Choice functions and extensive operators*

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Abstract

The paper puts forth a theory of choice functions in a neat way connecting it to a theory of extensive operators and neighborhood systems. We consider four classes of heritage choice functions satisfying the conditions **M, N, W**, and **C**.

Key words: neighborhood system, pre-topology, matroid, anti-matroid, exchange and anti-exchange conditions, closure operator, direct image.

1 Introduction

One needs to adopt an appropriate viewpoint to enhance understanding. Here we propose to look at choice functions through the looking-glass of extensive operators in order to arrive at a better understanding of both theories.

A choice function (on a set X) is a mapping $c : 2^X \to 2^X$ such that $c(A) \subseteq A$ for every $A \subseteq X$. It assigns to a set A the subset $c(A)$ of 'chosen' elements from A. Choice Theory is interested in a 'rational' choice. A series of requirements of rationality have been proposed. The most famous of them are the heredity axiom **H**, the concordance axiom **C**, the outcast axiom **O**, and their combinations (see [2] and [15]).

On the other hand, different kinds of extensive operators have been investigated in pure mathematics, especially closure operators. The latter ones have appeared in Combinatorics, Algebra, Topology, Logic, and so on.

An idea to connect choice functions and extensive operators is not new. Given an extensive operator ε, one can get a choice function c: $c(A) = X - \varepsilon(X - A)$ (or the complementary choice function $\overline{c}(A) = A \cap \varepsilon(X - A)$). This connection was used in [7] applying to closure operations and in [8] applying to monotone extensive operators. We propose in Section 3 another bijection between the set of choice functions and the set of extensive operators. Roughly speaking, the choice function $\overline{c}$ prescribes to choose elements $a \in A$ which are close to the complement $X - A$, while our choice function chooses elements $a \in A$ which are far away $A - a$. Here we have been guided by Koshevoy's construction [11] of path independent choice function as the set of extreme points of an anti-exchange closure operator; closed results was obtained in

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[4, 10]. It turns out that, under this bijection, the heredity property of choice functions corresponds to the monotonicity property of extensive operators. The concordance axiom converts into the additivity property (Section 8).

To get more subclasses of heredity choice functions, we split the outcast axiom **O** into two axioms **W** and **N** and introduce a new axiom **M**. Choice functions satisfying **H** and **W** correspond to closure operators and we call them *closed* choice functions, see Section 7. We consider also in Sections 5 and 6 choice functions satisfying **H** and **M**, and satisfying **H** and **N**. Three subclasses of closed choice functions satisfying axioms **N**, **C**, and **M** correspond to three well-established classes of closure operators - convex geometries, topologies and matroids (see Sections 10-12). A combinatorial counterpart of the axiom **N** is the anti-exchange condition. Plott (or path independent) choice functions correspond to anti-exchange closure operators known as convex geometries. If we add the concordance condition **C** to **H** and **W**, we obtain choice functions corresponding to additive closure operators, topologies, or pre-orders. Finally, the axiom **M** yields exchange closure operators, or matroids.

We introduce also in Section 4 a "topological" language which allows to discuss properties of heritage choice functions in terms of neighborhood systems (or of pre-topologies in the case of closed choice functions). Such topological terminology turns out to be very useful since it prompts habitual associations. Topological notions such as adherent and isolated points, open and closed sets and so on are relevant to properties of choice functions. For example, the axiom **C** implies that the corresponding pre-topology is actually a topology, whereas the axiom **N** means some version of separability axiom, and the axiom **M** means a kind of symmetry.

In the process of reformulation of Choice Theory in the language of extensive operators we bring a number of results already known under the same or an equivalent form, of course. We do not attempt to find original sources of them, referring instead to the book [2] and to [15]. The same is true for many notions. In particular, the links between neighborhood systems, extensive and contracting operators have appeared before in Choice Theory with the hyper-relations (see [2]) or the extended partial orders (see [17]), and in the foundations of General Topology with especially the Frechet V-spaces. We establish these relations in a neat style.

Another novelty of our study is an operation of inversion on the set of choice functions (Section 9). The inversion sends the class of heritage choice functions to itself and is an involution. The axiom **N** is auto-inverse, whereas the axioms **W** and **M** change places. In particular, we obtain a new interesting class of choice functions which are inverse to Plott functions.

In the last two sections of the paper we elaborate upon the notion of direct image as a power tool to construct and compare operators, choice functions and neighborhood systems given on different sets. On this way we consider a universal topological development of any pre-topology.

2 Contracting and extensive operators

Fix a set X. For the sake of simplicity we shall assume that X is a finite set; however, many notions and statements can be carried through to more general cases. Letters $a, b, x, ...$ denote elements of X, letters $A, B, ...$ denote subsets of X, that are elements of the set 2^X. $\overline{A}$ denotes the complementary subset, that is $\overline{A} = X - A$. Subsets of 2^X are denoted by calligraphic letters like $\mathcal{N}$ or $\mathcal{P}$.

Definition. An *operator* on X is a mapping $f : 2^X \to 2^X$.

To see an analogy with the classical notion of operator, one has to consider elements of 2^X as functions from X to the set $\{0, 1\}$; an operator transforms functions into functions.

The set of operators possesses many natural structures: the operation of composition, (coordinate-wise) operations $\cup$ and $\cap$, the natural order ($f \leq g$ if $f(A) \subseteq g(A)$ for all $A \subseteq X$). Though, we do not consider general operators. We shall be interested in either contracting or extensive operators.

Choice functions

Definition. An operator $f : 2^X \to 2^X$ is called *contracting* (or a *choice function*) if $f(A) \subseteq A$ for any A.

The poset $\mathbf{CF}(X)$ (or simply $\mathbf{CF}$) of contracting operators functions on X (with respect to the natural order) is a Boolean lattice; the complementation $\bar{f}$ to an operator f is given by the rule: $\bar{f}(A) = A - f(A)$. The identity operator $\mathbf{1}$ (choosing A from every A) is the maximal element of $\mathbf{CF}$. The "empty" choice function $\mathbf{0}$ (choosing $\emptyset$ from any set) is the minimal element of $\mathbf{CF}$.

In social sciences, contracting operators are called *choice functions*. One says that elements from $f(A)$ *are chosen* by the choice function f from the agenda-set A. Moreover, the choice of "best" alternative is of interest. There are two approaches to formalize the vague notion of the "best", internal and external. Due to the first approach, one takes some structure on X, a kind of a preference, and constructs explicitly a choice function using this structure. A typical example: let $\preceq$ be a reflexive binary relation on X; we choose elements which are maximal with respect to $\preceq$:

$$\mathrm{Max}_{\preceq}(A) = \{a \in A, \text{ such that } a \preceq x \text{ for } x \in A \text{ implies } a = x\}.$$

Choice functions of such a form are called *rationalizable by* the relation $\preceq$.

Due to the second approach, one imposes some consistency axioms on choice functions. For example, rationalizable (by reflexive binary relations) choice functions satisfies the following *heredity* property (which has many other names; see [2]):

H If $A \subseteq B$ then $f(B) \cap A \subseteq f(A)$.

In other words, if $a \in A$ has been chosen in a bigger set B then it has to be chosen in A.

We consider the heredity as a minimal rationality requirement and shall investigate mainly heritage choice functions. The set **Her** of heritage choice functions is stable

with respect to $\cap$ and $\cup$. Therefore **Her** is a distributive sublattice in the lattice **CH**; for more properties of this lattice see [14].

Another property of rationalizable choice functions is the following *no-dummy* property. An item $x \in X$ is *dummy* for a choice function f if $f(\{x\}) = \emptyset$. The no-dummy property asserts that the set of dummies is empty. This requirement also is important for Choice Theory but it is rather a normalization than a consistency requirement.

Rationalizable choice functions have (or can have) other properties; we shall consider them further. We distinguish four axioms (classes) and their combinations. To give an idea about them, let us return to reflexive binary relations. There are three main subclasses of binary relations: symmetric ($x \succeq y$ implies $y \succeq x$), transitive ($x \succeq y$ and $y \succeq z$ imply $x \succeq z$), and anti-symmetric ($x \succeq y$ and $y \succeq x$ imply $x = y$). The interrelations between these classes of reflexive binary relations are depicted in the following diagram:

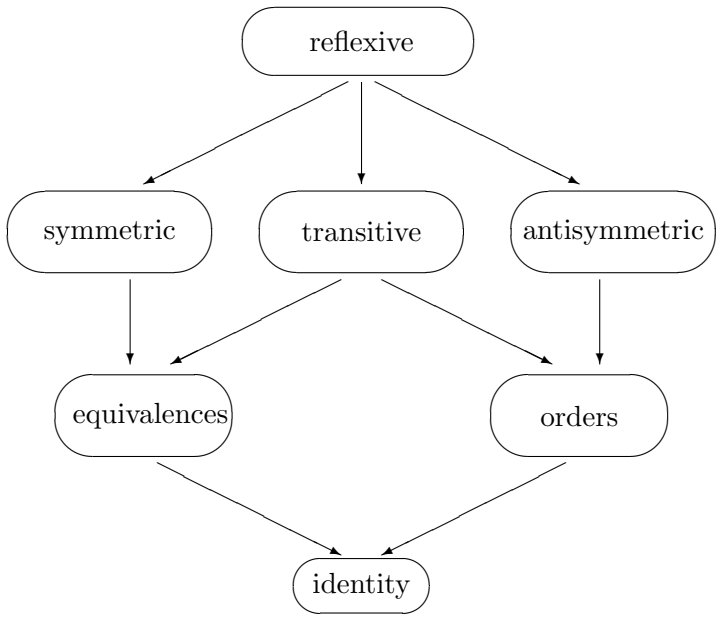


Figure 1

Three (future) subclasses of heritage choice functions, which we shall study in the paper, generalize these three classes (symmetric, transitive, and anti-symmetric); the fourth class of 'concordant' choice functions is formed by the properly rationalizable choice functions.

