1$. Then $V_i(j)$ consists of two nodes $v = v_{n-1}^{n-j}(j)$ and $v' = v_{n-1}^{n-j+1}(j)$. Put $k := n - j$. Then the edge $e = e^{\mathrm{SE}}(v)$ goes from Z_1^k to $Z_3^k = \{x_k^k(n-k+1)\}$, and the edge $e' = e^{\mathrm{SW}}(v')$ goes from $Z_3^{k+1} = \{x_{k+1}^{k+1}(n-k)\}$ to Z_4^{k+1}. By (36), we have $\partial f(e) = (a_k + \Delta_k^+) - a_k = \Delta_k^+$ and $\partial f(e') = a_{k+1} - (a_{k+1} + \Delta_k^-) = -\Delta_k^-$. Since at least one of Δ_k^+, Δ_k^- is zero, we conclude that at least one of e, e' is tight. So (6)(iii) is valid again.

Next we prove assertion (ii) in the lemma. (The idea is roughly as follows. For each k, compare the function $f_{a,\Delta}^k =: g$ with the function h on V^k taking the constant value a_k. By (34), $g = h + \Delta_k^+ \lambda^k + \Delta_{k-1}^- \rho^k$. In other words, g is obtained from $f_{\check{v}[a]|V^k}$ by adding Δ_k^+ times the "left rectangle function" λ^k, followed by subtracting $|\Delta_{k-1}^-|$ times the "right rectangle function" ρ^k. A crucial observation is that adding λ^k corresponds to applying the operator string $S_{n-1,k}$ (or shifting by kth unit base vector in the upper principal lattice $\Pi^\uparrow[a]$), while subtracting ρ^k corresponds to applying $S_{n-1,k-1}^{-1}$ (or shifting by minus $(k-1)$th unit base vector in $\Pi^\uparrow[a]$). This is because the substrings w in $S_{n-1,k}$ correspond to the SW-NE paths in L^k, and the substrings in $S_{n-1,k-1}$ to similar paths in $R^k \simeq L^{k-1}$).

Now we give a more careful and formal description. We use induction on

$$\eta(\Delta) := \Delta_1 + \cdots + \Delta_{n-1}. \tag{38}$$

In view of (37), $\eta(\Delta) \ge -a_2 - \cdots - a_n$. Suppose that this turns into equality. Then $\Delta_k = -a_{k+1} \le 0$ for $k = 1, \dots, n-1$, and $f = f_{a,\Delta}$ takes the following values within each V^k

(cf. (36)): $f(v) = a_k$ if $v \in P^k$, and $f(v) = 0$ if $v \in V^k - P^k$. This f is the minimal feasible function whose values in level n correspond to a; that is, v_f is the source of $K^{\uparrow}[a]$. Then v_f is the minimal vertex $\check{v}[a, 0]$ in $\Pi^{\uparrow}[a]$, and its deviation in $\Pi^{\uparrow}[a]$ is just Δ, as required. This gives the base of our induction.

Now consider an arbitrary Δ satisfying (37). Let k be such that $\Delta_k < c_k - a_k$ (if any) and define $\Delta'_k := \Delta_k + 1$ and $\Delta'_i := \Delta_i$ for $i \neq k$. Then $\eta(\Delta) < \eta(\Delta')$. We assume by induction that assertion (ii) is valid for $f_{a,\Delta}$, and our aim is to show validity of (ii) for $f_{a,\Delta'}$.

In what follows f stands for the former function $f_{a,\Delta}$.

Let v' be the vertex with the deviation Δ' in $\Pi^{\uparrow}[a]$. Both v_f and v' are principal vertices of the subcrystal $K^{\uparrow}[a]$, and the coordinate of v' in $\Pi^{\uparrow}[a]$ is obtained from the one of v_f by increasing its kth entry by 1. According to Proposition 2 (with n is replaced by $n-1$), v' is obtained from v_f by applying the operator string

$$S_{n-1,k} = w_{n-1,k,n-k} \cdots w_{n-1,k,1}, \tag{39}$$

where $w_{n-1,k,j} = F_j \cdots F_{j+k-1}$ (cf. (12)). In light of this, we have to show that

when (the sequence of moves corresponding to) $S_{n-1,k}$ is applied to f, the resulting feasible function is exactly $f_{a,\Delta'}$. (40)

For convenience, mth term F_{j+m-1} in the substring $w_{n-1,k,j}$ will be denoted by $\phi(j, m)$, $m = 1, \ldots, k$. So $w_{n-1,k,j} = \phi(j, 1)\phi(j, 2) \cdots \phi(j, k)$.

