Izvestiya: *Mathematics* **72**:4 689–716

Izvestiya RAN: *Ser. Mat.* **72**:4 67–96 DOI 10.1070/IM2008v072n04ABEH002415

The Robinson–Schensted–Knuth correspondence and the bijections of commutativity and associativity

V. I. Danilov and G. A. Koshevoi

Abstract. The bijections of associativity and commutativity arise from symmetries of the Littlewood–Richardson coefficients. We define these bijections in terms of arrays and show that they coincide with analogous bijections defined in terms of discretely concave functions using the octahedron recurrence as well as with bijections defined in terms of Young tableaux. The main ingredient in the proof of their coincidence is a functional version of the Robinson–Schensted–Knuth correspondence.

§ 1. Introduction

Let n be a fixed integer. A *partition* (with n parts) is an n-tuple $\lambda=(\lambda_1,\dots,\lambda_n)$ of decreasing non-negative integers: $\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_n\geqslant 0$. To each partition λ we associate the so-called Schur polynomial s_λ (see [1]–[3]), a symmetric polynomial in n variables. The polynomials s_λ with different λ form an additive basis in the ring of symmetric polynomials. Hence the product $s_\lambda s_\mu$ of Schur polynomials is a linear combination of Schur polynomials:

$$s_\lambda s_\mu=\sum_\nu c^\nu_{\lambda,\mu}s_\nu.$$

The corresponding structure constants $c^\nu_{\lambda,\mu}$ (which are actually non-negative integers) are called the *Littlewood–Richardson numbers* (*coefficients*). These numbers arise in many problems (the decomposition of tensor products of representations into irreducible components, the product in the Schubert ring, the description of the spectra of sums of Hermitian operators; see [4] for details) and have been studied in many papers. Since the ring of symmetric functions is commutative and associative, the Littlewood–Richardson coefficients satisfy the identities of commutativity

$$c^\nu_{\lambda,\mu}=c^\nu_{\mu,\lambda} \tag{1}$$

and associativity

$$\sum_\sigma c^\sigma_{\lambda,\mu}c^\pi_{\sigma,\nu}=\sum_\tau c^\tau_{\mu,\nu}c^\pi_{\lambda,\tau}. \tag{2}$$

The combinatorial approach suggests looking for finite sets behind integers and natural bijections of these sets behind equations of integers. This is especially fruitful in the case of the above formulae.

This paper was written with the partial financial support of the President's programme for the support of leading scientific schools (grant no. NSh-6417.2006.6) and the Russian Foundation for Basic Research (grants no. 047.011.2004.017-NWO and 05-01-02805-NCNILa).

AMS 2000 *Mathematics Subject Classification.* 05E10, 52C07, 26B25.

Littlewood and Richardson stated (but did not prove[1]) a combinatorial rule for finding the numbers $c^{\nu}_{\lambda,\mu}$. This amounts to interpreting $c^{\nu}_{\lambda,\mu}$ as the number of elements of a finite set $\mathrm{LR}(\nu\setminus\lambda,\mu)$ consisting of Young tableaux of a special form, the so-called Littlewood–Richardson tableaux (see [1] and § 9).

A relation between Young tableaux and functions on the triangular grid Δ_n was found in [3], [5], [6]. It enables one to interpret $c^{\nu}_{\lambda,\mu}$ as the number of integer points in the convex polygon $\mathrm{DC}_n(\lambda,\mu;\nu)$ consisting of discretely concave functions (with given boundary conditions). Another approach was suggested in [3], where this number is interpreted as the number of elements in the set $\mathrm{SP}_{\mathbb{Z}}(\lambda,\mu;\nu)$ of so-called standard pairs (see § 4).

Equation (1) suggests the existence of a natural (or canonical) bijection between the sets $\mathrm{SP}_{\mathbb{Z}}(\lambda,\mu;\nu)$ and $\mathrm{SP}_{\mathbb{Z}}(\mu,\lambda;\nu)$ (and similarly for DC and LR). We gave such a bijection in [3]. A bijection in terms of functions was constructed in [7] at more or less the same time. In terms of Young tableaux, several such bijections were constructed in [8], and the question of their coincidence was also raised there.

Equation (2) similarly suggests the existence of a natural bijection between the sets

$$\coprod_{\sigma}\bigl(\mathrm{SP}_{\mathbb{Z}}(\lambda,\mu;\sigma)\times\mathrm{SP}_{\mathbb{Z}}(\sigma,\nu;\pi)\bigr),\qquad \coprod_{\tau}\bigl(\mathrm{SP}_{\mathbb{Z}}(\mu,\nu;\tau)\times\mathrm{SP}_{\mathbb{Z}}(\lambda,\tau;\pi)\bigr).$$

Such a bijection was constructed in [9] in terms of functions on the triangular grid using the so-called octahedron recurrence. However, the role of this recurrence remained unclear. In [3] we suggested a version of the associativity bijection in terms of arrays. This version looks more natural and well motivated because arrays possess a crystalline structure, they admit tensor products (as do representations), and the array bijection of associativity just reflects the associativity of the tensor product. Conversely, the authors of [7] used functional bijections of commutativity and associativity to define a tensor product in the category HIVES.

In this paper we prove the coincidence of the bijections (of commutativity and associativity) constructed in three ways and three different terminologies. Of course, this implies the existence of a natural identification between the sets SP, DC and LR, which will also be proved. As a by-product, we show that the octahedron recurrence realizes the Robinson–Schensted–Knuth (RSK) correspondence in terms of functions.

Our exposition runs as follows. In § 2 we recall basic notions and facts from the theory of arrays. In § 3 we define the commutativity and associativity bijections in terms of arrays. In §§ 4 and 5 we explain the relation between arrays and functions on two-dimensional and three-dimensional grids. In § 6 we use the octahedron recurrence to define the associativity bijection and prove a functional version of the RSK correspondence. The coincidence of the functional and array bijections is proved in § 7. In § 8 we define the commutativity bijection in terms of functions and prove that it coincides with the array version. In § 9 we prove that this bijection coincides with two fundamental symmetries of Pak and Vallejo [8].

[1] There are now about ten different proofs of this rule. The search for them turned out to be very fruitful and led to the appearance of many interesting theories and techniques, including the theory of arrays used below.

A preliminary version of this paper can be found in [10].

The authors are grateful to A. M. Vershik and I. Pak for useful discussions and to the referee for many remarks. The second author also acknowledges support from the Russian Science Support Foundation.

§ 2. Arrays

Let m and n be non-negative integers. An *array* (of size $n \times m$) is a family $a = (a(i,j))$ of non-negative numbers, where $i = 1, \dots, n$, $j = 1, \dots, m$. The *ith column sum* (resp. the *jth row sum*) of the array is defined to be $\sum_j a(i,j)$ (resp. $\sum_i a(i,j)$). Of course, one can interpret arrays as matrices, but we shall never multiply arrays. It is more appropriate to regard them as masses disposed at the points (i,j) of the plane.

We state the necessary facts (taken from [3]) on arrays.

1. For any $j = 1, \dots, m-1$ and $t > 0$ there are operations D_j^t on the set of arrays. These operations depend only on rows j and $j+1$, and D_j^t transfers some mass (of total amount at most t) from the $(j+1)$st row to the jth row. More precisely, consider a number sequence $\beta_0, \beta_1, \dots, \beta_n$, where $\beta_0 = 0$, $\beta_1 = a(1, j+1)$ and

$$\beta_i = a(1, j+1) + \dots + a(i, j+1) - a(1,j) - \dots - a(i-1, j)$$

for $1 < i \leqslant n$. Let i^* be the index of the first maximum in the sequence $\beta_1, \dots, \beta_n$. If t is sufficiently small (to be exact, $t \leqslant \beta_{i^*} - \max(\beta_0, \beta_1, \dots, \beta_{i^*-1})$), then the operation D_j^t decreases $a(i^*, j+1)$ by t, increases $a(i^*, j)$ by t and does not change other elements of the array. The action of D_j^t for arbitrary t is defined by the semigroup rule

$$D_j^t D_j^s = D_j^{t+s}.$$

This definition implies that D_j^t does not change the column sums. Moreover, the arrays $D_j^t(a)$ stabilize for sufficiently large t (depending on a). We denote the sufficiently large power of D_j by $\mathbf{D}_j$. If the array a is integer-valued (that is, all the $a(i,j)$ are integers), then the array $\mathbf{D}_j a$ is also integer-valued.

2. An array a is said to be $\mathbf{D}_j$-*tight* if the operation $\mathbf{D}_j$ acts trivially on a (that is, $\mathbf{D}_j a = a$ or, equivalently, $D_j^t(a) = a$ for some $t > 0$). Of course, this implies that $\beta_i \leqslant 0$ for all i, that is,

$$a(1, j+1) + \dots + a(i, j+1) \leqslant a(1,j) + \dots + a(i-1, j) \tag{3}$$

for all i.

An array a is said to be $\mathbf{D}$-*tight* if it is $\mathbf{D}_j$-tight for every $j = 1, \dots, m-1$. By inequalities (3), such an array has zero elements above the diagonal: $a(i,j) = 0$ for $j > i$. By making appropriate condensations (that is, acting by $\mathbf{D}_j$), one can transform any array a to a $\mathbf{D}$-tight array. This $\mathbf{D}$-tight array is uniquely determined ([3], Proposition 5.9) and is denoted by $\mathbf{D}a$. The column sums of a and $\mathbf{D}a$ coincide. The row sums of $\mathbf{D}a$ form a non-increasing sequence $\lambda_1 \geqslant \dots \geqslant \lambda_m \geqslant 0$ according to (3). The m-tuple $\lambda = (\lambda_1, \dots, \lambda_m)$ is called the $\mathbf{D}$-*shape* of a.

3. Given an array a, we denote the centrally symmetric array (of the same size) by $*a$:

$$*a(i,j) = a(n+1-i, m+1-j).$$

Using this involution, we can define operations U_j^t by

$$U_j^t(a) = *\big(D_{m-j}^t(*a)\big).$$

The operations U_j^t and D_j^t are almost inverse to each other. More precisely, if $D_j^{t'}(a)$ is different from $D_j^t(a)$ for all $t' < t$, then $U_j^t D_j^t(a) = a$ ([3], Proposition 3.5) and similarly for D_j^t and U_j^t.

As above, we can define an upper-condensation operator $\mathbf{U}$. Then $\mathbf{DU}a = \mathbf{D}a$.

4. Passing to the transposed arrays, we can define the operations L_i^t, R_i^t and the operators $\mathbf{L}$ and $\mathbf{R}$ of left and right condensation. The n-tuple of column sums of $\mathbf{L}a$ is called the $\mathbf{L}$-*shape* of a.