Extensive operators

Definition. An operator ε is called *extensive* if $A \subseteq \varepsilon(A)$ for all A.

We denote extensive operators by symbols $\varepsilon, \mu, \sigma, \ldots$. The set of extensive operators is denoted **Ext** or **Ext**(X). Since the union and the intersection of extensive operators is extensive one, **Ext** is a (Boolean) lattice.

Monotone operators form the most important subclass of extensive operators. An operator μ is *monotone* if $A \subseteq B$ implies $\mu(A) \subseteq \mu(B)$. The set of monotone operators

is denoted $\mathbf{Mon}(X)$. Since the monotonicity is conserved by the operations $\cup$ and $\cap$, the set **Mon** is a distributive sublattice in **Ext**. As we shall see, the monotonicity condition is an analog of the heredity condition for choice functions.

The condition $\mu(\emptyset) = \emptyset$ is an analog of the no-dummy condition. Further we shall consider four main subclasses of monotone extensive operators. But firstly we construct a bijection between the sets **Ext** and **CF**.

3 The fundamental bijection

Here we define a transformation (the fundamental bijection) from choice functions to extensive operators and vice versa.

Let c be a choice function on X, and let ε be an extensive operator on X. Say that they *correspond* to each other, if for every A and every $x \notin A$

$$x \in c(A \cup x) \Leftrightarrow x \notin \varepsilon(A).$$

An equivalent reformulation: for every A and $a \in A$

$$a \in c(A) \Leftrightarrow a \notin \varepsilon(A - a).$$

It is clear that c defines ε (which is denoted as c^*) and ε defines c (which is denoted as ε^*). This proves

Theorem 1. *The correspondence is an antitone bijection between the set* $\mathbf{Ext}(X)$ *and* $\mathbf{CF}(X)$.

It is clear that $c^*(\emptyset)$ is the set of dummies of any choice function c.

The fundamental bijection transforms monotone extensive operators to heritage choice functions and vice versa.

Proposition 1. *A choice function h is heritage if and only if the operator $\mu = h^*$ is monotone.*

Proof. Suppose h is a heritage choice function, $A \subseteq B$ and $x \in h^*(A)$; we have to show that $x \in h^*(B)$. If $x \in B$ then it is obvious. Suppose now that $x \notin B$; then $x \in h^*(A)$ means that $x \notin h(A \cup x)$. According to the heredity of h, $x \notin h(B \cup x)$ and consequently $x \in h^*(B)$.

Conversely, suppose μ is a monotone extensive operator. We have to show that the choice function $h = \mu^*$ is heritage. Let $A \subseteq B$ and $a \in A \cap h(B)$; we have to check that $a \in h(A)$. Since $a \in h(B) = \mu^*(B)$ we have $a \notin \mu(B - a)$. By the monotonicity of μ, $a \notin \mu(A - a)$ and belongs to $h(A)$. $\square$

Thus, monotone extensive operators and heritage choice functions are cryptomorphic notions. In the next section we introduce in the play a third cryptomorphic notion, that of neighborhood systems.

4 Neighborhood systems

Here we introduce a 'topological' language for heritage choice functions and monotone operators.

Definition. A *neighborhood system* on X is a family $\mathcal{N} = (\mathcal{N}_x, x \in X)$ of subsets $\mathcal{N}_x \subseteq 2^X$ satisfying two axioms:

NS1. $x \in U$ for every $U \in \mathcal{N}_x$.

NS2. If $U \in \mathcal{N}_x$ and $U \subseteq V$ then $V \in \mathcal{N}_x$.

Terminology: elements of X are called *points*; instead of $U \in \mathcal{N}_x$ we say also that U is a *neighborhood* of the point x. If $\mathcal{N}_x = \emptyset$ we say that the point x is *dummy*.

In other words, to any point $x \in X$, we assign a filter $\mathcal{N}_x$ of neighborhoods of the point. For example, if we have a topology on X we can speak on neighborhoods. However such topological neighborhood systems satisfy two additional axioms **NS3** and **NS4**. We shall explain this below. The set of neighborhood systems on X is denoted $\mathbf{NS}(X)$. Ordered by inclusion, it is a distributive lattice.

Let $\mathcal{N} = (\mathcal{N}_x,\ x \in X)$ be a neighborhood system on X.

Definition. A point x is *adherent* for a set $A \subseteq X$ if every neighborhood of this point intersects A.

In other words, $\overline{A} = X - A$ is not a neighborhood of x. Note that any dummy point is adherent for every set (even empty). Let $\mathrm{Adh}(A)$ (or $\mathrm{Adh}_{\mathcal{N}}(A)$) denote the set of adherent points for A. It is obvious that the operator $\mathrm{Adh}_{\mathcal{N}}$ is extensive (due to NS1) and monotone. Thus we obtain an antitone mapping

$$\mathrm{Adh} : \mathbf{NS}(X) \to \mathbf{Mon}(X).$$

Definition. A point $a \in A$ is called *isolated* in A if some neighborhood of a does not intersect the set $A - a$.

Note that a dummy point is never isolated. By definition, the set $\mathrm{Iso}_{\mathcal{N}}(A)$ consists of all isolated points of A. So that we obtain a choice function $\mathrm{Iso}_{\mathcal{N}}$. If a point $a \in A$ is isolated in a bigger set B then it is isolated in A. Hence $\mathrm{Iso}_{\mathcal{N}}$ is a heritage choice function. Thus, we have a monotone mapping

$$\mathrm{Iso} : \mathbf{NS}(X) \to \mathbf{Her}(X).$$

Proposition 2. *The mappings* Adh *and* Iso *are bijections, and the following diagram is commutative*

$$\begin{tikzcd}
 & \mathbf{NS} \arrow[dl, "\mathrm{Adh}"'] \arrow[dr, "\mathrm{Iso}"] & \\
\mathbf{Mon} \arrow[rr, leftrightarrow] & & \mathbf{Her}
\end{tikzcd}$$

Proof. 1) Commutativity follows from a simple remark: a point x is adherent for a set A if and only is x is not isolated in the set $A \cup x$.

2) The operator $\mathrm{Adh}_{\mathcal{N}}$ determines $\mathcal{N}$. Indeed,

$$x \notin \mathrm{Adh}_{\mathcal{N}}(A) \Leftrightarrow \overline{A} \in \mathcal{N}_x.$$

This proves that the mapping Adh is injective.

3) The previous formula prompts an inversion to the mapping Adh. Let μ be a monotone extensive operator. A set U is said to be an *μ-neighborhood* of a point x if $x \notin \mu(\overline{U})$. It is clear that axioms NS1 and NS2 are fulfilled for the system $\mathcal{N}$ of μ-neighborhoods.

We assert that $\mathrm{Adh}_{\mathcal{N}} = \mu$. Indeed, for any A,

$$x \in \mathrm{Adh}_{\mathcal{N}}(A) \Leftrightarrow \overline{A} \notin \mathcal{N}_x \Leftrightarrow x \in \mu(\overline{\overline{A}}) = \mu(A).$$

This proves that Adh is a surjective mapping.

4) Theorem 1 together with 1)–3) establish bijectivity of the mapping Iso. The inverse mapping also can be explicitly defined. Let h be a heritage choice function. A set U is said to be an *h-neighborhood* of a point x if $x \in U$ and $x \in h(\overline{U} \cup x)$. The set of h-neighborhoods forms the neighborhood system $\mathrm{Iso}^{-1}(h)$ $\square$

Note that a point x is dummy for a neighborhood system $\mathcal{N}$ if and only if it is dummy for the corresponding choice function $\mathrm{Iso}(\mathcal{N})$.

Remark. In [8] Echenique considered another bijection between neighborhood systems and heritage choice functions. Let h be a heritage choice functions. Due to our construction, a set U is an h-neighborhood of x (that is $U \in \mathcal{N}_x$) if $x \in U$ and $x \in h(\overline{U} \cup x)$, while due to the Echenique's construction, a set V is an neighborhood of x if $x \in V$ and $x \notin h(V)$.

In the following four sections we introduce four classes of neighborhood systems. These classes will be distinguish by corresponding axioms. We find counterparts of these axioms in terms of (heritage) choice functions and (monotone) extensive operators; many of them will be our old acquaintances from Figure 1.

5 Exchange property

Let $\mathcal{N} = (\mathcal{N}_x, x \in X)$ be a neighborhood system. Consider the following *symmetry axiom*

S Suppose that U is a neighborhood of points x and y. If $U - x$ is a neighborhood of y then $U - y$ is a neighborhood of x.