We distinguish between two cases: $\Delta_k \geq 0$ and $\Delta_k < 0$.

Case 1 [$\Delta_k \geq 0$]. An essential fact is that the number $k(n-k)$ of operators in $S_{n-1,k}$ is equal to the number of nodes in the left rectangle L^k of G^k. Moreover, the level of each node $x_m(j)$ of L^k is equal to the "color" of the operator $\phi(j, m)$ (indeed, $x_m(j) = v^k_{j+m-1}(j)$ and $\phi(j, m) = F_{j+m-1}$).

Let $f^{j,m}$ denote the current function on $V(G)$ just before the application of $\phi(j, m)$ (when the process starts with $f = f_{a,\Delta}$). We assert that

for each m, the application of $\phi(j, m)$ to $f^{j,m}$ increases the value at the node $x^k_m(j)$ by 1, (41)

whence (40) will immediately follow.

In order to show (41), we first examine tight edges and the slacks $\varepsilon(v)$ of the nodes v in levels $< n$ for the initial function f. One can see from (36) that

for $k' = 1, \ldots, n$, each node v of the subgraph $G^{k'}$ has at least one entering edge (i.e., $e^{\mathrm{SW}}(v)$ or $e^{\mathrm{NW}}(v)$) which is f-tight, except, possibly, for the nodes $v^{k'}_{k'}(1)$, $v^{k'}_{k'-1}(1)$, $v^{k'}_n(n-k'+1)$, $v^{k'}_{n-1}(n-k'+1)$ (42)

indicated by stars in Figure 1.

Claim. For $k' = 1, \ldots, n$ and a node v of $G^{k'}$ at a level $< n$,

(a) if $v \neq v^{k'}_{k'}(1), v^{k'}_{k'-1}(1)$, then $\varepsilon(v) = 0$;

(b) if $v = v^{k'}_{k'}(1)$, then $\varepsilon(v) = c_{k'} - a_{k'} - \Delta^+_{k'} \geq 0$;

(c) if $v = v^{k'}_{k'-1}(1)$, then $\varepsilon(v) = -\Delta^-_{k'} \geq 0$.

Proof of Claim. Let $v = v^{k'}_i(j)$ and $i < n$. By (8), the slack $\varepsilon(v)$ is equal to $f(w) + f(z) - f(u) - f(v)$, where $w := v^{k'}_{i-1}(j-1)$, $z := v^{k'}_{i+1}(j)$, $u := v^{k'}_i(j-1)$ (these vertices belong to the extended graph $\overline{G}^{k'}$). We consider possible cases and use (36).

(i) If w, z, u are in $G^{k'}$, then $\partial f(w, v) = \partial f(u, z)$.

(ii) If both v, w are in the piece $Z^{k'}_1$ of $G^{k'}$, then $f(w) = f(v)$ and $f(u) = f(z) = c_{k'}$.

(iii) If $j = 1$ and $i \leq k' - 2$, then $f(v) = f(z)$ and $f(u) = f(w) = c_{k'}$. So in these cases we have $\varepsilon(v) = 0$, yielding (a).

(iv) Let $v = v^{k'}_{k'}(1)$. Then $f(v) = a_{k'} + \Delta^+_{k'}$ and $f(u) = f(w) = f(z) = c_{k'}$. This gives $\varepsilon(v) = c_{k'} - a_{k'} - \Delta^+_{k'}$, yielding (b).

(v) Let $v = v^{k'}_{k'-1}(1)$. Then $f(v) = a_{k'} + \Delta^+_{k'} + \Delta^-_{k'-1}$, $f(z) = a_{k'} + \Delta^+_{k'}$ and $f(u) = f(w) = c_{k'}$. This gives $\varepsilon(v) = -\Delta^-_{k'-1}$, yielding (c).