A key result of the theory of arrays states that the operations L_i^t and D_j^s (and hence the operators $\mathbf{L}$ and $\mathbf{D}$) commute ([3], Theorem 4.2). It follows that the $\mathbf{D}$-shape and $\mathbf{L}$-shape of any array a coincide with each other and with the shape of the diagonal array $\mathbf{DL}a = \mathbf{LD}a$. Thus we can speak simply of the *shape* of a and denote it by $\mathrm{sh}(a)$.

5. Another important corollary of the commutation theorem is the bijection theorem (or modified RSK correspondence; see [3], Theorem 6.2). *Suppose that d is a $\mathbf{D}$-tight array, l is an $\mathbf{L}$-tight array, and their shapes coincide* ($\mathbf{D}l = \mathbf{L}d$). *Then there is a unique array a such that $d = \mathbf{D}a$ and $l = \mathbf{L}a$.*

§ 3. Array bijections of associativity and commutativity

As in the introduction, we fix n. Given any arrays a and b of size $n \times n$, we write $a \otimes b$ for the array of size $2n \times n$ defined by

$$(a \otimes b)(i,j) = \begin{cases} a(i,j) & \text{if } i = 1,\dots,n, \\ b(i-n,j) & \text{if } i = n+1,\dots,2n. \end{cases}$$

Hence the arrays a and b are simply juxtaposed, first a then b.

Definition 1. A pair (a,b) of arrays of size $n \times n$ is called a *standard pair* if

1) a and b are $\mathbf{L}$-tight,

2) $a \otimes b$ is $\mathbf{D}$-tight (so that a is also $\mathbf{D}$-tight).

We say that $\mathrm{sh}(a)$ (resp. $\mathrm{sh}(b)$, $\mathrm{sh}(a \otimes b)$) is the *starting* (resp. *intermediate*, *final*) shape of this pair. Note that $\mathrm{sh}(a \otimes b)$ equals the row weight (the vector of row sums) of $a \otimes b$ because of condition 2). The set of all standard pairs with a given triple λ, μ, ν of shapes is denoted by $\mathrm{SP}(\lambda,\mu;\nu)$. The subset of integer-valued standard pairs is indicated by a subscript $\mathbb{Z}$. The set $\mathrm{SP}_{\mathbb{Z}}(\lambda,\mu;\nu)$ is finite and its cardinality equals the Littlewood–Richardson number $c_{\lambda,\mu}^{\nu}$ (see [3], § 12.4).

To define the commutativity bijection between $\mathrm{SP}(\lambda,\mu;\nu)$ and $\mathrm{SP}(\mu,\lambda;\nu)$, it is convenient to use a slightly different set of pairs: *anti-standard pairs*. These are pairs (a,b) of arrays such that

1) a is $\mathbf{R}$-tight, b is $\mathbf{L}$-tight,

2) $a \otimes b$ is $\mathbf{D}$-tight.

The shapes of such a pair are defined as before. The set of anti-standard pairs is denoted by $\mathrm{ASP}(\lambda, \mu; \nu)$. There is a canonical bijection

$$\mathrm{SP}(\lambda, \mu; \nu) \widetilde{\to} \mathrm{ASP}(\lambda, \mu; \nu), \qquad (a, b) \mapsto (\mathbf{R}a, b).$$

Its inverse is the map given by $(a, b) \mapsto (\mathbf{L}a, b)$.

We now define the commutativity bijection (the *commutor*)

$$\mathrm{Com}\colon \mathrm{ASP}(\lambda, \mu; \nu) \to \mathrm{ASP}(\mu, \lambda; \nu)$$

as follows. Let $(a, b) \in \mathrm{ASP}(\lambda, \mu; \nu)$ be an anti-standard pair. Then $\mathrm{Com}(a, b) = \mathbf{D}(*(a \otimes b)) = \mathbf{D}(*b \otimes *a)$.

It is easy to see that the pair $(b', a') = \mathbf{D}(*b \otimes *a)$ is anti-standard of the required type. Indeed, we have $b' = \mathbf{D}(*b)$. Since b is $\mathbf{L}$-tight ($b = \mathbf{L}b$), the array $*b = \mathbf{R}(*b)$ is $\mathbf{R}$-tight. Since the operators $\mathbf{D}$ and $\mathbf{R}$ commute, b' is also $\mathbf{R}$-tight. The shape of b' is equal to the shape of $*b$ and the shape of b, that is, $\mathrm{sh}(*b) = \mu$. The intermediate shape is similarly found to be $\mathrm{sh}(*a) = \mathrm{sh}(a) = \lambda$. The final shape is given by $\mathrm{sh}(*(a \otimes b)) = \mathrm{sh}(a \otimes b) = \nu$.

Example 1. Consider the following concatenated array (a, b):

0	0	0	1	2	1
0	0	2	1	2	0
0	2	1	3	0	0

It is an anti-standard pair with shapes $\lambda = (3, 2, 0)$, $\mu = (5, 4, 1)$ and $\nu = (6, 5, 4)$. The reverse array $*(a \otimes b) = *b \otimes *a$ is given by

0	0	3	1	2	0
0	2	1	2	0	0
1	2	1	0	0	0

Its $\mathbf{D}$-condensation is easily seen to be of the form

0	0	1	1	2	0
0	1	3	1	0	0
1	3	1	1	0	0

Reversing this array and condensing it down, we get the initial array

0	0	0	1	2	1
0	0	2	1	2	0
0	2	1	3	0	0

Of course, this is not accidental. The following general assertion holds.

Lemma 1. *The commutor* Com *is involutory, that is, the composite*

$$\mathrm{ASP}(\lambda, \mu; \nu) \xrightarrow{\mathrm{Com}} \mathrm{ASP}(\mu, \lambda; \nu) \xrightarrow{\mathrm{Com}} \mathrm{ASP}(\lambda, \mu; \nu)$$

is the identity map.

Proof. Put $(b', a') = \mathrm{Com}(a, b) = \mathbf{D}(*(a \otimes b))$. Then

$$\mathrm{Com}(b', a') = \mathbf{D}(*(b' \otimes a')) = \mathbf{D}\big(*\mathbf{D}(*(a \otimes b))\big) = \mathbf{D}(\mathbf{U}(a \otimes b)).$$

According to [3], § 5.10 (see also fact 3 in § 2), the last array is equal to $\mathbf{D}(a \otimes b)$ and hence to $a \otimes b$ because the array $a \otimes b$ is $\mathbf{D}$-tight.

Lemma 1 implies that Com is a bijection (which preserves integer-valued arrays). The corresponding bijection between $\mathrm{SP}(\lambda, \mu; \nu)$ and $\mathrm{SP}(\mu, \lambda; \nu)$ is also referred to as the *commutor*.

To define the associativity bijection, it is convenient to introduce standard triples of arrays. A triple (a, b, c) is said to be *standard* if

1) a, b and c are $\mathbf{L}$-tight,

2) the concatenated array $a \otimes b \otimes c$ is $\mathbf{D}$-tight.

One can obtain two standard pairs from any standard triple. This can be done in two different ways.

First, take the pairs (a, b) and $(\mathbf{L}(a \otimes b), c)$. Clearly, they are standard. It is also clear that *the final shape of the first pair coincides with the starting shape of the second*.

Second, take the pair $(a, \mathbf{L}(b, c))$ and a pair (b', c') such that $b' \otimes c' = \mathbf{D}(b, c)$. The first pair $(a, \mathbf{L}(b, c))$ is obviously standard. For the second pair, the $\mathbf{L}$-tightness of b' and c' follows from the $\mathbf{L}$-tightness of b and c because $\mathbf{D}$ and $\mathbf{L}_i$ commute, and the $\mathbf{D}$-tightness of $a \otimes \mathbf{L}(b, c)$ follows from the same commutation relation. Moreover, *the intermediate shape* $\mathrm{sh}(\mathbf{L}(b \otimes c))$ *of the first pair coincides with the final shape* $\mathrm{sh}(\mathbf{D}(b \otimes c))$ *of the second.*

Both of these ways lead to one-to-one maps. This is obvious for the first map; let us verify it for the second. Consider any standard pairs (a, l) and (b', c') such that l and $b' \otimes c'$ have the same shape. This condition means that $\mathbf{D}l = \mathbf{L}(b' \otimes c')$. By the bijection theorem (see fact 5 in § 2) there are arrays b and c such that $l = \mathbf{L}(b \otimes c)$ and $b' \otimes c' = \mathbf{D}(b \otimes c)$.

Putting all this together, we get the associativity bijection

$$\coprod_{\sigma}\big(\mathrm{SP}(\lambda, \mu; \sigma) \times \mathrm{SP}(\sigma, \nu; \pi)\big) \to \coprod_{\tau}\big(\mathrm{SP}(\mu, \nu; \tau) \times \mathrm{SP}(\lambda, \tau; \pi)\big). \tag{4}$$

It remains to note that this bijection preserves integer-valued arrays. The relation between our constructions and representation theory is explained in [3].

§ 4. The relation between arrays and functions

Let a be an array of size $n \times m$. It determines a function $f = f_a$ on the *rectangular grid* $\{0, 1, \dots, n\} \times \{0, 1, \dots, m\}$ by the law

$$f_a(i, j) = \sum_{i' \leqslant i, j' \leqslant j} a(i', j').$$

This is the total mass situated to the left of and lower than (i, j). Therefore we denote the function f_a symbolically by $\iint a$. This function vanishes on the left and lower boundaries of the rectangle. For all other values of (i, j) we obviously have

$$f(i, j) - f(i - 1, j) - f(i, j - 1) + f(i - 1, j - 1) = a(i, j).$$

Thus we see that $a(i,j)$ is something like a mixed derivative ($a = \partial\partial f$) of the function f. Since $a(i,j) \geqslant 0$, the function $\iint a$ is supermodular. We recall that a function f is said to be *supermodular* if the following inequalities hold for all i, j:

$$f(i,j) - f(i-1,j) - f(i,j-1) + f(i-1,j-1) \geqslant 0. \tag{5}$$

The value of $\iint a$ at a point (n,j) on the right-hand side of the rectangle is equal to the sum of row sums of a over rows $1,\dots,j$. For brevity, we say that the increments of $\iint a$ on the right-hand side of the rectangle are equal to the row sums. In the same vein, the increments of $\iint a$ on the upper face of the rectangle are equal to the column sums.

Now suppose that the array a is **D**-tight or, equivalently, that inequalities (3) hold. In terms of the function $f = \iint a$, this is equivalent to the validity of the inequalities

$$f(i-1,j) + f(i,j) - f(i-1,j-1) - f(i,j+1) \geqslant 0 \tag{6}$$

for all i, j. The corresponding inequalities

$$f(i,j) + f(i,j+1) - f(i-1,j) - f(i+1,j+1) \geqslant 0 \tag{7}$$

for all i, j describe the functions $f = \iint a$ related to **L**-tight arrays a.