As we shall see, **S** is a generalization of symmetry of (reflexive) binary relation. This requirement **S** has counterparts in terms of choice functions and extensive operators. Let μ be a monotone extensive operator; consider the following *exchange axiom*

Exc Suppose $x, y \notin \mu(A)$. If $x \in \mu(A \cup y)$ then $y \in \mu(A \cup x)$.

In terms of choice functions we introduce the *matroidal axiom*

M If $x \in f(A \cup x)$ then $f(A) \subseteq f(A \cup x)$.

The condition **M** means that deleting a 'good' item does not make 'bad' items 'good'. Note that deleting some 'bad' item can transform another 'bad' item into a 'good' one.

Lemma 1. *For a heritage choice function f, the axiom* **M** *is equivalent to either the following axioms*

M′ *If $x \in f(A \cup x)$ then $f(A \cup x) = f(A) \cup x$.*

M″ $f(A - f(A)) = \emptyset$.

Proof. **M** implies **M′** due to the heredity of f.

We can rewrite **M′** as a statement: $x \in f(A)$ implies that $f(A - x) = f(A) - x$. By induction we obtain **M″**.

Let us show that **M″** implies **M**. Indeed, suppose that $x \in f(A \cup x)$. Let $B = A \cup x$; then $B - f(B) \subseteq A$. Due to heredity, $f(A) \cap (B - f(B)) \subseteq f(B - f(B)) = \emptyset$ (according to **M**). Hence $f(A) \subseteq f(B) = f(A \cup x)$ and $f(A) \cup x \subseteq f(A \cup x)$. □

Proposition 3. *Let $\mathcal{N}$ be a neighborhood system, $\mu = Adh(\mathcal{N})$ and $f = Iso(\mathcal{N})$. The following assertions are equivalent*[1]*:*

1) $\mathcal{N}$ satisfies the symmetry axiom **S**.

2) The operator μ satisfies the exchange axiom **Exc**.

3) The choice function f satisfies **M**.

Proof. 1) ⇒ 2). Let $x, y \notin \mu(A)$ and let $x \in \mu(A \cup y)$. We have to show that $y \in \mu(A \cup x)$. Put $U = X - A$. The relation $x \notin \mu(A)$ means that U is a neighborhood of x; similarly U is a neighborhood of y. Suppose that y is not in $\mu(A \cup x)$. This means that there exists a neighborhood V of the point y which does not intersect $A \cup x$, that is $U - x$ is a neighborhood of y. Due to the axiom **S**, $U - y$ is a neighborhood of x, that is $x \notin \mu(A \cup y)$, what contradicts to the supposition $x \in \mu(A \cup y)$.

A reversing of this arguing proves 2) ⇒ 1).

1) ⇒ 3). Suppose that $x \in f(A \cup x)$ and $y \in f(A)$. We have to show that $y \in f(A \cup x)$; here we can assume that $x \notin A$. Since $x \in f(A \cup x)$ we have that $\overline{A}$ is a neighborhood of x. $y \in f(A)$ means that $y \cup \overline{A}$ is a neighborhood of y. Applying **S** to $U = y \cup \overline{A}$, we obtain that $U - x = y \cup \overline{(A \cup x)}$ is a neighborhood of y. That is $y \in f(A \cup x)$.

A reversing of this arguing proves 3) ⇒ 1). □

6 Anti-exchange property

It is known that interesting topologies satisfy separation conditions. Here we consider a very weak separation condition like the Kolmogorov condition (T_0). A neighborhood system $\mathcal{N}$ is called a *Kolmogorov* neighborhood system if the following axiom is fulfilled:

K Suppose that U is a neighborhood of two distinct points x and y. Then either $U - x$ is a neighborhood of y or $U - y$ is a neighborhood of x.

[1] the equivalence of 2) and 3) was proven in [4].

Let us introduce the following *anti-exchange axiom* for a monotone operator μ:

AExc Suppose a and b do not belong to $\mu(A)$, whereas $a \in \mu(A \cup b)$ and $b \in \mu(A \cup a)$. Then $a = b$.

A choice-functional counterpart of this axiom is the following condition on a choice function f:

N Suppose $a \in f(A \cup a)$ and $b \in f(A \cup b)$. Then $f(A \cup a \cup b)$ contains a or b.

(A comment. Here we can assume that a and b are different elements of $\overline{A}$. We can interpret the relation $a \notin f(A \cup a \cup b)$ as $b \succeq a$; similarly, $b \notin f(A \cup a \cup b)$ means $a \succeq b$. Therefore the axiom **N** asserts 'non-equivalence' of items or anti-symmetry of the relation $\succeq$.)

Proposition 4. *Let $\mathcal{N}$ be a neighborhood system, $\mu = Adh_{\mathcal{N}}$, and $f = Iso_{\mathcal{N}}$. The following three assertions are equivalent:*
1) *$\mathcal{N}$ is a Kolmogorov neighborhood system.*
2) *The extensive operator μ satisfies the anti-exchange axiom* **AExc**.
3) *The choice function f satisfies* **N**.

Proof. 1) $\Rightarrow$ 2). Suppose a, b are different elements, and a and b do not belong to $\mu(A)$. The last means that $U = \overline{A}$ is a neighborhood of a and b. Due to **K**, either $U - a$ is a neighborhood of b (and then $b \notin \mu(A \cup a)$) or $U - b$ is a neighborhood of a (and then $a \notin \mu(A \cup b)$).

2) $\Rightarrow$ 3). It is obvious since $a \in f(A \cup a)$ means $a \notin \mu(A)$ and so on.

3) $\Rightarrow$ 1). Suppose U is a neighborhood of two distinct points x and y, and $A = X - U$. Then $x \in f(A \cup x)$ and $y \in f(A \cup y)$. Due to **N**, either $x \in f(A \cup x \cup y)$ (which means that $U - y$ is a neighborhood of x) or $y \in f(A \cup x \cup y)$ (which means that $U - x$ is a neighborhood of y). $\square$

7 Closure operators and pre-topologies

Now we consider a topological counterpart to the most famous class of extensive operators, namely, the class of closure operators. A *closure operator* is a monotone extensive and idempotent operator σ (that is $\sigma(\sigma(A)) = \sigma(A)$).

For example, the identity operator **1** is a closure operator. Another example: the constant operator with the value X ($\sigma(A) = X$ for all A). Closure operators can be characterized by the following *transitivity property*:

T If $b \in \sigma(A)$ then $\sigma(A \cup b) \subseteq \sigma(A)$.

Lemma 2. *An extensive monotone operator is a closure operator if and only if it has the property* **T**.

Proof. Suppose that σ is idempotent, and $b \in \sigma(A)$. Then $A \cup b \subseteq \sigma(A)$. Due to monotonicity $\sigma(A \cup b) \subseteq \sigma(\sigma(A)) = \sigma(A)$. Therefore $\sigma(A \cup b) \subseteq \sigma(A)$.

Conversely, let $\sigma(A)$ be extensive monotone operator satisfying **T**. By induction, we have the following stronger property: if F is a finite subset of $\sigma(A)$ then $\sigma(A \cup F) \subseteq$

$\sigma(A)$. Since $\sigma(A)$ is finite[2], $\sigma(\sigma(A)) = \sigma(A \cup \sigma(A)) \subseteq \sigma(A)$. The inverse inclusion is obvious. $\square$

The poset of closure operators on X is denoted $\mathbf{Clo}(X)$. It is well-known that the set $\mathbf{Clo}(X)$ is stable with respect to $\cap$. As a consequence we obtain that, for any operator ε, there exists the least closure operator σ such that $\varepsilon \leq \sigma$; it is sufficient to take the intersection of all closure operators containing ε. We call this closure operator σ as the *closure* of ε and denote it by $\mathrm{cl}(\varepsilon)$. In particular, the poset $\mathbf{Clo}(X)$ is a lattice. The meet $\wedge$ coincides with the intersection $\cap$. The join $\vee$ is given by the formula:

$$\vee_i \sigma_i = \mathrm{cl}(\cup_i \sigma_i).$$

For properties of this lattice see [5] (see also Section 11). Any monotone operator can be represented as union of closure operators, see Section 12.

To describe closure operators in terms of neighborhood systems, we recall some topological notions. Fix a neighborhood system $\mathcal{N}$ and consider a subset $A \subseteq X$.

Definition. A point $a \in A$ is *interior* in A if A is a neighborhood of a. $\mathrm{int}(A)$ denotes the set of interior points of A. A set A is *open* if $A = \mathrm{int}(A)$.

Let us consider the following axiom for neighborhood systems:

NS4 If $U \in \mathcal{N}_x$ then $\mathrm{int}(U) \in \mathcal{N}_x$.

An equivalent reformulation: every neighborhood of any point contains an open neighborhood of the point. In particular, every minimal neighborhood is open.

Finally, we introduce a counterpart of this axiom in terms of choice functions.

Definition. A heritage choice function f is called *closed* if it satisfies the following axiom

W If $a \in f(A \cup a)$ and $b \notin f(A \cup b)$ then $a \in f(A \cup a \cup b)$.