This claim and the relations $\Delta^-_k = 0$ and $\Delta_k < c_k - a_k$ enable us to estimate the total slacks $\varepsilon_i(j)$ for f at the multinodes $V_i(j)$ with $i < n$:

(i) the edge $e^{\mathrm{SW}}\left(v^{k+1}_k(1)\right)$ is f-tight, $\varepsilon\left(v^k_k(1)\right) > 0$, and $\varepsilon(v) = 0$ for the other nodes v in $V_k(1)$; so $\varepsilon_k(1) > 0$;

(ii) if $i \neq k, n$, then $\varepsilon\left(v^i_i(1)\right), \varepsilon\left(v^{i+1}_i(1)\right) \geq 0$ and $\varepsilon(v) = 0$ for the other nodes v in $V_i(1)$; so $\varepsilon_i(1) \geq 0$;

(iii) if $i \neq n$ and $j > 1$, then $\varepsilon(v) = 0$ for all nodes v in $V_i(j)$; so $\varepsilon_i(j) = 0$. (43)

Now we are ready to prove (41). When dealing with a current function $f^{j,m}$ and seeking for the node at level $j+m-1$ where the operator $\phi(j, m)$ should act to increase $f^{j,m}$, we can immediately exclude from consideration any node v that has at least one *tight* entering edge (in view of the monotonicity condition (6) (i)).

Due to (42) and (43)(i), for the initial function $f = f^{1,k}$, there is only one node in level k that has no tight entering edge, namely, $v^k_k(1)$. So, at the first step of the process, the first operator $\phi(1, k)$ of $S_{n-1,k}$ acts just at $v^k_k(1)$, as required in (41).

Next consider a step with $f' := f^{j,m}$ for $(j, m) \neq (1, k)$, assuming validity of (41) on the previous steps.

(A) Let $j = 1$ (and $m < k$). For $v := v^k_m(1)$ and $z := v^k_{m+1}(1)$, we have $f'(v) = f(v) \leq f(z) = f'(z) - 1$. So the unique edge $e = (z, v)$ entering v is not f'-tight. By (42), there

are at most two other nodes in level m that may have no tight entering edges for f (and therefore, for f'), namely, $v_m^m(1)$ and $v_m^{m+1}(1)$. Then $\phi(1,m)$ must act at v, as required in (41) (since the nontightness of the SW-edge e of v implies that none of the nodes $v_m^{m'}(1)$ in $V_m(1)$ preceding v (i.e., with $m' < k$) can be the switch-node).

(B) Let $j > 1$. Comparing f' with f in the node $v := x_m^k(j) = v_{j+m-1}^k(j)$ and its adjacent nodes, we observe that v has no f'-tight entering edge and that $\varepsilon_{f'}(v) > 0$. Also for any other node v' in level $j+m-1$, one can see that if v' has a tight entering edge for f, then this property holds for f' as well, and that $\varepsilon_f(v') \geq \varepsilon_{f'}(v') \geq 0$. Using this, properties (42), (43) (iii), and condition (11), one can conclude that the total and reduced slacks for f' at the multinode $V' := V_{j+m-1}(j)$ are positive, that V' is the active multinode for f' in level $j+m-1$, and that $\phi(j,m)$ can be applied only at v, yielding (41) again.

Thus, (40) is valid in Case 1.

Case 2 [$\Delta_k < 0$]. We assert that in this case the string $S_{n-1,k}$ acts within the right rectangle R^{k+1} of the subgraph G^{k+1} (note that R^{k+1} is of size $k \times (n-k)$). More precisely,

$$\begin{aligned}&\text{each operator } \phi(j,m) \text{ modifies the current function by}\\ &\text{increasing its value at the node } x_m^{k+1}(j) \text{ by 1.}\end{aligned} \tag{44}$$

Then the resulting function in the process is just $f_{a,\Delta'}$ (in view of $(\Delta')_k^- = \Delta_k^- + 1$), yielding (40).

To show (44), we argue as in the previous case and use (42) and the previous claim. Since $\Delta_k < 0$, part (i) in (43) for the initial function f is modified as follows:

$$\begin{aligned}&\text{for } j = 1,\dots,n-k, \text{the SW-edge of each node}\\ &x_k^{k+1}(j) = v_{j+k-1}^{k+1}(j) \text{ is not } f\text{-tight}, \varepsilon\left(v_k^{k+1}(1)\right) > 0,\\ &\varepsilon\left(v_k^k(1)\right) \geq 0, \text{and } \varepsilon(v) = 0 \text{ for the other nodes } v\\ &\text{in } V_k(1); \text{ so } \varepsilon_k(1) > 0,\end{aligned} \tag{45}$$

while properties (ii) and (iii) preserve.

By (42) and (45), there are only two nodes in level k that have no f-tight entering edges, namely, $v_k^k(1)$ and $v_k^{k+1}(1)$. Also $e = e^{SW}(v_k^{k+1}(1))$ is not tight. So, at the first step, $\phi(1,k)$ must act at $v_k^{k+1}(1)$, as required in (44) (since the nontightness of e implies that the node $v_k^k(1)$ preceding $v_k^{k+1}(1)$ cannot be the switch-node in $V_k(1)$).