To describe these inequalities in a more uniform manner, we divide up the rectangle $[0,n]\times[0,m]$ by the three families of straight lines $x = i$, $y = j$, $x - y = k$, where i, j, k are integers. The rectangle splits into unit triangles with vertices at integer points. Neighbouring triangles are grouped into unit rhombuses as shown in Fig. 1.

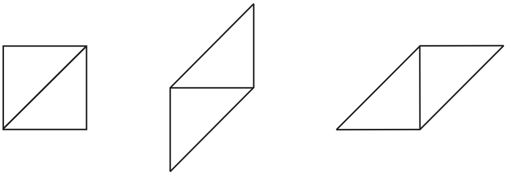

Figure 1

We can interpret (5)–(7) as *rhombic inequalities*: for every rhombus in Fig. 1, the sum of the values at the endpoints of the diagonal shown is less than or equal to the sum of the values at the other two vertices. The inequalities (5) of supermodularity correspond to the square on the left in Fig. 1, the inequalities (6) of **D**-tightness correspond to the rhombus in the middle, and the inequalities (7) of **L**-tightness correspond to the rhombus on the right.

To make the next step towards visual understanding, we extend the function f (defined on the rectangular grid) to a function $\tilde{f}$ defined on the whole rectangle $[0,n]\times[0,m]$. The extension is simply by affinity on each unit triangle. Then every rhombic inequality means that the restriction of $\tilde{f}$ to the corresponding rhombus is concave. If inequalities (5) and (6) hold, we see that $\tilde{f}$ is concave on every vertical strip of width 1. Then the function f is said to be *vertically strip-concave*

(VS-*concave*). In the same vein, if inequalities (5) and (7) hold, then $\tilde{f}$ is concave on every horizontal strip of width 1, and f is said to be *horizontally strip-concave* (HS-*concave*). We see that VS-concave (resp. HS-concave) functions are closely related to **D**-tight (resp. **L**-tight) arrays. Finally, if the inequalities (5)–(7) all hold, then $\tilde{f}$ is concave on the whole rectangle $[0,n]\times[0,m]$. Then the function f is said to be *discretely concave*.

For reasons which will soon be clarified, we are also interested in functions defined on *triangular grids* (of size n). Such a grid is denoted by Δ_n and is defined as the set of integer points in the triangle with vertices $(0,0)$, $(0,n)$ and (n,n) (Fig. 2). A function f on Δ_n is said to be *discretely concave* if the rhombus inequalities (5)–(7) hold for every unit rhombus in this triangle.

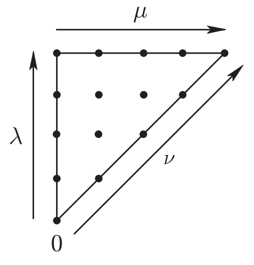

Figure 2

Let f be a discretely concave function on Δ_n. Then its restrictions to the left side, the upper side and the hypotenuse (more precisely, to the integer points of these sides) are discretely concave in the sense that their successive increments decrease. More precisely, the *increments* on the left side are defined as the n-tuple of numbers

$$\lambda(1)=f(0,1)-f(0,0),\ \lambda(2)=f(0,2)-f(0,1),\ \ldots,\ \lambda(n)=f(0,n)-f(0,n-1),$$

and they satisfy

$$\lambda(1)\geqslant\lambda(2)\geqslant\cdots\geqslant\lambda(n).$$

One similarly defines n-tuples μ (where $\mu(i)=f(i,n)-f(i-1,n)$ for $i=1,\ldots,n$) and ν (where $\nu(k)=f(k,k)-f(k-1,k-1)$ for $k=1,\ldots,n$). They are also monotone. These n-tuples are called the *increments* of f on the corresponding sides of the triangular grid (or simply the triangle). They remain unchanged if we add a constant to f. Therefore we usually normalize the function f by requiring that it vanishes at the lower vertex.

We write $\mathrm{DC}_n(\lambda,\mu;\nu)$ for the set of all discretely concave functions on the triangular grid Δ_n with increments λ, μ, ν. This set is a convex polytope in the space of all functions on Δ_n. Discretely concave functions on triangular grids play an important role in the study of the spectra of Hermitian matrices (see [6], [11]). For our purposes, it is important to know that the sets SP (introduced above) are naturally identified with DC_n. Let us explain this in more detail.

Let (a,b) be a standard pair of arrays. We consider the function $f=\iint(a\otimes b)$ on the rectangular grid $\{0,\ldots,2n\}\times\{0,\ldots,n\}$, that is, on the lattice of integer points in the rectangle $[0,2n]\times[0,n]$.

Since $a \otimes b$ is **D**-tight, the function f is VS-concave on the whole rectangle $OBDE$ (Fig. 3). Since b is **L**-tight, the function f is HS-concave on the square $ACDE$.

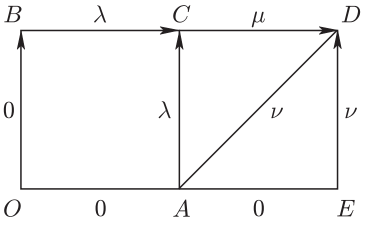

Figure 3

Therefore f is discretely concave on this square. In particular, its restriction to the triangular grid ACD is also discretely concave.

We now examine the increments of this function on the sides of the triangle ACD. The increments on AC are the row-sums of a, that is, the starting shape of (a, b). The increments on CD are the column-sums of b, which are equal to the shape of b (that is, to the intermediate shape of (a, b)) since b is **L**-tight. The increments on ED are equal to the shape of $a \otimes b$, that is, to the final shape of (a, b). Since b is **L**-tight, all the elements of b below the diagonal are equal to 0. Therefore the increments on ED are equal to the increments on the hypotenuse AD. Hence f is discretely concave on the triangular grid ACD (which carries the determining part of f) and has increments $\mathrm{sh}(a)$, $\mathrm{sh}(b)$ and $\mathrm{sh}(a \otimes b)$ on this grid. Thus we have defined a map

$$\mathrm{SP}(\lambda, \mu; \nu) \to \mathrm{DC}_n(\lambda, \mu; \nu).$$

This map is obviously linear and sends integer arrays to integer-valued functions. Moreover, it is invertible: given a function f in $\mathrm{DC}_n(\lambda, \mu; \nu)$, we can take $\partial\partial f$ for b and the diagonal array $\mathrm{diag}(\lambda)$ for a. This proves the following proposition.

Proposition 1. *The map* $\mathrm{SP}(\lambda, \mu; \nu) \to \mathrm{DC}_n(\lambda, \mu; \nu)$ *is bijective* (*and is called the canonical bijection*).

In what follows we use this canonical identification to prove the coincidence of the array bijections of commutativity and associativity with their functional counterparts. To explain the construction of the functional bijections, we use the so-called *octahedron recurrence* as a main ingredient. This recurrence is defined in three-dimensional space. Therefore we consider functions defined on the three-dimensional lattice $\mathbb{Z}^3$.

§ 5. Functions on the three-dimensional grid

Here we consider functions defined at the points of the three-dimensional lattice $\mathbb{Z}^3$. We cut the space $\mathbb{R}^3$ by four families of 'modular' planes of the form $x = i$, $y = j$, $z = k$ and $x + y + z = l$, where i, j, k, $l \in \mathbb{Z}$. Then the space is divided into tetrahedra and octahedra of unit size. The octahedra all have the same structure and are obtained from each other by integer shifts. Each octahedron has three diagonals (parallel to the vectors $(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)$) and three corresponding pairs of opposite vertices.

The diagonal parallel to $(-1,1,1)$ is said to be *principal*, and the vector $(-1,1,1)$ is called the *propagation vector*.

Definition 2. A function F on the three-dimensional lattice is said to be *polarized* if, for every unit octahedron, the sum of the values of F at the vertices of the principal diagonal is equal to the maximum of the sum of the values of F at the vertices of the other two, that is, if

$$\begin{aligned} &F(i,j,k) + F(i+1,j-1,k-1) \\ &\quad = \max\bigl(F(i,j-1,k) + F(i+1,j,k-1), F(i,j,k-1) + F(i+1,j-1,k)\bigr) \end{aligned} \tag{8}$$

for all i, j, k.

The notion of a polarized function makes sense for functions defined at integer points of any convex polyhedron composed of tetrahedra and octahedra of unit size. An example is given by the set $\Delta_n(OXYZ)$ of all integer points (x,y,z) in the tetrahedron of size n defined by $x,y,z \geqslant 0$, $x+y+z \leqslant n$ (Fig. 4)

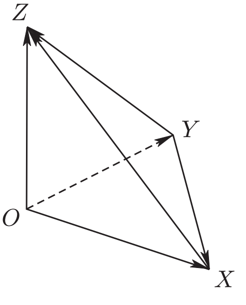

Figure 4

Let F be a polarized function on the grid $\Delta_n(OXYZ)$. It is easy to see that F is uniquely determined by its values on the *floor* $\Delta_n(OXY)$ and the *front face* $\Delta_n(OXZ)$. Indeed, consider a point $(i,1,1)$ of the grid. It belongs to the unit octahedron with vertices $(i,1,1)$, $(i,0,1)$, $(i,1,0)$, $(i+1,0,1)$, $(i+1,1,0)$ and $(i+1,0,0)$. All these vertices except for $(i,1,1)$ lie either on the floor or on the front face. Hence we know the values of F at these vertices. Since F is polarized, we have

$$F(i,1,1) + F(i+1,0,0) = \max\bigl(F(i,0,1) + F(i+1,1,0), F(i,1,0) + F(i+1,0,1)\bigr),$$

which determines the value at $(i,1,1)$. Repeating this argument for points of the form $(i,2,1)$ with $0 \leqslant i \leqslant n-3$, we find the values of F at these points, and similarly for all points $(i,j,1)$ with $i+j \leqslant n-1$. This determines the values of F at all points with $k=1$. It remains to repeat this argument for $k = 2,3,\dots$.

We see that one can arbitrarily prescribe the values of the function on the floor and front face and then, using the polarization conditions (8), extend these values to all points of the grid $\Delta_n(OXYZ)$. This propagation of the function is known as the *octahedron recurrence* (OR). In some sense, this is a tropicalization of a method (suggested by Dodgson and Jacobi) for the calculation of determinants (see [9], [12], [13]).

We would like to mention that the property of being polarized may be regarded as a kind of concavity of F on unit octahedra. Fig. 5 shows such a unit octahedron with vertices $\mathbf{0}$, a, a', b, b' and $\mathbf{1}$. The vertices of the principal diagonal are $\mathbf{0}$ and $\mathbf{1}$.