(An intuition behind this requirement is as follows. The relation $a \in f(A \cup a)$ means that a has no rival in A. $b \notin f(A \cup b)$ means that b has a rival in A. Hence b is not a rival for a, and a is chosen from $A \cup a \cup b$ as before. Of course, this is only a vague heuristic argument.)

It is easy to see that a union of closed choice functions is closed as well. The 'empty' choice function $\mathbf{0}$ is closed. As a consequence, for any choice function c, there exists the greatest closed choice function f such that $f \leq c$.

Proposition 5. *Let $\mathcal{N}$ be a neighborhood system, $\mu = Adh(\mathcal{M})$, and $f = Iso(\mathcal{N})$. The following assertions are equivalent*:

1) *$\mathcal{N}$ satisfies the axiom* **NS4**;

2) *μ is a closure operator;*

3) *f is a closed choice function.*

The equivalence 2) and 3) as well as the condition **W** were obtained in earlier work of K.Ando [4].

[2]here we use the finiteness of the set X.

Proof. 1) $\Rightarrow$ 2). Show that σ satisfies **T**. Let b be an adherent point for A and x be an adherent point for $A \cup b$. The latter means that any neighborhood U of x meets $A \cup b$. Due to **NS4**, we can assume that U is open. If U contains the point b then it is a neighborhood of b as well, and therefore it meets A. Otherwise U meets A. In the both cases U meets A and consequently x is an adherent point for A. By Lemma 2, σ is a closure operator.

2) $\Rightarrow$ 3). Let us check the axiom **W** for the choice function f. Suppose that $a \in f(A \cup a)$ and $b \notin f(A \cup b)$. The first means that $a \notin \sigma(A)$, the second means that $b \in \sigma(A)$. Due to **T** we have $\sigma(A \cup b) \subseteq \sigma(A)$. Hence $a \notin \sigma(A \cup b)$ as well, what means that $a \in f(A \cup b \cup a)$.

3) $\Rightarrow$ 1). We have to show that any neighborhood U of a point x contains an open neighborhood of x. Here we can assume that the neighborhood U of the point x is minimal. Let $A = X - U$; since x is an isolated point in $A \cup x$, $x \in f(A \cup x)$.

Let y be another point of U. Because of minimality of U, x is not isolated point of $A \cup x \cup y$, that is $x \notin f(A \cup x \cup y)$. The property **W** implies then $y \in f(A \cup y)$, that is y is isolated in $A \cup y$. This means that U is a neighborhood of y. Thus, any point of U is interior and U is open. $\square$

Pre-topologies

There is a convenient way to speak about neighborhood systems $\mathcal{N}$ satisfying the axiom **NS4**. Above we have introduced the notion of an open subset with respect to $\mathcal{N}$. It is obvious that the union of open subsets is open. This prompts the following

Definition. A set $\mathcal{P}$ of subsets in X is called a *pre-topology* on X if it is stable with respect to unions. Elements of $\mathcal{P}$ are called *open* sets.

In particular, the empty set is open. We do not assume that X is open, however; some elements can be dummy. Complements to open sets can be considered as 'closed' sets; the collection of 'closed' sets is stable with respect to intersections and is a *closure system*. Thus, pre-topologies and closure systems (as well as neighborhood systems satisfying **NS4**) are cryptomorphic notions.

Any neighborhood system $\mathcal{N}$ provides us with the pre-topology $\mathcal{P} = \mathcal{PT}(\mathcal{N})$ consisting of $\mathcal{N}$-open sets. Conversely, let $\mathcal{P}$ be a pre-topology. We say that V is a *$\mathcal{P}$-neighborhood* of a point x if there exists an open set U such that $x \in U \subseteq V$. The resulting neighborhood system $\mathcal{N} = \mathcal{NS}(\mathcal{P})$ satisfies **NS4**. Moreover, we have inclusion $\mathcal{NS}(\mathcal{PT}(\mathcal{N})) \subseteq \mathcal{N}$, and $\mathcal{NS}(\mathcal{PT}(\mathcal{N})) = \mathcal{N}$ if and only if $\mathcal{N}$ satisfies **NS4**.

In particular, one can assign a (closed) choice function $f = \mathrm{Iso}_{\mathcal{P}}$ to every pre-topology $\mathcal{P}$. Namely,

$$f(A) = \{a \in A,\ A \cap U = \{a\} \text{ for some } U \in \mathcal{P}\}.$$

Conversely, let h be a heritage choice function. Call a set U *h-open* if $x \in h(\overline{U} \cup x)$ for every $x \in U$. We assert that *the set $\mathcal{P}(h)$ of h-open sets forms a pre-topology.* Indeed, suppose that (U_i) is a family of h-open sets; we have to show that $U = \cup_i U_i$ is also h-open. Let $x \in U$; then $x \in U_i$ for some i and $x \in h(\overline{U_i} \cup x)$. Since $\overline{U} \subseteq \overline{U_i}$ and h is heritage we conclude that $x \in h(\overline{U} \cup x)$.

If we now define f as $\mathrm{Iso}_{\mathcal{P}(h)}$ then, obviously, $f \leq h$. Moreover, f is the largest closed choice function such that $f \leq h$. Indeed, if g is a closed choice function and $g \leq h$ then $\mathcal{P}(g) \subseteq \mathcal{P}(h) = \mathcal{P}(f)$, from where $g \leq f$. In other words, f is nothing but the closure $cl(h)$ of the heritage choice function h.

To sum up, we have a natural monotone bijection between the lattice **PreTop** of pre-topologies and the lattice **CloCF** of closed choice functions. Pre-topologies give a convenient and economic way to construct and discuss closed choice functions (or closure operators).

8 Additive operators

Additive (extensive) operators form the fourth great class of monotone operators. For any monotone operator μ we have inequality

$$\mu(A) \cup \mu(B) \subseteq \mu(A \cup B).$$

Definition. An extensive operator α is *additive* if, for all A and B,

$$\alpha(A \cup B) = \alpha(A) \cup \alpha(B).$$

Note that any (extensive) additive operator is monotone. The identity operator $\mathbf{1}$ is (the least) additive operator. The set of additive operators is denoted **Add**. This set is stable with respect to $\cup$. On the contrary, the class **Add** is not stable with respect to $\cap$. Moreover, the following statement holds:

Lemma 3. *Every monotone extensive operator can be represented as intersection of additive operators.*

Proof. Let μ be a monotone extensive operator. For every $B \subseteq X$ define the operator μ_B by the rule:

$$\mu_B(A) = \begin{cases} \mu(B), \text{ if } A \subseteq B, \\ X \text{ otherwise.} \end{cases}$$

It is clear that μ_B is extensive and additive operator. On the other hand, $\mu = \cap_B \mu_B$. Indeed, for every A,

$$(\bigcap_B \mu_B)(A) = \bigcap_B \mu_B(A) = \bigcap_{A \subseteq B} \mu_B(A) = \bigcap_{A \subseteq B} \mu(B) = \mu(A),$$

because, due to monotonicity of μ, $\mu(A) \subseteq \mu(B)$ for $A \subseteq B$. $\square$

A counterpart of additivity in terms of choice functions is the following *concordance* property

C $f(A) \cap f(B) \subseteq f(A \cup B)$ for every A, B.

A heritage choice function c is *concordant* if it satisfies **C**.

Additive operators (and concordant choice functions) have a nice descriptions in terms of neighborhood systems. Introduce the following axiom.

NS3. For every point x, the set $\mathcal{N}_x \subseteq 2^X$ is stable with respect to intersection.

An equivalent formulation of this axiom: every non-dummy point x has a unique minimal neighborhood U_x.

Proposition 6. *Let $\mathcal{N}$ be a neighborhood system, $\mu = Adh_{\mathcal{N}}$, and $f = Iso_{\mathcal{N}}$. The following three assertions are equivalent:*
1) *$\mathcal{N}$ satisfies the axiom* **NS3**.
2) *The operator μ is additive.*
3) *The choice function f is concordant.*

Proof. 1) $\Rightarrow$ 2). We have to show that $\mu(A \cup B) \subseteq \mu(A) \cup \mu(B)$. Suppose that $x \notin \mu(A) \cup \mu(B))$. This mean that there exists a neighborhood U of x which does not intersect A, and another neighborhood V of the same point x which does not intersect B. Due to **NS3**, the intersection $U \cap V$ is a neighborhood of x as well, and it does not intersect $A \cup B$. Thus, $x \notin \mu(A \cup B)$.

2) $\Rightarrow$ 1). Suppose that U and V are two neighborhoods of a point x; we have to show that $U \cup V$ is a neighborhood as well. Let $A = \overline{U}$ and $B = \overline{V}$. Then $x \notin \mu(A)$ and $x \notin \mu(B)$. Due to the additivity, $x \notin \mu(A \cup B)$, that is there exists a neighborhood W of the point x which does not intersect the set $A \cup B = \overline{(U \cap V)}$. Therefore $W \subseteq U \cap V$ and. due to **NS2**, $U \cap V$ is a neighborhood of x.

1) $\Rightarrow$ 3). Suppose that $x \in f(A)$ and $x \in f(B)$; we have to show that $x \in f(A \cup B)$. $x \in f(A)$ means that $U \cap A = \{x\}$ for some a neighborhood U of x; similarly, $V \cap B = \{x\}$ for some a neighborhood V of x. Due to **NS3**, $U \cap V$ is a neighborhood of x. Since $U \cap V$ intersects $A \cup B$ only at the point x, we conclude that $x \in f(A \cup B)$.