The fact that $\phi(1,m)$ with $m < k$ acts at $v_m^{k+1}(1)$ is shown by arguing as in (A) above. And for $j > 1$, to show that $\phi(j,m) = F_{j+m-1}$ acts at $x_m^{k+1}(j) = v_{j+m-1}^{k+1}(j)$, we argue as in (B) above. Here, when $m = k$, we also use the fact that the edge $e^{SW}(x_k^{k+1}(j))$ is not f-tight (by (45)), whence both edges entering $x_k^{k+1}(j)$ are not tight for the current function. So (44) is always valid.

Thus, we have the desired property (40) in both Cases 1 and 2, and statement (ii) in Lemma 13 follows.

This completes the proof of relation (21), yielding Theorem 8. □

5. Illustrations

In this concluding section, we give two illustrations to the above assembling construction for A-crystals. The first one refines the interrelation between upper and lower subcrystals in an arbitrary A_2-crystal; this can be compared with the explicit construction (the so-called "sail model") for A_2-crystals in [3]. The second one visualizes the subcrystals structure for one instance of A_3-crystals, namely, $K(1,1,1)$.

5.1. A_2-Crystals. The subcrystals structure becomes simpler when we deal with an A_2-crystal $K = K(c_1, c_2)$. In this case the roles of upper, lower, and middle subcrystals are played by 1-paths, 2-paths, and vertices of K, respectively, where by an i-path we mean a maximal path of color i.

Consider an upper subcrystal in K. This is a 1-path $P = (v_0, v_1, \dots, v_p)$ containing exactly one principal vertex $\check{v}[a]$ of K (the heart of P); here v_i stands for ith vertex in P, $a = (a_1, a_2) \in \mathbb{Z}_+^2$, and $a \leq c$. Let $\check{v}[a] = v_h$. Formulas (15) and (16) give

$$|P| = p = c_1 - a_1 + a_2, \qquad h = a_2. \tag{46}$$

Fix a vertex $v = v_i$ of P. It belongs to some 2-path (lower subcrystal) $Q = (u_1, u_2, \dots, u_q)$. Let $v = u_j$ and let $\check{v}[b] = u_{\overline{h}}$ be the principal vertex of K occurring in Q (the heart of Q). The vertex v forms a middle subcrystal of K; its deviations from the heart of P and from the heart of Q are equal to $i-h =: \delta$ and $j - \overline{h} =: \overline{\delta}$, respectively. By (20) in Theorem 7, we have $\overline{\delta} = -\delta$. Then we can compute the coordinates b by the use of (21) and, further, apply (17) and (18) to compute the length of Q and the locus of its heart. This gives the following:

$$\begin{aligned}&\text{(i) if } \delta \geq 0 \ \left(\text{i.e., } a_2 \leq i \leq c_1 - a_1 + a_2\right), \text{then}\\ &\quad b_1 = a_1 + \delta = a_1 + i - a_2, \qquad b_2 = a_2,\\ &\quad |Q| = c_2 - b_2 + b_1 = c_2 - 2a_2 + a_1 + i,\\ &\quad \text{and } |Q| - \overline{h} = |Q| - b_1 = c_2 - a_2;\\ &\text{(ii) if } \delta \leq 0 \ \left(\text{i.e., } 0 \leq i \leq a_2\right), \text{then } b_1 = a_1,\\ &\quad b_2 = a_2 + \delta = a_2 + \left(i - a_2\right) = i,\\ &\quad |Q| = c_2 - b_2 + b_1 = c_2 - i + a_1,\\ &\quad \text{and } \overline{h} = b_1 = a_1.\end{aligned} \tag{47}$$

Using (46) and (47), one can enumerate the sets of 1-paths and 2-paths and properly intersect corresponding pairs, obtaining the A_2-crystal $K(c)$. It is rather routine to check that the resulting graph coincides with the one generated by the *sail model* from [3]. Next we outline that construction.