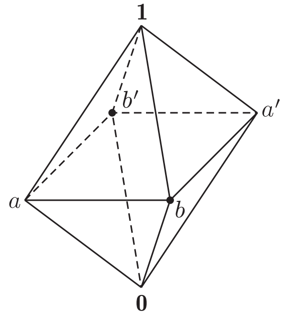

Figure 5

The polarization condition (8) means that

$$F(\mathbf{1}) + F(\mathbf{0}) = \max\bigl(F(a) + F(a'), F(b) + F(b')\bigr).$$

Suppose that the maximum is attained at $F(b) + F(b')$. Then we have $F(\mathbf{0}) + F(\mathbf{1}) = F(b) + F(b')$ and hence F can be extended to an affine function on the half-octahedron $a\mathbf{01}bb'$ as well as to an affine function on the half-octahedron $a'\mathbf{01}bb'$. It is easy to see that the resulting piecewise-linear function $\widetilde{F}$ on the whole octahedron is concave.

Thus the property of being polarized means that F can be extended to a function which is concave on each unit octahedron and affine on each of the two rhombuses containing the principal diagonal (the propagation vector). It follows that this property is not affected if we add a linear function to F. Moreover, we can add to F any function of the form $\alpha(x) + \beta(y) + \gamma(z) + \delta(x + y + z)$.

Extending a polarized function F to all unit octahedra in a piecewise-linear manner (as above) and to all unit tetrahedra by affinity, we get a function $\widetilde{F}$ on the whole simplex $OXYZ$. The requirement that $\widetilde{F}$ be concave may be regarded as a kind of discrete concavity for F. This requirement may easily be restated in terms of the initial function F. We recall that the definition of two-dimensional discretely concave functions involves three series of rhombic inequalities. Proceeding in a similar way, we have four series of 'modular' planes ($x = i$, $y = j$, $z = k$, $x + y + z = l$) and rhombuses of three kinds on each plane. The corresponding rhombic inequality must hold for each of these rhombuses: the sum of the values of the function on the 'short' diagonal must be less than or equal to the sum of values on the 'long' diagonal.

Definition 3. A function F on the three-dimensional grid $\Delta_n(OXYZ)$ is said to be *polarized discretely concave* if it is polarized and the rhombic inequalities hold for all 'modular' rhombuses in the simplex $OXYZ$.

Theorem 1. *Let F be a polarized function on the grid $\Delta_n(OXYZ)$. If the restrictions of F to the floor $\Delta_n(OXY)$ and front face $\Delta_n(OXZ)$ are discretely concave, then F is polarized discretely concave.*

We prove Theorem 1 in the appendix (§ 10). Note that this theorem is equivalent to Corollary 1 (also proved in [7]).

Corollary 1. *If a polarized function is discretely concave on the floor and front face of the tetrahedron, then it is discretely concave on the other two faces.*

Indeed, any rhombus on these faces is a rhombus in the three-dimensional grid $\Delta_n(OXYZ)$, and we get the corresponding rhombic inequality.

By adding appropriate separable functions, we get Corollaries 2–4 below (which are not used in this paper).

Corollary 2. *If a polarized function F is discretely concave on the floor and* HS-*concave on the front face, then it is* HS-*concave on the other two triangles.*

Corollary 3. *If a polarized function F is* VS-*concave on the front face and* HS-*concave on the floor* (*where the interval XY is regarded as horizontal*), *then it is* VS-*concave on the back face $\Delta(OYZ)$.*

Corollary 4. *If a polarized function F is discretely concave on the floor and* VS-*concave on the front face, then it is discretely concave on the back face.*

We shall also use a version of the OR for functions on the prism $\Delta_n(OXY) \times \{0, 1, \dots, m\}$. Let F be a function defined at the integer points of this prism. We say that F is *polarized* if it satisfies the polarization condition on all unit octahedra lying in the prism and is separable (with respect to the variables x and z) on the 'non-modular' face $\Delta_n(XY) \times \{0, 1, \dots, m\}$. In other words, for every unit square in this face, the sums of values of the function on the two diagonals coincide:

$$F(i, j, k) + F(i + 1, j - 1, k + 1) = F(i + 1, j - 1, k) + F(i, j, k + 1).$$

Such a function is said to be *polarized discretely concave* if the rhombic inequalities hold for all rhombuses lying in the prism.

Corollary 5. *Let F be a polarized function on the prism $\Delta_n(OXY) \times \{0, 1, \dots, m\}$. If the restrictions of F to the floor $\Delta_n(OXY) \times \{0\}$ and front face $\Delta_n(OX) \times \{0, 1, \dots, m\}$ are discretely concave, then F is polarized discretely concave.* (*In particular, it is discretely concave on the back face $\Delta_n(OY) \times \{0, 1, \dots, m\}$ and the roof $\Delta_n(OXY) \times \{m\}$.*)

Proof. We easily see that it suffices to consider the case $m = 2$. However, we start with the case $m = 1$. The idea is to enlarge the base to size $n + 1$ by adding new points $(n + 1, 0, 0), \dots, (0, n + 1, 0)$. By prescribing sufficiently small values of F at these points, we see that the extended function is discretely concave on the 'enlarged' floor $\Delta_{n+1}(OXY) \times \{0\}$ and the 'enlarged' front face. Let $\widetilde{F}$

be the polarized function with these initial data. By Theorem 1, $\widetilde{F}$ is polarized discretely concave. We claim that its restriction to the prism is also polarized. To establish this, we must verify that it coincides with F on the non-modular face $\Delta_n(XY) \times \{0,1\}$, but this follows since the values of $\widetilde{F}$ at the new points are sufficiently small. Therefore F and $\widetilde{F}$ coincide on the prism. Since $\widetilde{F}$ is discretely concave, so is F.

We now consider the case $m = 2$. We must verify the rhombic inequalities for all rhombuses in the prism. We first consider the rhombuses of vertical size 2. Such a rhombus is easily seen to lie in a tetrahedron of size $n + 1$. But then the corresponding rhombic inequality follows since it holds for $\widetilde{F}$. The remaining rhombuses lie either in the prism $\Delta_n(OXY) \times \{0,1\}$ or in the prism $\Delta_n(OXY) \times \{1,2\}$. If the rhombus lies in the first 'lower' prism, then the rhombic inequality follows from the result for the case $m = 1$. In particular, F is discretely concave on the triangle $\Delta_n(OXY) \times \{1\}$. Applying the case $m = 1$ to the prism $\Delta_n(OXY) \times \{1,2\}$, we see that the rhombic inequalities hold for all rhombuses in this prism.

§ 6. The functional form of the associativity bijection

Consider the three-dimensional grid $\Delta_n(OXYZ)$. Suppose that functions F_Z and F_Y are defined on the floor $\Delta_n(OXY)$ and front face $\Delta_n(OXZ)$ and their restrictions to the edge $\Delta_n(OX)$ coincide. Consider the polarized extension of F to the whole three-dimensional grid (which, as proved above, is unique). Let F_X and F_O be the restrictions of F to the back face $\Delta_n(OYZ)$ and inclined face $\Delta_n(XYZ)$. We get again two functions that coincide on the edge $\Delta_n(YZ)$. (Of course, this operation is invertible: one can start with F_X and F_O and get F_Z and F_Y.)

This transformation enables us to define the functional form of the associativity bijection. Namely, we place a discretely concave function with increments λ, μ and σ on the floor $\Delta_n(OXY)$ as shown in Fig. 6, and a discretely concave function

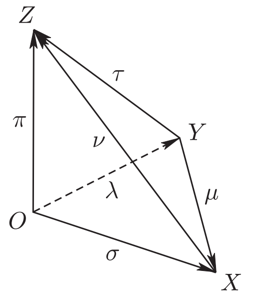

Figure 6

with increments σ, ν and π on the front face $\Delta_n(OXZ)$. Since the increments coincide on the edge $\Delta_n(OX)$, we can assume that the functions coincide there. Using Theorem 1 (or rather Corollary 1), we get discretely concave functions with increments μ, ν, τ and λ, τ, π on the remaining two faces, and conversely. Thus we get the *functional bijection of associativity*

$$\coprod_{\sigma} \mathrm{DC}(\lambda,\mu;\sigma)\times \mathrm{DC}(\sigma,\nu;\pi)\to \coprod_{\tau} \mathrm{DC}(\mu,\nu;\tau)\times \mathrm{DC}(\lambda,\tau;\pi). \tag{9}$$

This is precisely the bijection constructed in [9].

We aim at showing that this bijection becomes the array bijection of associativity (studied in § 3) if we canonically identify DC with SP. The constructions of this identification and of the associativity bijection lead us to the problem of comparing the polarization condition and the array condensation operations.

It is convenient to consider the prism $OAECBD$ shown in Fig. 7 and the integer

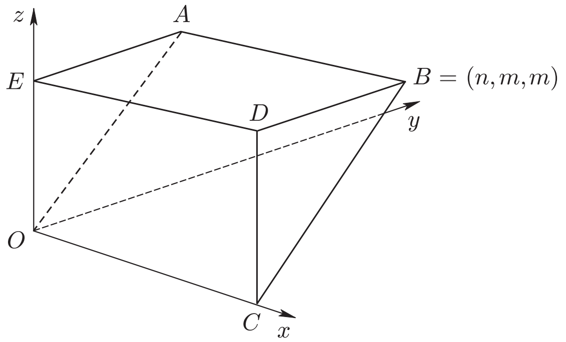

Figure 7

points inside it. The modular planes are parallel to the planes $x = 0$, $z = 0$, $x = y$ and $z = y$. We note that the face $OEDC$ of the prism lies in the non-modular plane $y = 0$. The propagation vector $(1, 0, 1)$ belongs to this non-modular plane, as expected. To pass to the previous notation, take the vectors $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 0)$ for the basis. Then the propagation vector is $(1, 0, 1) = -(0, 1, 0) + (0, 0, 1) + (1, 1, 0)$, and condition (8) is rewritten as

$$F(x,y,z)+F(x+1,y,z+1)=\max\bigl(F(x,y,z+1)+F(x+1,y,z), F(x,y-1,z) + F(x+1,y+1,z+1)\bigr). \tag{10}$$

The initial data will be specified on the triangle OAE and the inclined rectangle $OABC$. These data are extended to the non-modular rectangle by separability: we assume that the value of the function at a point $(i, 0, k)$ equals the sum of its values at $(i, 0, 0)$ and $(0, 0, k)$. Using the OR, we successively extend these initial data to all integer points of the prism. In particular, we get the output data on the upper rectangle $EABD$ and the right triangle CDB.

Now let a be an array of size $n \times m$. We specify the zero function on the triangle OAE (of course, at the integer points of this triangle) and the function $\iint a$ on the inclined rectangle $OABC$. More precisely, the value at (i, j, j) is declared to be $\iint a(i, j)$. (Therefore the function is equal to zero on the non-modular rectangle $OEDC$.) Using the OR, we get functions on the upper rectangle $EABD$ and the right triangle CDB. We see from Corollary 5 that the function on the upper rectangle is VS-concave and the function on the triangle CDB is HS-concave. We claim that the former function equals $\iint \mathbf{D}a$ and the latter equals the restriction of $\iint \mathbf{L}a$ to this triangle.

More precisely, the following theorem holds.