3) $\Rightarrow$ 1). Reverse the previous arguing. $\square$

As a corollary of Proposition 6 and Lemma 3 we obtain the following (well-known in Choice Theory) assertion: *Any heritage choice function can be represented as a union of concordant choice functions.*

Dominance relation

As for the case of closure operators, there is a structure on the set X which generates (arbitrary) neighborhood systems satisfying the axiom **NS3** and the corresponding additive operators and concordant choice functions. This structure is a binary relation $\prec$ on X. A point x with $x \prec x$ is a dummy; D denotes the set of dummies.

Let us say how this structure generates: a) a neighborhood system $\mathcal{N} = (\mathcal{N}_x)$, b) an additive operator $\mu = Adh(\mathcal{N})$, and c) a concordant choice function $c = Iso(\mathcal{N})$.

a) A dummy-point has no neighborhood. For a non-dummy point x, let $U_x = \{x\} \cup \{y \in X,\ x \prec y\}$. A neighborhood of such a point is a subset containing U_x. Thus, U_x is the minimal neighborhood, hence the obtained neighborhood system $\mathcal{N}$ satisfies **NS3**.

b) $\mu(A) = D \cup A \cup \{x,\ x \prec a \text{ for some } a \in A\}$.

c) $c(A) = A - dom(A)$, where $dom(A) = \{x,\ x \prec a \text{ for some } a \in A\}$. In other words, $c(A)$ consists of non-dominated items. Note that every dummy is dominated. It is easy to see that c is a concordant choice function.

Every additive operator (or concordant choice function) is generated by an appropriate $\prec$. In terms of choice function c this relation $\prec$ is constructed as follows. For $x, y \in X$

$$x \prec y \Leftrightarrow x \notin c(\{x, y\}).$$

The obtained binary relation $\prec$ is called in Choice Theory a *revealed preference*.

Note also that one could consider a reflexive variant of the previous structure. The reflexive variant consists of a pair $(D, \preceq)$, where D is a set (of dummies) and $\preceq$ is a reflexive binary relation on X such that $x \preceq y$ implies $x \notin D$. The structures $\prec$ and $(D, \preceq)$ are related as follows:

$$x \in D \Leftrightarrow x \prec x; \quad \text{for } x \notin D \quad x \preceq y \Leftrightarrow \text{ if } x = y \text{ or } x \prec y.$$

In terms of the reflexive variant $(D, \preceq)$ the corresponding choice function c is given by the rule (compare with the formula for $Max_{\preceq}$ in the no-dummy case from Section 2.1):

$$c(A) = \{a \in A - D,\ a \preceq x \text{ for } x \in A \text{ implies } x = a\}.$$

Let us explain how the conditions of exchange, anti-exchange and closedness of the corresponding extensive operator μ are formulated in terms of $(D, \preceq)$.

1) The operator μ satisfies **Exc** iff the relation $\preceq$ is symmetric on $X - D$.

2) The operator μ satisfies **AExc** iff the relation $\preceq$ is anti-symmetric on $X - D$.

In particular, μ satisfies both **Exc** and **AExc** iff $\preceq$ is identity on $X - D$. In this case $\mu(A) = D \cup A \cup \{x,\ x \preceq a \text{ for some } a \in A \cap D\}$.

3) μ is closure operator iff the relation $\preceq$ is given on the set $X - D$ and is transitive, that is a pre-order on $X - D$. The corresponding neighborhood system satisfies **NS3** and **NS4**; the corresponding pre-topology is a topology on $X - D$.

9 The inversion

Here we consider a remarkable involution on the sets **NS**, **Mon**, and **Her**.

Let $\mathcal{N} = (\mathcal{N}_x, x \in X)$ be a neighborhood system on X. A subset $V \subseteq X$ is called an *inverse neighborhood* of a point x if $x \in V$ and $V - x$ intersects every neighborhood of the point x. It is clear that the set $\mathcal{N}_x^\circ$ of inverse neighborhoods of x satisfies the axioms **NS1** and **NS2**, and we yield *the inverse* neighborhood system $\mathcal{N}^\circ = (\mathcal{N}_x^\circ, x \in X)$. For example, $\{x\}$ is a neighborhood of x if and only if $\mathcal{N}_x^\circ = \emptyset$.

Lemma 4. *The inversion $\mathcal{N} \mapsto \mathcal{N}^\circ$ is an antitone involution of the set* **NS**.

Proof. The first assertion is obvious. Let us prove that $^\circ$ is involution, that is $\mathcal{N}^{\circ\circ} = \mathcal{N}$. The inclusion $\supseteq$ is trivial, and we shall prove the opposite inclusion. Suppose that $U \in \mathcal{N}_x^{\circ\circ}$. Then $U - x$ does not intersect $\overline{U} \cup x$, therefore $\overline{U} \cup x$ is not in $\mathcal{N}_x^\circ$. Hence the set $\overline{U}$ does not intersect some set $V \in \mathcal{N}_x$, $\overline{U} \cap V = \emptyset$, that is $V \subseteq U$. Due to **NS2**, $U \in \mathcal{N}_x$. $\square$

In term of (heritage) choice functions the inversion is given by the formula:

$$x \in f^\circ(A \cup x) \Leftrightarrow x \notin f(\overline{A}).$$

In particular, $x \in f^\circ(X) \Leftrightarrow f(\{x\}) = \emptyset$. If $f = Iso_{\mathcal{N}}$ for a neighborhood system $\mathcal{N}$ then $x \notin f(\overline{A})$ means that $x \cup A$ is not a neighborhood of x, that is x is not an interior point in $x \cup A$. Thus,

$$f^\circ(A) = A - int_{\mathcal{N}}(A) =: Bou_{\mathcal{N}}(A),$$

or

$$(Iso_{\mathcal{N}})^\circ = Bou_{\mathcal{N}}.$$

In terms of (monotone) extensive operators the inversion is given by the formula:

$$x \in \mu^\circ(A) \Leftrightarrow x \notin \mu(\overline{A} - x) \text{ or } x \in A.$$

Note that there exists auto-inverse neighborhood systems, monotone operators and heritage choice functions. Further we shall discuss properties of the inversion in terms of choice functions.

Proposition 7. *The axiom* **N** *is auto-inverse.*

Prove. We have to show that f° satisfies **N** provided f satisfies **N**. Suppose that $a \in f^\circ(A \cup a)$ and $b \in f^\circ(A \cup b)$; we have to prove that a or b belongs to $f^\circ(A \cup a \cup b)$. (Here we can assume that $a, b \notin A$.) $a \in f^\circ(A \cup a)$ means that $a \notin f(\overline{A})$; similarly, $b \notin f(\overline{A})$. $a \notin f^\circ(A \cup a \cup b)$ means that $a \in f(\overline{A \cup b}) = f(\overline{A} - b) = f(B \cup a)$, where $B = \overline{(A \cup a \cup b)} = \overline{A} - a - b$. Similarly, $b \in f(B \cup b)$. Due to **N**, a (or b) belongs to $f(B \cup a \cup b) = f(\overline{A})$, what contradicts to $a \notin f(\overline{A})$.

Proposition 8. *The inversion swaps the axioms* **W** *and* **M**.

Proof. Suppose that f° satisfies **M**; we have to show that f satisfies the axiom **W**. Suppose, on the contrary, that $a \in f(A \cup a)$, $b \notin f(A \cup b)$ and $a \notin f(A \cup a \cup b)$; we can assume here that $a, b \notin A$. The relation $a \in f(A \cup a)$ is equivalent to $a \notin f^\circ(\overline{A})$, the second relation $b \notin f(A \cup b)$ is equivalent to $b \in f^\circ(\overline{A})$, the third one is equivalent to $a \in f^\circ(\overline{A} - b)$. Due to the axiom **M**, the second and third inclusions imply that $a \in f^\circ(\overline{A})$, what contradicts to the first relation.

The inverse statement is proven similarly. □

Corollary. *A neighborhood system $\mathcal{N}$ satisfies* **S** *if and only if the inverse neighborhood system $\mathcal{N}^\circ$ is a pre-topology.*

As a consequence we obtain a characterization of choice functions satisfying the axioms **H** and **M**. Such choice functions take the form $\mathrm{Bou}_{\mathcal{P}}$, where $\mathcal{P}$ is a pre-topology.

Finally, the concordance axiom **C** transforms under the inversion to the following (strange from the point of view of rationality) axiom

C° $\quad f(A \cap B) \subseteq f(A) \cup f(B)$.

A choice function inverse to a concordant choice function can be non-concordant.

In Sections 10-12 we shall consider heritage choice functions (or the corresponding monotone operators) which satisfy two (or three) conditions among **M, W, N**.

10 Exchange closure operators

Matroids are closure operators (or the corresponding closure systems) which satisfy the exchange axiom **Exc**. We call the corresponding choice functions matroidal.

Definition. A choice function is called *matroidal* if it satisfies the axioms **H, W**, and **M**.

Now we give a description of matroids (and matroidal choice functions) in terms of pre-topologies.