Given $c \in \mathbb{Z}_+^2$, the A_2-crystal $K(c)$ is produced from two particular two-colored graphs R and L, called the *right sail* of size c_1 and the *left sail* of size c_2, respectively. The vertices of R correspond to the vectors $(i,j) \in \mathbb{Z}^2$ such that $0 \leq j \leq i \leq c_1$, and the vertices of L to the vectors $(i,j) \in \mathbb{Z}^2$ such

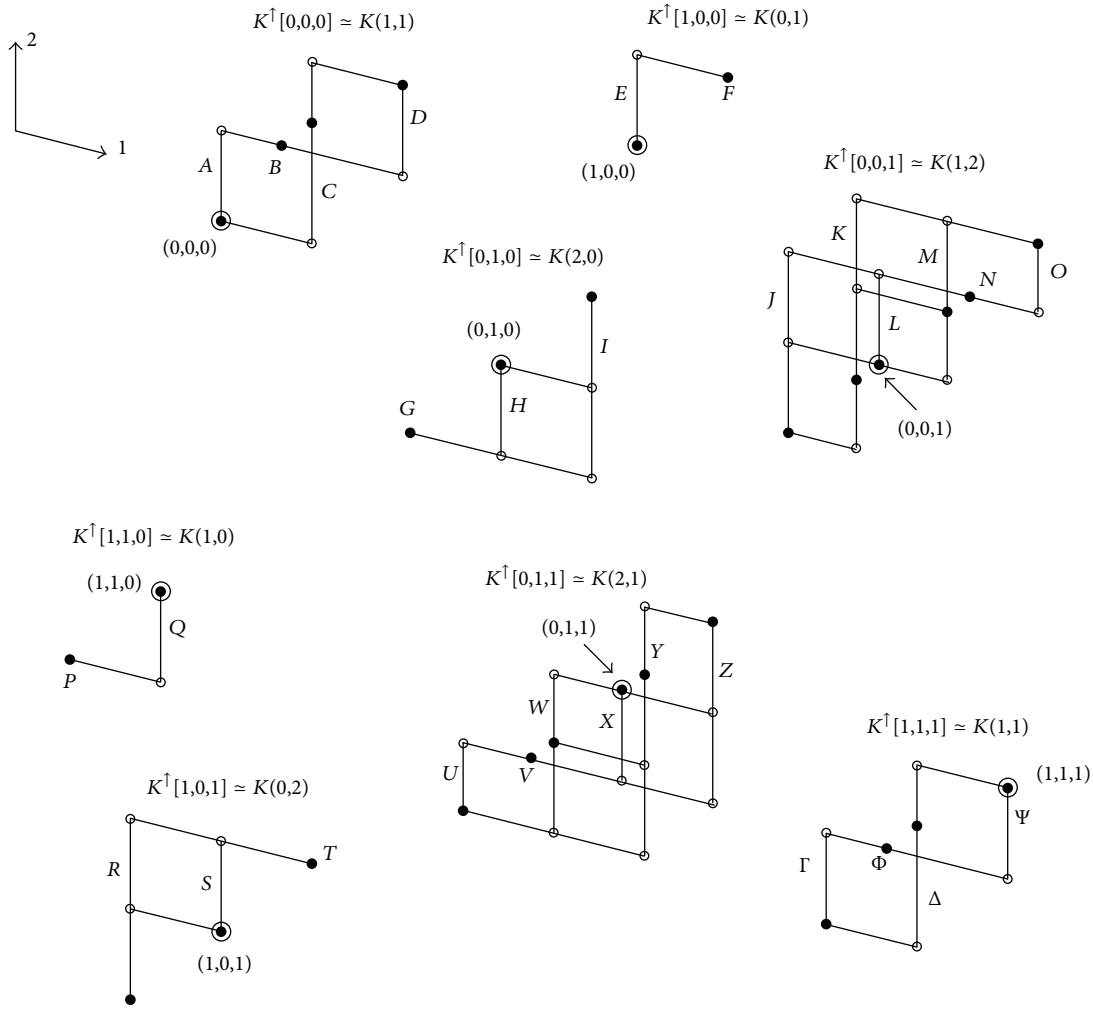
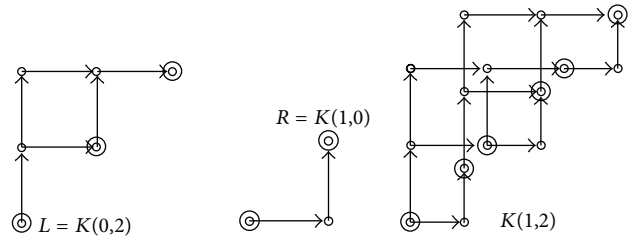


FIGURE 2: The upper subcrystals in $K(1, 1, 1)$ (see Section 5).

that $0 \le i \le j \le c_2$. In both R and L, the edges of color 1 are all possible pairs of the form $((i, j), (i + 1, j))$, and the edges of color 2 are all possible pairs of the form $((i, j), (i, j + 1))$. (Observe that both R and L satisfy axioms (A1)–(A4), R is isomorphic to $K(c_1, 0)$, L is isomorphic to $K(0, c_2)$, and their critical vertices are the "diagonal vertices" (i, i).)