Theorem 2. *Let F be a polarized function on the prism $OAECBD$. Suppose that F vanishes on the triangle OEA and equals $\iint a$ on $OABC$, that is, $F(i,j,j) = (\iint a)(i,j)$. Then $F(i,j,m) = (\iint \mathbf{D}a)(i,j)$ for all i,j and $F(n,j,k) = (\iint \mathbf{L}a)(j,k)$ for $0 \leqslant j \leqslant k \leqslant m$.*

Proof. We first consider the case of a two-row array, when $m = 2$. The lower (resp. upper) row is filled with the numbers $a(i,1)$ (resp. $a(i,2)$). Therefore

$$\begin{aligned} f(i,0) &= 0, \\ f(i,1) &= a(1,1) + \cdots + a(i,1), \\ f(i,2) &= f(i,1) + a(1,2) + \cdots + a(i,2). \end{aligned}$$

This is the function $\iint a$. We put $a' = \mathbf{D}a$ and $f' = \iint a'$. The function f' coincides with f on the lower boundary (where they both vanish) and on the upper boundary for $j = 2$. Changes occur only for $j = 1$ because some mass is transferred from the second layer to the first. It is shown in formula $(***)$ of [3] that $f'(i,1) = f(i,1) + \max(\beta_1, \ldots, \beta_i)$, where

$$\begin{aligned} \beta_i &= a(1,2) + \cdots + a(i,2) - a(1,1) - \cdots - a(i-1,1) \\ &= f(i,2) - f(i,1) - f(i-1,1) + f(i-1,0). \end{aligned}$$

We have $f(i,2) = f'(i,2)$, $\max(\beta_1, \ldots, \beta_i) = \max\bigl(\max(\beta_1, \ldots, \beta_{i-1}), \beta_i\bigr)$ and

$$\max(\beta_1, \ldots, \beta_{i-1}) = f'(i-1,1) - f(i-1,1).$$

Using these equations, we get

$$\begin{aligned} f'(i,1) = f(i,1) + \max[&f'(i-1,1) - f(i-1,1), f'(i,2) \\ &- f(i,1) - f(i-1,1) + f(i-1,0)] \\ = \max[&f'(i-1,1) + f(i,1), f'(i,2) + f(i-1,0)] - f(i-1,1). \end{aligned}$$

This is precisely formula (10) for the OR since $F(i,1,1) = f(i,1)$ and $F(i,1,2) = f'(i,1)$.

We can now use induction to show that the function at the kth level (that is, at height $z = k$ in the prism $OEACDB$) is equal to $\iint \mathbf{D}(a_k)$, where the array a_k is obtained from a by omitting the upper rows numbered $k+1, \ldots, m$.

Indeed, suppose that this holds for k. Let us verify it for $k+1$. We have a $\mathbf{D}$-tight array $d_k = \mathbf{D}(a_k)$ beneath the row $a(\cdot, k+1)$ of the old array. Performing the octahedron recurrence at this level corresponds to performing the product of the operations $\mathbf{D}_1, \ldots, \mathbf{D}_k$. But this is precisely what is given by the formula in § 6.5 of [3]. In the end, we get the function $\iint \mathbf{D}a$ at the upper base. This proves the first part of the theorem.

It remains to show that we get $\iint \mathbf{L}a$ on the right side. But we have just seen that the values on the interval $(n,\cdot,k)$ are equal to the values of the function $\iint d_k = \iint \mathbf{D}(a_k)$. Hence $\int d_k$ is the integral of the shape of d_k (the integral of a number sequence $(\lambda_1,\lambda_2,\dots)$ is the sequence $(0,\lambda_1,\lambda_1+\lambda_2,\dots)$) or, equivalently, the integral of the shape of a_k (see [3], § 5.10). But the shape of a_k equals that of $\mathbf{L}(a_k)=\mathbf{L}(a)_k$, and the shape of an $\mathbf{L}$-tight array coincides with the sequence of its column-sums. This proves the second assertion of the theorem.

In the following sections we apply Theorem 2 to the bijections of associativity and commutativity.

§ 7. Coincidence of the associativity bijections

Theorem 3. *If we identify* SP *with* DC, *then the associativity bijection* (4) *coincides with the associativity bijection* (9).

Proof. Take a standard pair (b,c) of arrays and the corresponding function $\iint b\otimes c$ on the rectangular grid of size $2n\times n$. We place this grid on the inclined rectangle $OABC$ of the prism $OAECBD$ (Fig. 8) and thus define a function on $OABC$.

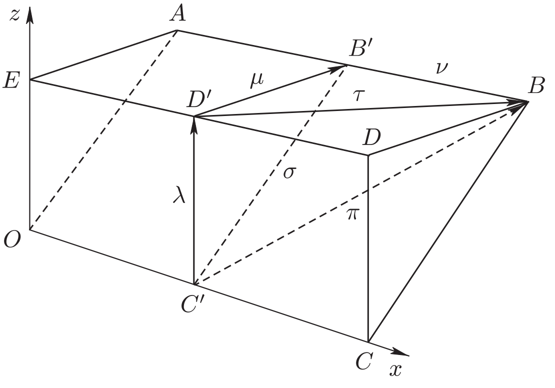

Figure 8

First, we are interested in the restrictions of the polarized function to the triangles $C'D'B'$, $D'B'B$, $B'BC'$ and $BC'D'$, that is, to the faces of the tetrahedron $C'D'B'B$. The point is that the octahedron recurrence from the definition of the bijection (9) occurs just on this tetrahedron (with the same propagation vector).

The first face is the triangle $C'D'B'$. By Theorem 2 the polarized function equals $\iint \mathbf{L}b=b$ on this face.

On the second face $D'B'B$, our function equals the restriction of $\iint(b'\otimes c')$ to this triangle (or to the square $D'B'BD$), where we write $b'\otimes c'$ as an abbreviation for $\mathbf{D}(b\otimes c)$. Hence this restriction equals $\iint c'$ up to adding the function $\int b'$ of the variable y.

On the third (inclined) face $B'BC'$ of the tetrahedron, our function equals the restriction of $\iint(b\otimes c)$ to this piece of the inclined rectangle. As in the previous case, we easily see that this function equals the sum of $\iint c$ and a one-dimensional function (the integral of the row-weight of b). For future use, we mention that the row-weight of b equals the difference between the row-weights of $a \otimes b$ and a, or the difference between the shapes of $\mathbf{L}(a, b)$ and a.

The structure of the polarized function on the fourth face $BC'D'$ of the tetrahedron $C'D'B'B$ is non-obvious. The answer is that *our function on $BC'D'$ essentially coincides with the same function on BCD, that is, with $\iint \mathbf{L}(b \otimes c)$.* More precisely, the value of our function at a point $(n+i, i, k)$ equals the value of $\iint \mathbf{L}(b \otimes c)$ at (i, k).

We postpone the proof of this auxiliary assertion till the end of the proof of the theorem. We now complete the comparison of the resulting function and the function obtained from the associativity bijection (9). On the first and third faces, we have the following initial data: the functions $\iint b$ on the first face and $\iint c$ (up to adding a one-dimensional function) on the third. On the second and fourth faces, our functions are $\iint c' + \int b'$ and $\iint \mathbf{L}(b \otimes c)$ respectively. These are basically the same functions as in (9).

To get precisely the same functions, we must add the function $\int a$ or $\int \lambda$ of the third (vertical) coordinate to our polarized three-dimensional function on the prism $OEACDB$. The result is as follows. On the first face, we get the discretely concave function $\int \lambda + \iint b$ with boundary increments λ, μ and σ, where σ is the shape of $a \otimes b$. On the third face, we get the function $\iint c + \bigl(\int l - \int a\bigr) + \int a = \iint c + \int l$. Since $l = \mathbf{L}(a \otimes b)$, we have $\int l = \int \sigma$. Hence our function on the third face is discretely concave with boundary increments σ, ν and π, where π is the shape of (a, b, c). On the second face, our function is given by $\iint c' + \int b' = \iint c' + \int \mu$ (because $b' = \mathbf{D}b$ has shape μ). Hence it is discretely concave with boundary increments μ, ν and τ, where τ is the shape of $b \otimes c$. On the fourth face, our function equals $\iint \mathbf{L}(b \otimes c) + \int \lambda$. This is the discretely concave function corresponding to the standard pair $\bigl(a, \mathbf{L}(b \otimes c)\bigr)$ with boundary increments λ, τ and π.

It remains to note that the octahedron recurrence is not affected by adding the function $\int \lambda$ of the vertical coordinate. Hence we see that the bijection (4) coincides with the bijection (9).

We now prove the auxiliary assertion. To do this, we note that the three-dimensional function (after adding $\int \lambda$) is polarized discretely concave on the pyramid $C'D'DCB$ (Corollary 5). Moreover, it is constant on every interval that lies in the base of the pyramid and is parallel to the x-axis. Hence it is constant on any interval parallel to the x-axis. The theorem is proved.

§ 8. The functional form of the commutativity bijection

In this section we describe the commutativity bijection in functional terminology. It will then be clear that this bijection coincides with the commutativity bijection constructed in [7].

In § 3 we constructed the commutativity bijection

$$\mathrm{Com}\colon \mathrm{ASP}(\lambda, \mu; \nu) \to \mathrm{ASP}(\mu, \lambda; \nu).$$

Using the canonical identifications of ASP with SP and SP with DC, we get the bijection

$$\mathrm{Com}'\colon \mathrm{DC}_n(\lambda,\mu;\nu) \to \mathrm{DC}_n(\mu,\lambda;\nu).$$

Let f be a discretely concave function with boundary increments λ, μ, ν. We consider the $\mathbf{L}$-tight array $b = \partial\partial f$ (situated above the diagonal). Its vector of column-sums (I-weight) equals μ and its vector of row-sums (J-weight) equals $\nu - \lambda$. Using only λ, we construct the unique $\mathbf{DR}$-tight array $a = \mathbf{R}(\mathrm{diag}(\lambda))$ of shape λ. The resulting (anti-standard) pair (a, b) is precisely that corresponding to the function f.