Definition. A pre-topology is called *symmetric* if it satisfies the following axiom:

S′ Let U be an open set containing points x and y. If $U - y$ is a neighborhood of x then $U - x$ is a neighborhood of y.

Lemma 5. *A pre-topology $\mathcal{P}$ is symmetric if and only if the corresponding neighborhood system $\mathcal{N} = \mathcal{NS}(\mathcal{P})$ satisfies* **S**.

Proof. It is clear that **S** implies **S′**. Let us show the opposite. Suppose that a pre-topology $\mathcal{P}$ satisfies **S′**. Let U be a neighborhood of x and y, and let $W = \mathrm{int}(U)$. Obviously, x and y belong to W.

Suppose now that $U - y$ is a neighborhood of x. This means that there exists an open set V such that $x \in V \subseteq U - y$. It is clear that $V \subseteq W$, hence $W - y$ is a neighborhood of x. Due to **S′**, $W - x$ (and hence $U - x$) is a neighborhood of y. □

Matroidal choice functions have the form $\mathrm{Iso}_{\mathcal{P}}$, where $\mathcal{P}$ is a symmetric pre-topology. In this case the inverse pre-topology $\mathcal{P}^\circ$ also symmetric, and the same choice function can be realized as $\mathrm{Bou}_{\mathcal{P}^\circ}$. Thus, matroidal choice functions can be identify with pairs $(\mathcal{P}, \mathcal{Q})$ of reciprocally inverse pre-topologies. The inversion sends $(\mathcal{P}, \mathcal{Q})$ to $\mathcal{Q}, (\mathcal{P})$.

Facts and notions of Matroid Theory (see, for example, [1]) can be interpreted in terms of matroidal choice functions. For example, a set $I \subseteq X$ is *independent* if $I = f(I)$. A maximal (by inclusion) independent set is called a *basis* of the matroid. It is well-known that the set of complements to bases forms the set of bases of the *dual* matroid. We assert the following

Proposition 9. *If a choice function f corresponds to a matroid then the inverse choice function f° corresponds to the dual matroid.*

For this we have to show that if B is a basis (with respect to f) then its complementation $\overline{B}$ is a basis (with respect to f°).

Suppose that B is a basis with respect to f. This means that a) $b \in f(B)$ for every $b \in B$ and b) $a \notin f(B \cup a)$ for every $a \in \overline{B}$. a) is equivalent to a'): $b \notin f^\circ(\overline{B} \cup b)$ for every $b \in B$. b) is equivalent to b'): $a \in f^\circ(\overline{B})$ for every $a \in \overline{B}$. Thus we obtain that $\overline{B}$ is a basis with respect to f°. □

In conclusion of Section we give a description of matroidal choice functions which satisfy axiom **N**, that is choice functions satisfying four axioms **H, N, M, W**.

Definition. A choice function f is called *dichotomous* if $f(A) = A \cap f(X)$.

A dichotomous choice function divides all items on two parts: acceptable (that is belonging to $f(X)$) and non-acceptable. From every agenda-set A, it chooses all

acceptable items. It is easy to check that every dichotomous choice function satisfies the axioms **H, N, M, W** (as well as **C**). We assert that the inverse is also true (another characterization of dichotomous choice functions is given in [3]).

Theorem 2. *A choice function f satisfies the axioms* **H, N, M, W** *if and only if it is dichotomous.*

Proof. Let A be an arbitrary subset in X. Consider an auxiliary set $B = A \cup \overline{f(X)}$. Due to **M** (and heredity), $f(B) = B \cap f(X) = A \cap f(X)$. Therefore $f(B) \subseteq A \subseteq B$. Due to the Outcast axiom (see the next section), $f(A) = f(B) = A \cap f(X)$. $\square$

Corollary. *Axioms* **H, N, M, W** *imply* **C**.

11 Kolmogorov pre-topologies and Plott functions

Now we consider closure operators satisfying the anti-exchange axiom **AExc**. The corresponding closure systems are called *convex geometries*. The corresponding neighborhood systems satisfy axioms **K** and **NS4**. In order to work with pre-topologies, we consider two pre-topological versions of the axiom **K**:

K′ If U is a non-empty open set. Then $U - x$ is open for some $x \in U$.

K″ If U is a minimal neighborhood of a point x then the set $U - x$ is open.

Lemma 6. *For a pre-topology, the axioms* $\mathbf{K}, \mathbf{K}'$ *and* $\mathbf{K}''$ *are equivalent.*

Proof. $\mathbf{K} \Rightarrow \mathbf{K}''$. Let U be a minimal neighborhood of a point x. If $U = \{x\}$ then $U - x = \emptyset$ is open. Suppose, therefore, that y is another point in U. Due to minimality of U, $U - y$ is not a neighborhood of the point x. On the other hand, U is a neighborhood of x and y. Then **K** implies that $U - x$ is a neighborhood of y, that is $U - x$ is an open set.

$\mathbf{K}'' \Rightarrow \mathbf{K}'$. Let U be a non-empty open set. And let $V \subset U$ be a maximal open set in U different from U. Suppose $x \in U - V$ and U_x is a minimal neighborhood of x contained in U. Due to the axiom **K"**, the set $U_x - x$ is open as well. And we have the following chain of open sets

$$V \subseteq V \cup (U_x - x) \subset V \cup U_x \subseteq U.$$

From the maximality of V we conclude that $V = V \cup (U_x - x)$, $V \cup U_x = U$ and $V = U - x$.

$\mathbf{K}' \Rightarrow \mathbf{K}$. Let U be a neighborhood of distinct points x and y. Replacing U by $\mathrm{int}(U)$, we can assume that U is an open set. Due to $\mathbf{K}'$, there exists a point $z \in U$ such that $U - z$ is open. If z is x or y, all is proven. In the opposite case we replace U by $U - z$ and repeat the argument. $\square$

Example 1. Given a heritage choice function h, we introduce the notion of *accessible sets* in X. They are defined inductively:

1) the empty set $\emptyset$ is accessible;

2) if $x \notin h(E)$ and $E - x$ is an accessible set then E is an accessible set.

Let $\mathcal{E}$ be the set of accessible sets.

Theorem 3[3]. *The set $\mathcal{E}$ forms a Kolmogorov pre-topology on X.*

Proof. Let us check that a union of two accessible sets E and E' is an accessible set as well. We argue inductively on the size of E. If $E = \emptyset$ then the assertion is obvious. In the opposite case, let $x \in E$ be such an element that $x \notin h(E - x)$ and $E - x$ is accessible. By the induction, the set $(E - x) \cup E'$ is accessible. On the other hand, due to heredity, $x \notin f((E - x) \cup E')$. Therefore the set $(E - x) \cup E' \cup x = E \cup E'$ is accessible.

Since the empty set is accessible, we have proven that $\mathcal{E}$ is a pre-topology. The axiom $\mathbf{K}'$ follows from the definition of accessibility. $\square$

Now we consider choice functions corresponding to anti-matroids or Kolmogorov pre-topologies, that is choice functions satisfying the axioms **H, W, N**. Here it is convenient to introduce in the play the Outcast axiom

O If $x \notin f(A)$ then $f(A - x) \subseteq f(A)$.

In words: deleting a 'bad' item does not lead to appearance of new 'good' items. Together with **H**, the axiom **O** can be rewritten in a stronger form:

O′ If $f(B) \subseteq A \subseteq B$ then $f(B) = f(A)$.

Definition. A *Plott function* is a choice function satisfying the axioms **H** and **O**.

It is easy to see that a union of Plott functions is a Plott function. Intersection of Plott functions can be not a Plott function. Moreover, one can show that every heritage choice function h can be represented as intersection of Plott functions.

Lemma 7. *Axioms* **H** *and* **O** *are equivalent to axioms* **H, N** *and* **W**.

Proof. 1) **O** implies **N**. Indeed, suppose that $f(A \cup x \cup y)$ does not contain x and y. Due to **O**, $f(A \cup x)$ does not contain x; similarly, $f(A \cup y)$ does not contain y.

2) **H** and **O** imply **W**. Indeed, suppose $a \in f(A \cup a)$ and $b \notin f(A \cup b)$. Due to the heredity, $b \notin f(A \cup a \cup b)$. From **O** we conclude that $a \in f(A \cup a) \subseteq f(A \cup a \cup b)$.

3) **W** and **N** imply **O**. Indeed, suppose $a \in f(A \cup a)$ and $b \notin f(A \cup a \cup b)$; we have to show that $a \in f(A \cup a \cup b)$. If $b \in f(A \cup b)$ then (due to **N**) $a \in f(A \cup a \cup b)$. If $b \notin f(A \cup b)$ then (due to **W**) $a \in f(A \cup a \cup b)$. $\square$

Corollary. *A choice function is a Plott function if and only if it satisfies the axioms* **H, W, N**.

Corollary [11]. *A Plott function is a choice of extreme points of some convex geometry.*

Thus, we have the following cryptomorphic notions:

- Plott choice functions,
- anti-exchange closure operators,
- convex geometries,

[3]Actually, this statement was proven in [13] and [16].

- Kolmogorov pre-topologies.