In order to produce $K(c)$, take c_2 disjoint copies $R_1, \ldots, R_{c_2}$ of R and c_1 disjoint copies $L_1, \ldots, L_{c_1}$ of L, referring to R_j as jth right sail and to L_i as ith left sail. Let $D(R_j)$ and $D(L_i)$ denote the sets of diagonal vertices in R_j and L_i, respectively. For all $i = 1, \ldots, c_1$ and $j = 1, \ldots, c_2$, we identify the diagonal vertices $(i, i) \in D(R_j)$ and $(j, j) \in D(L_i)$. The resulting graph is just the desired $K(c)$. The edge colors of $K(c)$ are inherited from L and R. One checks that $K(c)$ has $(c_1 + 1) \times (c_2 + 1)$ critical vertices; they coincide with the diagonal vertices of the sails. The principal lattice of $K(c)$ is just constituted by the critical vertices.

The case $(c_1, c_2) = (1, 2)$ is drawn in the picture; here the critical (principal) vertices are indicated by big circles, 1-edges by horizontal arrows, and 2-edges by vertical arrows:



(48)

In particular, the sail model shows that the numbers of edges of each color in an A_2-crystal are the same. This implies a similar property for any A_n-crystal.

5.2. A_3-Crystal $K(1, 1, 1)$. Next we illustrate the A_3-crystal $K = K(1, 1, 1)$. It has 64 vertices and 102 edges, is rather puzzling, and drawing it in full would be cumbersome and take too much space; for this reason, we expose it by fragments;

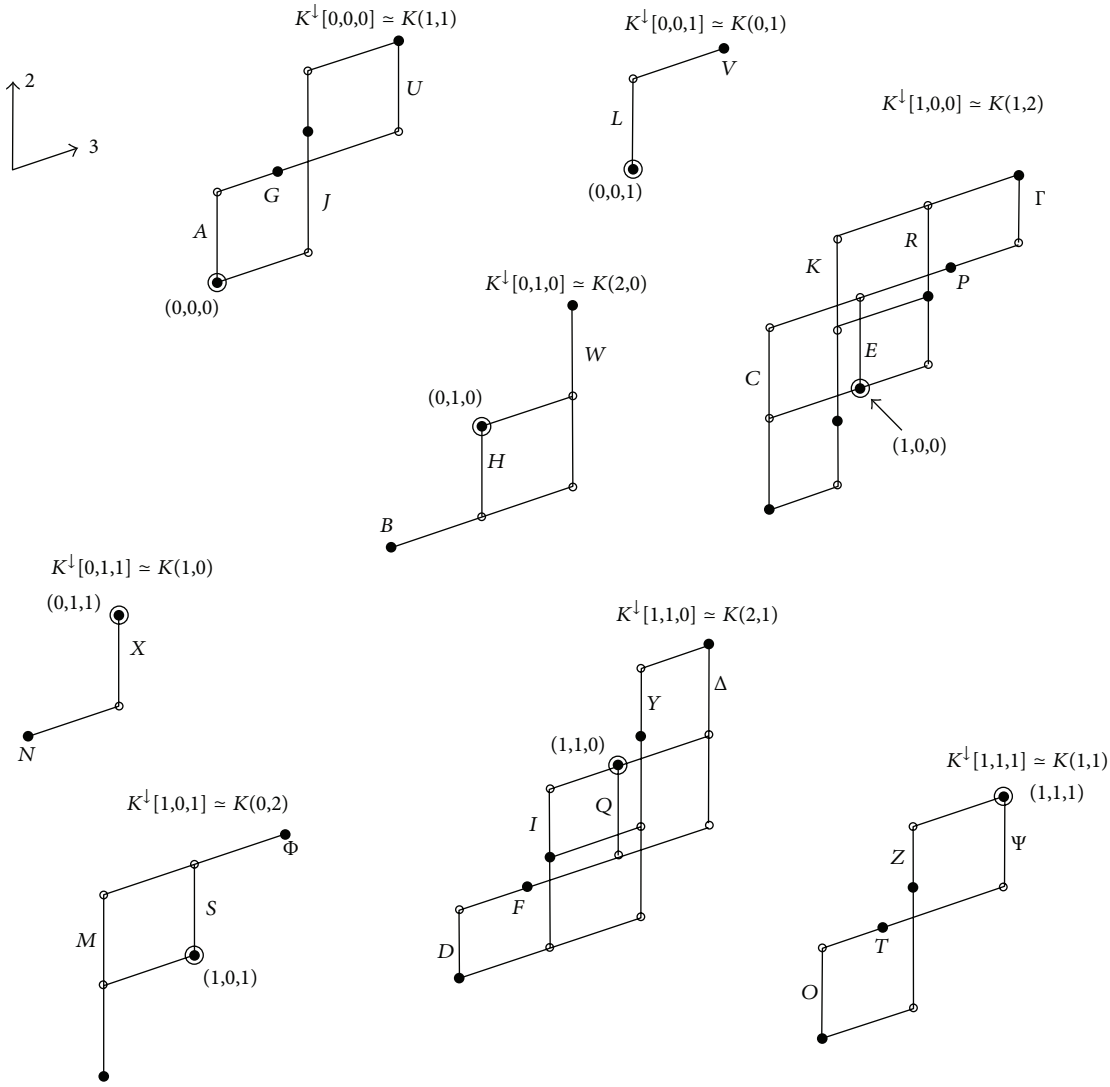
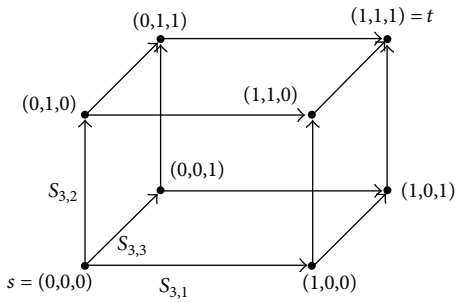