We now must define the bijection Com in terms of ASP. To do this, we form the (anti-standard) pair $(b', a') = \mathbf{D}(*b \otimes *a)$. (We know the structure of b': it is the unique $\mathbf{DR}$-tight array of shape μ. The column-weight of a' is also known: it is equal to the column-weight of $*a$. Since the weight of a equals $\lambda^{\mathrm{op}} = (\lambda_n, \ldots, \lambda_1)$, the weight of $*a$ equals λ. This can also be proved in another way: since $*a$ is $\mathbf{L}$-tight, its column-weight is equal to the shape of $*a$, which is in turn equal to the shape of a, that is, to λ.) To construct a function from this pair, we define

$$\mathrm{Com}'(f) = \iint a' + \int \mu.$$

By Theorem 2 we know the structure of condensation down in terms of functions. One must take the function g corresponding to the array $*b \otimes *a$, place it on the inclined face $OABC$ (Fig. 9), prescribe zero values on the faces OEA and $OEDC$

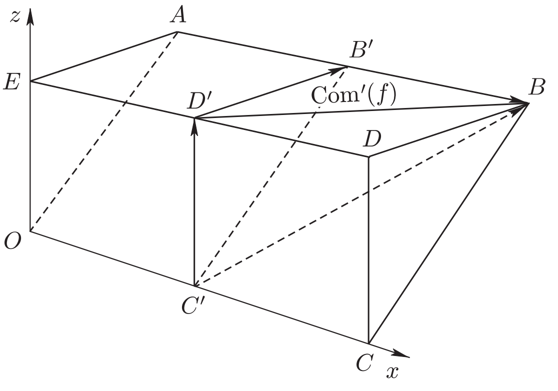

Figure 9

and perform the octahedron recurrence. Then the desired function will appear on the upper base $EABD$. More precisely, the function $\mathrm{Com}'(f)$ will be situated (defined) precisely on the triangle $D'B'B$ (the order of the vertices is important). Its type is determined by the boundary increments μ, λ, ν.

There arises the question of how to describe the function $g = \iint(*b \otimes *a)$. Clearly, it must be related to the initial function f in a simple way. More precisely, we shall establish its connection with the function $\iint(a \otimes b)$ on the doubled rectangle $2n \times n$. Since the restriction of g to the triangle $C'B'B$ equals f, we shall denote the function $\iint(a \otimes b)$ again by f. Its boundary increments are shown in Fig. 10.

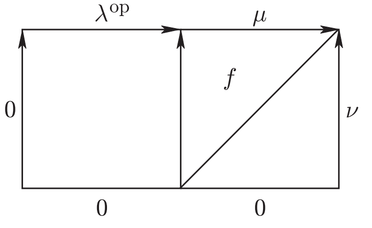

Figure 10

We now 'turn over' the function f, that is, we define a function $*f$ on the same rectangle by $(*f)(i,j) = f(2n-i, n-j)$. We almost get the function g. More precisely, $*f$ and g have the same $\partial\partial$ but different boundary increments (and values at $(0,0)$). This can be seen from Fig. 11, which depicts the function $*f$.

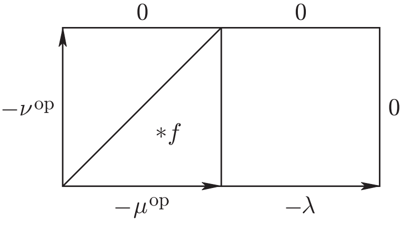

Figure 11

These boundary conditions can easily be corrected by adding an appropriate separable function (in the variables x and $z = y$). Thus we get

$$g = *f + \int_x (\mu^{\mathrm{op}}, \lambda) + \int_z \nu^{\mathrm{op}} - f(0,0) = *f + \int_x (\mu^{\mathrm{op}}, \lambda) + \int_z \nu^{\mathrm{op}} - |\nu|.$$

Here $(\mu^{\mathrm{op}}, \lambda) = (\mu_n, \ldots, \mu_1, \lambda_1, \ldots, \lambda_n)$, $|\nu| = \nu_1 + \cdots + \nu_n$. For future use, we mention that g (Fig. 12) is constant on the intervals $[(n+j, j), (2n, j)]$ because the array $*a$ is empty (zero) at the corresponding cells.

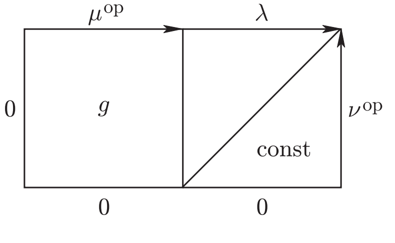

Figure 12

Thus we place the function g on the inclined rectangle $OABC$ (see Fig. 9) and perform the octahedron recurrence. We get a three-dimensional function G on the prism. The desired function $\mathrm{Com}'(f)$ is defined at the integer points of the triangle $D'B'B$. Its boundary increments are equal to μ, λ, ν, as is easily seen by considering the $\mathbf{D}$-tight array $\mathbf{D}(*b \otimes *a)$.

For reasons which will soon be clarified, we are also interested in the values of G on the vertical face $C'D'B'$. We know that the function $\iint \mathbf{L}(*b)$ is defined on this face and has boundary increments 0, μ and $\nu^{\mathrm{op}} - \lambda^{\mathrm{op}} = (\nu - \lambda)^{\mathrm{op}}$. Adding the 'one-dimensional' function $\int_z (-\nu^{\mathrm{op}})$, we put

$$h = \iint \mathbf{L}(*b) - \int_z \nu^{\mathrm{op}} + |\nu|. \tag{11}$$

We see that this function has increments $-\nu^{\mathrm{op}}$, μ and $-\lambda^{\mathrm{op}}$.

The following assertion describes the relation between this intermediate function h (Fig. 13) and the desired function $\mathrm{Com}'(f)$.

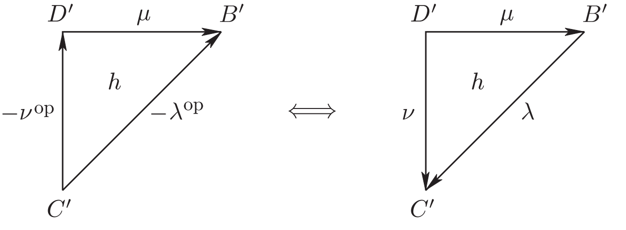

Figure 13

Proposition 2. 1) *The octahedron recurrence on a quarter of the octahedron* (*the Henriques–Kamnitzer construction* [7]) *transforms the function* $*f$ *on* $OB'C'$ *to* h. *In other words*, *we have* $h = HK(*f)$ *up to adding a constant.*

2) *The function* h *basically coincides with* $\mathrm{Com}'(f)$. *More precisely*, *if we regard* $\mathrm{Com}'(f)$ *as a function on the triangle* $D'B'B$, *then* $h(i,j) = \mathrm{Com}'(f)(n-j, n-j+i, n)$.

In other words, the function $\mathrm{Com}'(f)$ is obtained by 'rotating the function h anticlockwise through one third of a complete turn', as shown in Fig. 14.

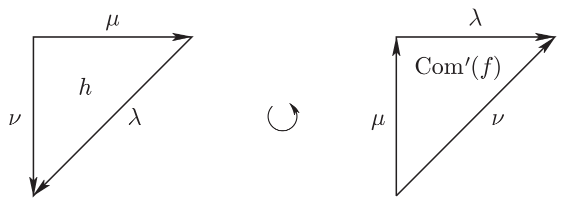

Figure 14

Corollary 6. *In terms of discretely concave functions, the commutor* Com' *coincides with the Henriques–Kamnitzer commutor.*

Indeed, Henriques and Kamnitzer ([7], Remark 4.5) assert that their commutor is obtained by performing the octahedron recurrence on a quarter of the octahedron and then rotating the picture through one third of a complete turn, just as shown in Fig. 14.

Example 2. We illustrate the action of the commutor in Example 1 of § 3. Consider the function g corresponding to the array. (We define g on the inclined face of the prism.) This function is easily seen to have the following values:

$$\begin{matrix} 0 & 1 & 5 & 10 & 13 & 15 & 15 \\ 0 & 1 & 5 & 7 & 9 & 9 & 9 \\ 0 & 1 & 3 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Performing the octahedron recurrence, we get the following function on the upper base of the prism:

$$\begin{matrix} 0 & 1 & 5 & 10 & 13 & 15 & 15 \\ 0 & 1 & 5 & 9 & 11 & 11 & 11 \\ 0 & 1 & 4 & 5 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

In particular, we get $\mathrm{Com}'(f)$:

$$\begin{matrix} 10 & 13 & 15 & 15 \\ 9 & 11 & 11 & \\ 5 & 6 & & \\ 0 & & & \end{matrix}$$

On the triangle $C'D'B'$ we get a function with values

$$\begin{matrix} 0 & 5 & 9 & 10 \\ 0 & 5 & 7 & \\ 0 & 4 & & \\ 0 & & & \end{matrix}$$

which becomes h after adding the vertical function $-\int \nu + |\nu|$:

$$\begin{matrix} 0 & 5 & 9 & 10 \\ 6 & 11 & 13 & \\ 11 & 15 & & \\ 15 & & & \end{matrix}$$

We see that h coincides with $\mathrm{Com}'(f)$ after a rotation through one third of a complete turn.

Proof of Proposition 2. 1) The function $\iint \mathbf{L}(*b)$ is obtained from the (initial piece of) the function g by means of the OR. But the OR commutes with the operation of adding a separable function (in the variables x and z). Therefore we subtract the separable function $\int_x(\mu^{\mathrm{op}},\lambda)+\int_z\nu^{\mathrm{op}}-|\nu|$ from g. Then we get $*f$ on the inclined rectangle. On the face $C'D'B'$ (see Fig. 8) we subtract the same function $\int_z\nu^{\mathrm{op}}$ (up to a constant) from $\iint\mathbf{L}(*b)$. As a result, we get h.

Thus, the OR on a quarter of the octahedron transforms $*f$ to h.

2) The function $\mathrm{Com}'(f)$ on the upper base is obtained from g by means of the OR. Thus adding the function $-\int_z\nu^{\mathrm{op}}+|\nu|$ to G changes nothing on the upper base $EABD$.

Consider the restriction of the three-dimensional function $G-\int_z\nu^{\mathrm{op}}+|\nu|$ to the tetrahedron $C'D'B'B$ (see Fig. 8). First, this function is polarized. Second, its value $\iint\mathbf{L}(*b)-\int_z\nu^{\mathrm{op}}+|\nu|=h$ on the face $C'D'B'$ is discretely concave (because it coincides, up to an affine function, with the function corresponding to the standard pair $\bigl(-\nu^{\mathrm{op}},\mathbf{L}(*b)\bigr)$). Third, it is discretely concave on the face $C'B'B$ and equals the constant $|\nu|$ on the edge $C'B$. Let us verify the last assertion.

We consider the function g on the square $C'B'BC$ (see Fig. 8). It is equal to the sum of $\iint(*a)$ and $\int_z(\nu^{\mathrm{op}}-\lambda^{\mathrm{op}})$. We are interested in the function $g-\int_z\nu^{\mathrm{op}}$, which equals $\iint(*a)-\int_z\lambda^{\mathrm{op}}$ by what has been said above. It remains to use the explicit form of $*a$ or a, which yields that a is **DR**-tight and hence $*a$ is **LU**-tight.

We can assume that λ has a very special form, namely $(1,\dots,1,0,\dots,0)=(1^k,0^{n-k})$. Then we can find the function $\iint(*a)-\int_z\lambda^{\mathrm{op}}$ explicitly and see that it equals $\min(k+i-j,0)$. Thus the assertion on discrete concavity becomes obvious. The values of this function on the interval $C'B'$ (that is, for $i=j$) are given by $\min(k,0)=0$ because $k\geqslant 0$.