To illustrate this claim, let us consider

Example 2. *The canonical closure operator.*

Suppose $X = 2^I$ for an auxiliary set I. Now we can speak about union $x \cup y$ of elements $x, y \in X$ as well as inclusion $x \subseteq y$. Interpreting $x \cup y$ as a (non-trivial) convex combination of points x and y, we can understand X as a convex space and consider the corresponding convex geometry. More precisely, for $A \subseteq X$, its *convex hull* $\mathrm{co}(A)$ consists of union of elements of A (note that $\emptyset$ is always in $\mathrm{co}(A)$). It is obvious that co is a closure operator.

Closed sets of this closure operator co (that is "convex" sets) are exactly pre-topologies on the set I. Thus, the lattice of convex sets is the lattice $\mathbf{PreTop}(I)$ of pre-topologies on I. We draw below this lattice for $I = \{i, j\}$:

$$\begin{tikzcd}
 & \{\emptyset, i, j, ij\} & \\
\{\emptyset, j, ij\} \arrow[ur] & & \{\emptyset, i, ij\} \arrow[ul] \\
\{\emptyset, j\} \arrow[u] & \{\emptyset, ij\} \arrow[ul] \arrow[ur] & \{\emptyset, i\} \arrow[u] \\
 & \{\emptyset\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We assert that *the closure operator* co *satisfies the anti-exchange axiom.* Indeed, suppose a and b do not belong to $\mathrm{co}(A)$, but $a \in \mathrm{co}(A \cup \{b\})$ and $b \in \mathrm{co}(A \cup \{a\})$. The relations $a \in \mathrm{co}(A \cup \{b\})$ and $a \notin \mathrm{co}(A)$ mean that $a = a \cup a_1 \cup ... \cup a_k$, where $a_1, ..., a_k \in A$, from where we conclude that $b \subseteq a$. Similarly, $b \in \mathrm{co}(A \cup \{a\})$ and $b \notin \mathrm{co}(A)$ imply that $a \subseteq b$. Hence $a = b$.

Since the lattice of closed sets of an anti-exchange closure operator is meet-distributive [9], we obtain the following fact (see also [5], Theorem 1): *the lattice* $\mathbf{PreTop}(I)$ *is meet-distributive* (hence, it is ranked). For a closed set $A \subseteq X$ (that is for a pre-topology on I) $\mathrm{rk}(A) = |A| - 1$. In particular, A is an atom of the lattice if $A = \{\emptyset, a\}$ for some $a \in X$.

Now we present the corresponding choice function co^*. By definition, $\mathrm{co}^*(A)$ consists of elements $a \in A$ such that $a \notin \mathrm{co}(A - a)$, that is a can not be represented as a union of other elements of A. Thus, $\mathrm{co}^*(A)$ consists of $\cup$-irreducible elements of A. One can say also that $\mathrm{co}^*(A)$ is the minimal base of the pre-topology $\mathrm{co}(A)$. Due to Proposition 9, the choice function co^* is a Plott function on X (actually, on $X - \emptyset$).

Being a closure operator on X, co defines a (*canonical*) pre-topology on $X = 2^I$. Open sets of this pre-topology are complements (in 2^X) to various pre-topologies on I. One can show that minimal neighborhoods of a point $x \in X$ have the form $U_{x,i}$, where $i \in x$ and

$$U_{x,i} = \{y \in X,\ i \in y \subseteq x\}.$$

The sets $U_{x,i}$ are $\cup$-irreducible elements of the canonical pre-topology on X and give meet-irreducible elements of the dual lattice of closure operators on X (see [5]).

12 Anti-Plott functions

Here we would like to discuss heritage choice functions satisfying axioms **M** and **N**. Due to Proposition 8, such functions are precisely the inversions of heritage choice functions satisfying axioms **W** and **N**, that are Plott functions. Thus, all properties of such *anti-Plott functions* can be extracted from properties of Plott functions.

1. The class of Plott functions is closed with respect to the union. Therefore the class of anti-Plott functions is closed with respect to the intersection.

2. Every Plott function is a union of the so called *linear* Plott functions. A linear Plott function is given by some simple (without repeats, see [6]) word $w = a_1a_2...a_s$ in the alphabet X. The corresponding linear Plott function chooses (from a subset A) the first letter of the word w which belongs to A (or chooses nothing, if $A \cap \{a_1, ..., a_s\} = \emptyset$).

An anti-linear function l_w° (the inversion to a linear Plott function l_w) performs as follows. Let a_t be the first letter of w which does not belong to A; then

$$l_w^\circ(A) = A - \{a_1, ..., a_{t-1}\}.$$

And every anti-Plott function can be represented as intersection of some anti-linear functions.

3. It is known in Choice Theory (see [2, 6]) that every heritage choice function h can be represented as intersection of Plott functions. Correspondingly, every heritage choice function is a union of anti-Plott functions. Let us make this assertion more explicit and precise. For this aim we define, for any pair (x, C) with $x \in C \subseteq X$, the *co-circuit* choice function $k^{x,C}$ by the following rule:

$$k^{x,C}(A) = \begin{cases} \{x\}, \text{ if } x \in A \subseteq C, \\ \emptyset \text{ otherwise.} \end{cases}$$

It is rather clear that $k^{x,C}$ is a concordant anti-Plott function. The following proposition can be found in [14] as well as the assertion that the co-circuit choice functions are nothing else that the join-irreducible elements of the lattice **Her**.

Proposition 10. *Let h be a heritage choice function. Then $h = \bigcup_{x \in h(C)} k^{x,C}$.*

4. Let us add to the axioms **H, M, N** the no-dummy requirement: $f(\{x\}) = x$ for every $x \in X$. Then the Plott function f° has the property: $f^\circ(X) = \emptyset$. The unique such a function is the empty choice function $\mathbf{0}$. Therefore $f = \mathbf{0}^\circ$ is the identity choice function $\mathbf{1}$.

5. For every heritage choice function h there exists its Plottization (see [6]), that is the maximal Plott function f such that $f \leq h$. Correspondingly, there exists the *anti-Plott hull* of h, that is the minimal anti-Plott function f such that $f \geq h$. Below we give an explicit construction of the anti-Plott hull.

We start with a reformulation of the axioms **N** and **M**. It is convenient to rewrite them as the following united axiom

NM if $a_i \in f(A \cup a_i)$ for $i = 1, ..., s$ then $f(A \cup \{a_1, ..., a_s\}) = f(A) \cup \{a_1, ..., a_s\}$.

Lemma 8. **NM** *is equivalent to the conjunction of the axioms* **N** *and* **M**.

Proof. The axiom **NM** for $s = 1$ is precisely the axiom **M**. The axiom **NM** for $s = 2$ implies **N**.

Conversely, prove at first that $\mathbf{NM}_2$ is true. Suppose that $a \in f(A \cup a)$ and $b \in f(A \cup b)$. Due to **N** one of a or b, for example, a, is in $f(A \cup a \cup b)$. Then, due to **M**, $f(A \cup a \cup b) = f(A \cup b) \cup a = f(A) \cup a \cup b$.

The general case we shall prove by induction on s. Let $B = A \cup a_1$. Due to $\mathbf{NM}_2$, $a_i \in f(B \cup a_i)$ for $i = 2, ..., s$. By the induction, $f(A \cup \{a_1, ...a_s\}) = f(B \cup \{a_2, ...a_s\} = f(B) \cup \{a_2, ...a_s\} = f(A) \cup a_1 \cup \{a_2, ..., a_s\}$. $\square$

Let h be a heritage choice function. In Example 1 from Section 11 we have introduced the notion of accessible sets. We have shown that the set of accessible sets is a Kolmogorov pre-topology.

Corollary 1. *For a set A, let A_0 be the maximal accessible set in A. Then the choice function f, $f(A) = A - A_0$, is an anti-Plott function.*

Indeed, A_0 is the interiority of A with respect to the Kolmogorov pre-topology $\mathcal{E}$.

Corollary 2. *If h is an anti-Plott function then $h(A) = A - A_0$ for all A.*

Indeed, since A_0 is the maximal accessible set in A then, for every $a \in A - A_0$, $a \in f(A_0 \cup a)$. The statement now follows from **NM**.

Thus, f is the anti-Plott hull of h.

13 Direct images of operators

Up to now we have deal with operators on a fixed set X. Now we shall consider and compare operators, choice functions and neighborhood systems given on different sets.

Let $\varphi : X \to Y$ be a mapping of sets. We shall say that a point $x \in X$ *is over* a point $y \in Y$ if $\varphi(x) = y$. Suppose that $\mathcal{N}$ is a neighborhood system on X. We define the neighborhood system $\varphi_*(\mathcal{N})$ on Y by the following rule:

A set V is a neighborhood of $y \in Y$ if $y \in V$ and V contains a set of the form $\varphi(U)$, where is a neighborhood of some point $x \in X$ lying over y.

In other words, V is a neighborhood of $y \in Y$ if $\varphi^{-1}(V)$ is a neighborhood of some x over y.

The corresponding direct images of monotone operators and heritage choice functions are given by the following rules (we leave proves to the reader). If μ is a monotone (and extensive) operator on X then

$$\varphi_*(\mu)(B) = \varphi_+(\mu(\varphi^{-1}(B))),$$

where $B \subseteq Y$ and φ_+ is the so called *full image*, that is $\varphi_+(A) = \{y \in Y, \varphi^{-1}(y) \subseteq A\}$. Obviously, $\varphi_*(\varepsilon)$ is a monotone extensive operator on Y. Since φ_+ commutes with $\cap$, φ_* commutes with $\cap$.