FIGURE 3: The lower subcrystals in $K(1,1,1)$ (see Section 5).

namely, we demonstrate all of its upper and lower subcrystals. We abbreviate notation $\breve{v}[(i,j,k)]$ for principal vertices to (i,j,k) for short. So the principal lattice consists of eight vertices $(0,0,0),\ldots,(1,1,1)$, as drawn in the picture (where the arrows indicate moves by principal operator strings $S_{3,k}$ as in (12)):



$$S_{3,1} = F_3F_2F_1$$
$$S_{3,2} = F_2F_3F_1F_2$$
$$S_{3,3} = F_1F_2F_3 \tag{49}$$

Thus, K has eight upper subcrystals $K^{\uparrow}[i,j,k]$ and eight lower subcrystals $K^{\downarrow}[i,j,k]$ (writing $K^{\bullet}[i,j,k]$ for $K^{\bullet}[(i,j,k)]$); they are drawn in Figures 2 and 3. Here the directions of edges of colors 1,2,3 are as indicated in the upper left corner. In each subcrystal, we indicate its critical vertices by black circles and the unique principal vertex of K occurring in it (the heart) by a big white circle. K has 30 middle subcrystals (paths of color 2), which are labeled as $A,\ldots,Z,\Gamma,\Delta,\Phi,\Psi$ (note that $B,F,G,N,P,T,V,$ and Φ consist of single vertices).

For each upper subcrystal $K^{\uparrow}[i,j,k]$, its parameter $c^{\uparrow}$ and heart locus $\hbar^{\uparrow}$, computed by use of (15) and (16), are as follows (where $\widetilde{K},\widetilde{s},$ and z denote the current subcrystal, its source, and its heart, resp.):

(i) for $K^{\uparrow}[0,0,0]$: $c_1^{\uparrow} = 1-0+0 = 1$, $c_2^{\uparrow} = 1-0+0 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 0$ (so $\widetilde{K}$ is isomorphic to $K(1,1)$ and z coincides with $\widetilde{s}$);

(ii) for $K^{\uparrow}[1,0,0]$: $c_1^{\uparrow} = 1-1+0 = 0$, $c_2^{\uparrow} = 1-0+0 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 0$;

(iii) for $K^{\uparrow}[0,1,0]$: $c_1^{\uparrow} = 1 - 0 + 1 = 2$, $c_2^{\uparrow} = 1 - 1 + 0 = 0$, $\hbar_1^{\uparrow} = 1$, and $\hbar_2^{\uparrow} = 0$ (so $\widetilde{K} \simeq K(2,0)$ and z is located at $S_{2,1}(\widetilde{s}) = F_2F_1(\widetilde{s})$);

(iv) for $K^{\uparrow}[0,0,1]$: $c_1^{\uparrow} = 1 - 0 + 0 = 1$, $c_2^{\uparrow} = 1 - 0 + 1 = 2$, $\hbar_1^{\uparrow} = 0$, and $\hbar_2^{\uparrow} = 1$ (so $\widetilde{K} \simeq K(1,2)$ and z is located at $S_{2,2}(\widetilde{s}) = F_1F_2(\widetilde{s})$);