Thus the polarized function $G-\int_z\nu^{\mathrm{op}}+|\nu|$ takes discretely concave values on the faces $C'D'B'$ and $C'B'B$ of the tetrahedron $C'D'B'B$. By Theorem 1 this function is polarized discretely concave. Since it is constant on the interval $C'B'$, it is also constant on any interval that lies inside the tetrahedron and is parallel to $C'B'$ or, equivalently, to the vector $(1,1,1)$. In particular, the value of this function at a point (n,i,j) is equal to its value at the point $(n,i,j)+(n-j)(1,1,1)=(2n-j,n+i-j,n)$. This proves assertion 2).

§ 9. Comparing the commutor with the fundamental symmetries of Pak and Vallejo

The paper [8] contains the construction of several bijections in terms of semi-standard Young tableaux (in what follows, simply tableaux). The authors call them *fundamental bijections*. These bijections act between $\mathrm{LR}(\nu\setminus\lambda,\mu)$ and $\mathrm{LR}(\nu\setminus\mu,\lambda)$, where λ, μ, ν are partitions with n parts, that is, n-tuples of non-increasing non-negative integers, and $\mathrm{LR}(\nu\setminus\lambda,\mu)$ is the set of Littlewood–Richardson tableaux of skew shape $\nu\setminus\lambda$ and weight μ (see [1]–[3]).

We claim that the fundamental symmetries ρ_1 and ρ_2 of Pak and Vallejo coincide with the commutor. More precisely, we use here the natural identification (bijection) between SP and LR.

Let us briefly describe this bijection (see [3], § 9 for more details on the relation between arrays and Young tableaux). Given an integer-valued array a with m rows, we canonically construct a 'tableau'. The jth row of the 'tableau' begins with $a(1,j)$ symbols 1, proceeds with $a(2,j)$ symbols 2 and so on. This results in a 'tableau' of irregular form. However, if a is **D**-tight, we get a semistandard Young tableau. Its shape coincides with that of a. Conversely, any semistandard tableau contracts to a **D**-tight array.

Let (a, b) be a standard pair of integer-valued arrays of size $n \times n$. We denote the starting, intermediate and final shapes of this pair by λ, μ and ν respectively. Since the array $a \otimes b$ is **D**-tight, it induces a semistandard tableau of shape ν. Moreover, the **D**-tight array a determines a subtableau of shape μ. The remaining part of the diagram (of skew shape $\nu \setminus \lambda$) is filled with a skew tableau of weight μ. Since b is **L**-tight, this skew tableau yields a dominating word (or Yamanuchi word; see [3], §§ 9.6, 9.7) and we get a Littlewood–Richardson tableau. The converse of this construction is obvious. Given a skew tableau of shape $\nu \setminus \lambda$, we fill the empty part of shape λ with the canonical tableau (λ_1 symbols 1 in the first row, λ_2 symbols 2 in the second row and so on; the alphabet filling the skew tableau is shifted by n). The result is an ordinary Young tableau, which yields the **D**-tight array $a \otimes b$. The array a corresponds to the canonical tableau and, therefore, is **L**-tight. If the initial tableau is a Littlewood–Richardson tableau, then the corresponding word $w(b)$ is dominating and the array b is also **L**-tight.

Example 3. Consider the pair (a, b) from Example 1 (see § 3), or rather the standard pair $(\mathbf{L}(a), b)$. The corresponding tableau is given by

$$\begin{array}{cccccc} 4 & 5 & 5 & 6 & & \\ 2 & 2 & 4 & 5 & 5 & \\ 1 & 1 & 1 & 4 & 4 & 4 \end{array}$$

We note that $w(b) = 4556455444$ is indeed a Yamanuchi word (in the alphabet $\{4, 5, 6\}$).

We recall that the first fundamental symmetry ρ_1 was defined in [8] in terms of the tableau-switching algorithm. In our example the symbols 1, 2, 3 must 'pass' through the symbols 4, 5, 6. This results in the following tableau (with another ordering of symbols: $4 < 5 < 6 < 1 < 2 < 3$):

$$\begin{array}{cccccc} 6 & 1 & 2 & 2 & & \\ 5 & 5 & 5 & 5 & 1 & \\ 4 & 4 & 4 & 4 & 4 & 1 \end{array}$$

This is precisely the tableau corresponding to the pair (b', a') in Example 2 (see § 8).

The appearance of (b', a') in the output is not accidental. We claim that the following proposition holds in general.

Proposition 3. *If we identify* $\mathrm{SP}_{\mathbb{Z}}(\lambda, \mu; \nu)$ *with* $\mathrm{LR}(\nu \setminus \lambda, \mu)$, *then the commutor* Com *coincides with the first fundamental symmetry* ρ_1.

Proof. Indeed, § 3.1 of [8] contains a formula for the tableau-switching algorithm as a composite of three Schützenberger involutions $\mathbf{S}_1$, $\mathbf{S}_{12}$, $\mathbf{S}_1$: starting with a standard pair (a, b), we first apply the Schützenberger involution $\mathbf{S}$ to a, then apply $\mathbf{S}$ to $\mathbf{S}a \otimes b$ and finally apply $\mathbf{S}$ to the first part of the resulting array.

Let us clarify the meaning of $\mathbf{S}a$. By the definition in [3], this is $\mathbf{D}(*a)$. Since a is $\mathbf{DL}$-tight, $*a$ is $\mathbf{R}$-tight. Therefore $\mathbf{D}(*a)$ is the (unique) $\mathbf{DR}$-tight array of shape λ. The array $\mathbf{R}a$ is also $\mathbf{DR}$-tight of shape λ. Hence in our case we have $\mathbf{S}a = \mathbf{D}(*a) = \mathbf{R}a$, that is, we have passed from the standard pair to an anti-standard one.

We now apply the involution $\mathbf{S} = \mathbf{S}_{12}$ to the array $\mathbf{R}a \otimes b$. The result coincides with $\mathbf{D}\bigl(*(\mathbf{R}a\otimes b)\bigr)$, as in our commutor Com. The next application of $\mathbf{S}_1$ is a passage from ASP to SP.

We briefly comment on the coincidence of the commutor and the second fundamental symmetry ρ_2 of Pak and Vallejo. This symmetry (to be precise, a map ρ_2') is defined as the composite (see [8])

$$\mathrm{LR}(\nu \setminus \lambda, \mu) \xrightarrow{\tau} T(\mu, \nu - \lambda) \xrightarrow{\mathbf{S}} T(\mu, (\nu - \lambda)^{\mathrm{op}}) \xrightarrow{\gamma^{-1}} \mathrm{LR}(\nu \setminus \mu, \lambda). \tag{12}$$

Here $T(\mu, \alpha)$ is the set of Young tableaux of shape μ and weight α. Let us interpret the composite (12) in terms of arrays. As already mentioned, each element of $\mathrm{LR}(\nu \setminus \lambda, \mu)$ may be regarded as an $\mathbf{L}$-tight array b such that $\mathrm{diag}(\lambda) \otimes b$ is $\mathbf{D}$-tight. The map τ transposes the array b, sending it to the $\mathbf{D}$-tight array b^T. The Schützenberger involution $\mathbf{S}$ sends b^T to $\mathbf{D}(*b^T)$. Again transposing this array, we get the $\mathbf{L}$-tight array $\mathbf{D}(*b^T)^T = \mathbf{L}(*b^{TT}) = \mathbf{L}(*b)$. We have already met this array (see (11)). Placing the diagonal array $-\nu^{\mathrm{op}}$ to the left of it, we get the $\mathbf{D}$-tight array $\mathrm{diag}(-\nu^{\mathrm{op}}) \otimes \mathbf{L}(*b)$. We also recall that $h = \iint \mathbf{L}(*b) - \int \nu^{\mathrm{op}} + |\nu|$.

The array $a' = \mathrm{Com}(b)$ determines the function $\tilde{h} = \mathrm{Com}'(f) = \iint a' + \int \mu$. Proposition 2 yields that this function is obtained from h by a simple change of coordinates: $h(i, j) = \tilde{h}(n - j, n - j + i)$. It remains to express the array $a' = \partial\partial\tilde{h}$ in terms of the array $\tilde{b} = \mathrm{Com}(b) = \partial\partial h$. More precisely, we shall express $\tilde{b}$ via a'. By definition,

$$\tilde{b}(i, j) = h(i, j) - h(i - 1, j) - h(i, j - 1) + h(i - 1, j - 1).$$

Expressing h in terms of $\tilde{h}$, we get

$$\begin{aligned}\tilde{b}(i, j) = \tilde{h}(n - j, n - j + i) &- \tilde{h}(n - j, n - j + i + 1)\\ &- \tilde{h}(n - j + 1, n - j + i + 1) + \tilde{h}(n - j + 1, n - j + i).\end{aligned}$$

The first two terms of this expression are

$$\mu_{n-j+i} + a'(1, n - j + i) + a'(2, n - j + i) + \cdots + a'(n - j, n - j + i).$$

The difference between the third and fourth terms is expressed in a similar way. We finally get

$$\begin{aligned}\tilde{b}(i, j) = {}&[\mu_{n-j+i} + a'(1, n - j + i) + a'(2, n - j + i) + \cdots + a'(n - j, n - j + i)]\\ &- [\mu_{n-j+i+1} + a'(1, n - j + i + 1) + a'(2, n - j + i + 1) + \cdots\\ &\qquad \cdots + a'(n - j + 1, n - j + i + 1)].\end{aligned}$$

This is precisely the formula used in [8] to define the map γ. We see that our commutor Com coincides with the map ρ_2' (after identifying $\mathrm{LR}(\nu \setminus \lambda, \mu)$ and $\mathrm{SP}_{\mathbb{Z}}(\lambda, \mu; \nu)$). Since the commutor is involutory (by Lemma 1), it coincides with its inverse. But the inverse of ρ_2' was shown in [8] to be equal to ρ_2. Hence the map ρ_2 equals ρ_2' and also coincides with our commutor. This proves the following theorem.

Theorem 4. *The commutor coincides with the fundamental symmetries ρ_1, ρ_2 and ρ_2' of Pak and Vallejo.*

§ 10. Appendix. Proof of Theorem 1

To prove Theorem 1, it will be convenient to assume that the initial data are given on the floor and inclined face of the simplex $\Delta_n(XYZ)$. The propagation vector is thus $(-1,-1,1)$. The proof involves no conceptual difficulties but consists of rather straightforward manipulations with inequalities. Before passing to the general case, we consider two simple particular cases.

Case $n = 2$. Let F be a polarized function on $\Delta_2(OXYZ)$. We must verify all the rhombic inequalities. Here all the rhombuses lie on the faces of the tetrahedron.