(v) for $K^{\uparrow}[1,1,0]$: $c_1^{\uparrow} = 1 - 1 + 1 = 1$, $c_2^{\uparrow} = 1 - 1 + 0 = 0$, $\hbar_1^{\uparrow} = 1$, and $\hbar_2^{\uparrow} = 0$;

(vi) for $K^{\uparrow}[1,0,1]$: $c_1^{\uparrow} = 1 - 1 + 0 = 0$, $c_2^{\uparrow} = 1 - 0 + 1 = 2$, $\hbar_1^{\uparrow} = 0$, and $\hbar_2^{\uparrow} = 1$;

(vii) for $K^{\uparrow}[0,1,1]$: $c_1^{\uparrow} = 1 - 0 + 1 = 2$, $c_2^{\uparrow} = 1 - 1 + 1 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 1$ (so $\widetilde{K} \simeq K(2,1)$ and z is located at $S_{2,2}S_{2,1}(\widetilde{s}) = F_1F_2F_2F_1(\widetilde{s})$);

(viii) for $K^{\uparrow}[1,1,1]$: $c_1^{\uparrow} = 1 - 1 + 1 = 1$, $c_2^{\uparrow} = 1 - 1 + 1 = 1$, and $\hbar_1^{\uparrow} = \hbar_2^{\uparrow} = 1$.

Since $K(1,1,1)$ is "symmetric," so are its upper and lower subcrystals; that is, each $K^{\downarrow}[i,j,k]$ is obtained from $K^{\uparrow}[k,j,i]$ by replacing color 1 by 3. In Figure 3, when writing $K^{\downarrow}[i,j,k] \simeq K(\alpha,\beta)$, the parameters α, β concern colors 3 and 2, respectively.

Now the desired $K(1,1,1)$ is assembled by gluing the fragments in Figures 2 and 3 along the 2-paths $A, \ldots, \Psi$.

Acknowledgments

This paper is supported by RFBR Grant 10-01-9311-CNRSL_a. A part of this research was done while Alexander V. Karzanov was visiting Equipe Combinatoire et Optimisation, Université de Paris, France and the Institut für Diskrete Mathematik, Universität Bonn, Germany.

References

[1] M. Kashiwara, "Crystalizing the q-analogue of universal enveloping algebras," *Communications in Mathematical Physics*, vol. 133, no. 2, pp. 249–260, 1990.

[2] M. Kashiwara, "On crystal bases," in *Representations of Groups (Banff, AB,1994)*, vol. 16, pp. 155–197, American Mathematical Society, Providence, RI, USA, 1995.

[3] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy, "Combinatorics of regular A_2-crystals," *Journal of Algebra*, vol. 310, no. 1, pp. 218–234, 2007.

[4] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy, "The crossing model for regular A_n-crystals," *Journal of Algebra*, vol. 320, no. 9, pp. 3398–3424, 2008.

[5] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, "Affine crystals and vertex models," in *Infinite Analysis, Part A, B*, vol. 16 of *Advanced Series in Mathematical Physics*, pp. 449–484, World Scientific Publishing, River Edge, NJ, USA, 1992.

[6] J. R. Stembridge, "A local characterization of simply-laced crystals," *Transactions of the American Mathematical Society*, vol. 355, no. 12, pp. 4807–4823, 2003.

[7] I. M. Gelfand and M. L. Tsetlin, "Finite-dimensional representations of the group of unimodular matrices," *Doklady Akademii Nauk SSSR*, vol. 71, pp. 825–828, 1950 (Russian).

[8] M. Kashiwara and T. Nakashima, "Crystal graphs for representations of the q-analogue of classical Lie algebras," *Journal of Algebra*, vol. 165, no. 2, pp. 295–345, 1994.

[9] P. Littelmann, "Paths and root operators in representation theory," *Annals of Mathematics*, vol. 142, no. 3, pp. 499–525, 1995.

[10] P. Littelmann, "Cones, crystals, and patterns," *Transformation Groups*, vol. 3, no. 2, pp. 145–179, 1998.

[11] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy, "On the combinatorial structure of crystals of types A,B,C," http://arxiv.org/abs/1201.4549.

[12] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy, "B_2 -crystals: axioms, structure, models," *Journal of Combinatorial Theory, Series A*, vol. 116, no. 2, pp. 265–289, 2009.