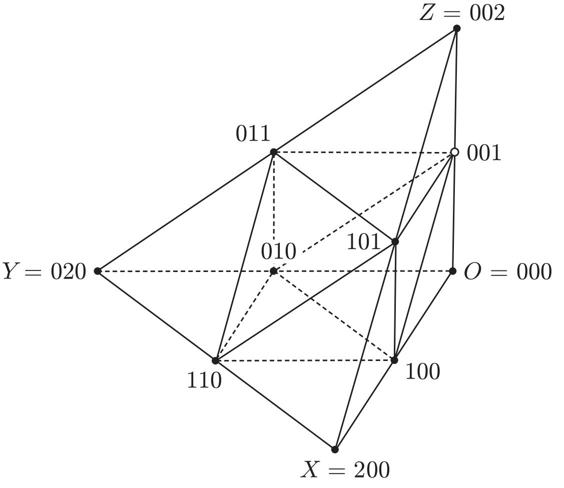

Figure 15

If they lie on the floor or on the inclined face, then the corresponding rhombic inequalities hold by definition. Hence it remains to consider the rhombuses lying on the vertical faces. Since these faces are symmetrically placed, it suffices to consider the rhombuses on the face OXZ (Fig. 15). There are three such rhombuses: the rhombus $000, 001, 101, 100$, the rhombus $200, 100, 001, 101$ and the rhombus $002, 001, 100, 101$. The last two have the same structure, and so we consider the first two.

For the first rhombus, we must show that

$$F(001) + F(100) \geqslant F(101) + F(000). \tag{13}$$

Since 001 and 110 form the principal diagonal of the octahedron, the polarization condition yields the inequality

$$F(001) + F(110) \geqslant F(101) + F(010). \tag{14}$$

Using the rhombic inequality for the rhombus $000, 100, 110, 010$, which lies on the floor, we have

$$F(100) + F(010) \geqslant F(000) + F(110). \tag{15}$$

Adding (14) and (15), we get (13).

For the second rhombus, we must show that

$$F(100) + F(101) \geqslant F(200) + F(001). \tag{16}$$

By the polarization condition, we have either

$$F(001) + F(110) = F(101) + F(010), \tag{17}$$

or

$$F(001) + F(110) = F(100) + F(010). \tag{18}$$

Suppose that (17) holds. Then we use the rhombus $200, 110, 010, 100$, which lies on the floor, and get the inequality

$$F(010) + F(200) \leqslant F(110) + F(100). \tag{19}$$

Adding (19) and (17), we get the desired inequality (16). Now suppose that (18) holds. Then the rhombus $200, 110, 010, 101$, which lies on the inclined face, yields the inequality

$$F(200) + F(010) \leqslant F(110) + F(101). \tag{20}$$

Adding (20) and (18), we get the desired inequality (16).

Thus we have verified the case $n = 2$.

Case $n = 3$. Here we consider the truncated pyramid of height 1 shown in Fig. 16. We assume that the rhombic inequalities hold for the rhombuses lying on the floor or on the inclined face. We claim that they hold for the rhombuses in the upper base. The upper base contains two essentially different rhombuses: $001, 101, 111, 011$ and $021, 111, 101, 011$. Thus we must verify two rhombic inequalities,

$$F(011) + F(101) \geqslant F(001) + F(111), \tag{21}$$

$$F(011) + F(111) \geqslant F(021) + F(101). \tag{22}$$

Let us prove inequality (21). Since 001 is a vertex of the principal diagonal $\{001, 110\}$, the sum $F(001) + F(110)$ equals either $F(011) + F(100)$ or $F(101) + F(010)$. Suppose that $F(001)+F(110) = F(101)+F(010)$. Since 011 and 120 form the principal diagonal of the octahedron, we have $F(011) + F(120) \geqslant F(020) + F(111)$. Finally, the rhombus $110, 120, 020, 010$, which lies on the floor, yields the inequality $F(020) + F(110) \geqslant F(010) + F(120)$. Adding these two inequalities and the equation $F(101) + F(010) = F(011) + F(110)$, we get (21). The case $F(001) + F(110) = F(011) + F(100)$ is treated in a symmetric way.

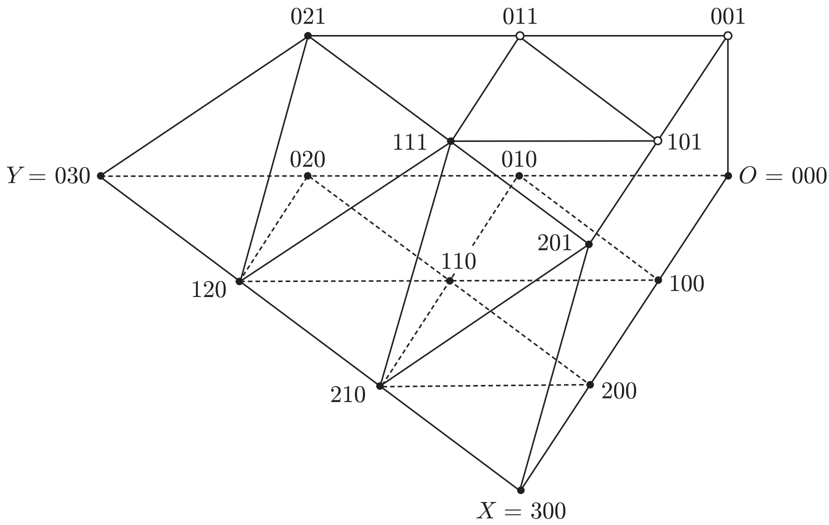

Figure 16

Let us establish (22). Since 011 and 120 are the vertices of the principal diagonal of an octahedron, we have the polarization inequality

$$F(011) + F(120) \geqslant F(021) + F(110).$$

On the other hand, the points 101 and 210 form a diagonal of the octahedron. Hence we have one of the following equations: either a) $F(101)+F(210) = F(200)+F(111)$, or b) $F(101) + F(210) = F(110) + F(201)$.

a) $F(200) + F(111) = F(101) + F(210)$. We have $F(110) + F(210) \geqslant F(120) + F(200)$ for a rhombus lying on the floor. Adding the formulae $F(011) + F(120) \geqslant F(021) + F(110)$, $F(200) + F(111) = F(101) + F(210)$ and $F(110) + F(210) \geqslant F(120) + F(200)$, we get (22).

b) $F(110) + F(201) = F(101) + F(210)$. Here we use the rhombic inequality $F(210)+F(111) \geqslant F(120)+F(201)$ for a rhombus lying on the inclined face. Since 011 and 120 are the vertices of the principal diagonal of an octahedron, we have the polarization inequality $F(011) + F(120) \geqslant F(021) + F(110)$. Adding the last two inequalities and equation b), we get (22).

Thus we have verified the case $n = 3$.

Using the above cases, we get the following assertion.

Lemma 2. *Suppose that F is a polarized function and its restrictions to the floor and inclined face are discretely concave. Then F becomes discretely concave after restriction to any plane parallel to the floor or to the inclined face.*

Using the case $n = 3$, we see that the rhombic inequalities hold for all horizontal rhombuses at height 1 that are tangent to the inclined face. By symmetry, these inequalities hold for all rhombuses that lie in the plane $x + y + z = n - 1$ and are tangent to the floor. Applying the case $n = 3$ to the smaller tetrahedron Δ_{n-1}, we get the rhombic inequalities for horizontal rhombuses at height 1 that are tangent to the wall $x+y+z=n-1$, and so on. We finally see that the inequalities hold for any horizontal rhombus at height 1. By symmetry, they hold for any rhombus lying in the plane $x+y+z=n-1$. But now our polarized function F is discretely concave on two walls, $x+y+z=n$ and $x+y+z=n-1$. By induction, it is discretely concave on any plane of the form $x+y+z=k$, $1\leqslant k\leqslant n$. We similarly see that it is discretely concave on any horizontal plane.

We can now prove Theorem 1 in the general case. Take any rhombus inside $\Delta_n(OXYZ)$ and circumscribe a tetrahedron of size 2 around it. By Lemma 2, the polarized function F is discretely concave on the floor and on the inclined face of this tetrahedron. Using the case $n=2$, we see that it is discretely concave in this tetrahedron and, therefore, the corresponding rhombic inequality holds.

Bibliography

[1] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Clarendon Press, Oxford 1979; Russian transl., Mir, Moscow 1985.

[2] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge Univ. Press, Cambridge 1997; Russian transl., MCCME, Moscow 2006.

[3] V. I. Danilov and G. A. Koshevoi, "Arrays and the combinatorics of Young tableaux", *Uspekhi Mat. Nauk* **60**:2 (2005), 79–142; English transl., *Russian Math. Surveys* **60**:2 (2005), 269–334.

[4] W. Fulton, "Eigenvalues, invariant factors, highest weights, and Schubert calculus", *Bull. Amer. Math. Soc.* (2) **37**:3 (2000), 209–249.

[5] A. S. Buch, "The saturation conjecture (after A. Knutson and T. Tao)", *Enseign. Math.* (2) **46**:1–2 (2000), 43–60.

[6] A. Knutson and T. Tao, "The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I: Proof of the saturation conjecture", *J. Amer. Math. Soc.* **12**:4 (1999), 1055–1090.

[7] A. Henriques and J. Kamnitzer, "The octahedron recurrence and gl_n crystals", *Adv. Math.* **206**:1 (2006), 211–249.

[8] I. Pak and E. Vallejo, *Reductions of Young tableau bijections*, `arXiv: math.CO/0408171`.

[9] A. Knutson, T. Tao, and C. Woodward, "A positive proof of the Littlewood–Richardson rule using the octahedron recurrence", *Electron. J. Combin.* **11**:1 (2004), paper R61 (electronic); `arXiv: math.CO/0306274`.

[10] V. Danilov and G. Koshevoy, *Arrays and the octahedron recurrence*, `arXiv: math.CO/0504299`.

[11] V. I. Danilov and G. A. Koshevoi, "Discrete convexity and Hermitian matrices", Tr. Mat. Inst. Steklova, vol. 241, Nauka, Moscow 2003, pp. 68–89; English transl., *Proc. Steklov Inst. Math.* **241** (2003), 58–78.

[12] D. E. Speyer, "Perfect matchings and the octahedron recurrence", *J. Algebraic Combin.* **25**:3 (2007), 309–348; `arXiv: math.CO/0402452`.

[13] S. Fomin and A. Zelevinsky, "The Laurent phenomenon", *Adv. in Appl. Math.* **28**:2 (2002), 119–144; `arXiv: math.CO/0104241`.

V. I. Danilov
Central Institute of Economics and Mathematics,
Russian Academy of Sciences
E-mail: `vdanilov43@mail.ru`

Received 1/APR/05
15/MAY/07
Translated by A. V. DOMRIN

G. A. Koshevoi
Central Institute of Economics and Mathematics,
Russian Academy of Sciences
E-mail: `koshevoy@cemi.rssi.ru`