Let now h be a heritage choice function on X. Its *direct image* (with respect to the mapping φ) is given by the following formula (where $B \subseteq Y$):

$$b \in \varphi_*(h)(B \cup b) \Leftrightarrow \text{ there exists } a \in X \text{ over } b \text{ such that } a \in h(\varphi^{-1}(B) \cup a)\}.$$

The direct image of choice functions commutes with $\cup$.

Theorem 4. *Suppose that a neighborhood system $\mathcal{N}$ on X satisfies* **NS4** *(that is a pre-topology). Then $\varphi_*(\mathcal{N})$ also is a pre-topology and a set $V \subseteq Y$ is open if and only if it has the form $\varphi(U)$ for an open set U in X.*

Proof. It is clear that, for an open U in X, the set $\varphi(U)$ is open in Y.

Suppose now that $V \subseteq Y$ is a neighborhood of a point y. And let U be a neighborhood of a point $x \in X$ over y such that $\varphi(U) \subseteq V$. Since $\mathcal{N}$ satisfies **NS4**, there exists an open neighborhood U' of the point x such that $U' \subseteq U$. Then $\varphi(U')$ is an open neighborhood of y, contained in V. Thus, $\varphi_*(\mathcal{N})$ satisfies **NS4**.

Let, finally, V be an open set in Y. For every $y \in V$, there exists a point x over y and an open neighborhood U_x of x such that $\varphi(U_x) \subseteq V$. It is clear that $U = \cup_x U_x$ is open set in X and $\varphi(U) = V$. $\square$

Corollary. *φ_* transforms closure operators in closure operators and closed choice functions in closed ones.*

Corollary. *The map $\varphi_* : \mathbf{PreTop}(X) \to \mathbf{PreTop}(Y)$ commutes with $\vee$.*

Proposition 14. *The direct image of a Kolmogorov pre-topology is a Kolmogorov pre-topology.*

(The corresponding fact for Plott choice functions was proven in [6], Proposition 8. Note that in the case of Plott functions our direct image coincides with that given in [6].)

Proof. Let $\mathcal{P}$ be a Kolmogorov pre-topology on X; we have to show that $\varphi_*(\mathcal{P})$ is Kolmogorov as well. Suppose V is open in Y, that is $V = \varphi(U)$, where $U \in \mathcal{P}$. One need to check that $V - y$ is open for some $y \in Y$. We shall assume that U is minimal open set with the property $V = \varphi(U)$. Since $\mathcal{P}$ is Kolmogorov pre-topology, there exists a point $x \in U$ such that $U' = U - x$ is open. Since U is minimal, $\varphi(U')$ differs from $V = \varphi(U)$, and the difference is the point $\varphi(x)$. Thus $\varphi(U') = V - \varphi(x)$ is open in Y. $\square$

14 Topological development of a pre-topology

The direct image is a powerful tool of constructing choice functions, pre-topologies or monotone operators. We take a simple object on a set X and a mapping $\varphi : X \to Y$; then the direct image gives us a (more complicate) object on the set Y. Moreover, we shall show that this tool is universal.

Let $\mathcal{N} = (\mathcal{N}_x,\ x \in X)$ be a neighborhood system on a set X. For $x \in X$ and $N \in \mathcal{N}_x$, we let $X_{x,N} = N$. We endow this set by a very simple neighborhood system: all points of $X_{x,N}$ except x are dummies, and $x \in X_{x,N}$ has a unique neighborhood

N. Finally, $\widehat{X}$ is the direct sum of $X_{x,N}$ as neighborhood spaces; $\widehat{X} = \coprod_{x,N\in\mathcal{N}_x} X_{x,N}$. There is the canonical mapping φ of $\widehat{X}$ in X; on a summand $X_{x,N}$ it coincides with the natural embedding of N in X.

It is easy to check that the direct image of the neighborhood system $\widehat{\mathcal{N}}$ is the initial $\mathcal{N}$. Actually, this construction coincides with that of Litvakov [12] and with a decomposition given by Proposition 9.

In the case of pre-topologies, one possible to give more economical and less tautological construction. Namely, we construct a *topological development* of a pre-topology. More precisely, for a pre-topology $\mathcal{P}$ on X, we construct canonically a new set $\widetilde{X}$, a topology $\widetilde{\mathcal{P}}$ on $\widetilde{X}$ and a mapping $\psi : \widetilde{X} \to X$ such that $\psi_*(\widetilde{\mathcal{P}}) = \mathcal{P}$.

For this we use minimal neighborhoods. Let $\mathcal{B}(x)$ denote the set of minimal neighborhoods of a point $x \in X$. A pair (x, U), where $x \in X$ and $U \in \mathcal{B}(x)$, is called a *super-point* of X. The set $\widetilde{X}$ consists of all super-points of X; in other words, $\widetilde{X}$ is a graph of of the correspondence $\mathcal{B}$. The mapping $\psi : \widetilde{X} \to X$ transforms a super-point (x, U) in the point x.

Now we define the topology $\widetilde{\mathcal{P}}$ on $\widetilde{X}$. Given a super-point $\widetilde{x} = (x, U)$, let $\widetilde{U}(\widetilde{x})$ denote a subset in $\widetilde{X}$ consisting of pairs $(x', U') \in \widetilde{X}$ such that $U' \subseteq U$. A subset $\widetilde{U} \subseteq \widetilde{X}$ is called *open* (and belonging to $\widetilde{\mathcal{P}}$) if $\widetilde{U}$ contains $\widetilde{U}(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{U}$. Clearly, $\widetilde{\mathcal{P}}$ is a pre-topology on $\widetilde{X}$.

Theorem 5. *$\widetilde{\mathcal{P}}$ is a topology on $\widetilde{X}$, and $\psi_*(\widetilde{\mathcal{P}}) = \mathcal{P}$.*

We need the following

Lemma 9. *Let $\widetilde{x} = (x, U)$ be a super-point. Then*
1) *the subset $\widetilde{U}(\widetilde{x})$ is open in $\widetilde{X}$;*
2) *$\psi(\widetilde{U}(\widetilde{x})) = U$.*

Proof of Lemma. 1) Suppose that a super-point $\tilde{y} = (y, V)$ is in $U(\tilde{x})$. This means that $V \subseteq U$. If a super-point (z, W) belongs to $\widetilde{V}(\widetilde{y})$ then $W \subseteq V \subseteq U$ and hence $(z, W) \in \widetilde{U}(\widetilde{x})$.

2) Obviously $\psi(\tilde{U}(\tilde{x})) \subseteq U$. Conversely, take an arbitrary point y of U. Let V be a minimal neighborhood of the point y such that $V \subseteq U$. Then the super-point $\tilde{y} = (y, V)$ belongs to $\tilde{U}(\tilde{x})$ and $\psi(\tilde{y}) = y$. $\square$

Proof of Theorem 5. Due to Lemma 9, 1), the set $\widetilde{U}(\widetilde{x})$ is a unique minimal (open) neighborhood of $\widetilde{x}$. This implies that $\widetilde{\mathcal{P}}$ is a topology.

Let us check now that $\psi_*(\widetilde{\mathcal{P}}) = \mathcal{P}$. Suppose that $\widetilde{U}$ is an open set in $\widetilde{X}$. Due to Lemma 9, 2), $\psi(\widetilde{U})$ is open in X. Conversely, suppose that U is an open set in X. Define $\widetilde{U}$ as the set of super-points (x, V) such that $V \subseteq U$ (compare with the definition of $\widetilde{U}(\widetilde{x})$. It is obvious that $\widetilde{U}(\widetilde{x} \subseteq \widetilde{U}$ for every $\widetilde{x} \in \widetilde{U}$, from where $\widetilde{U}$ is open. On the other hand, $\psi(\widetilde{U}) = U$. Indeed, if $x \in U$, then there exists a minimal neighborhood V of x which is contained in U; in this case $(x, V) \in \widetilde{U}$ and $\psi(x, V) = x$. $\square$

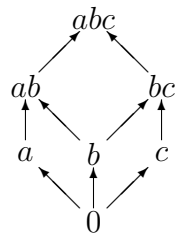
As consequences we get: *every closed choice function is the direct image of a choice function rationalized by a transitive and reflexive binary relation; every matroidal choice function is the direct image of a choice function rationalized by an equivalence relation.*

Proposition 11. *In the previous notations, if $\mathcal{P}$ is a Kolmogorov pre-topology then $\widetilde{\mathcal{P}}$ is a Kolmogorov topology.*

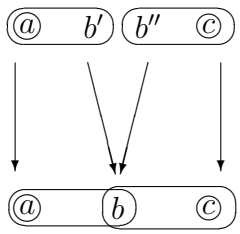
Proof. Suppose that $\widetilde{U}$ is a minimal (open) neighborhood of a super-point $\widetilde{x} = (x, U)$; as we know $\widetilde{U} = \widetilde{U}(\widetilde{x})$. We have to show that $\widetilde{U} - \widetilde{x}$ is an open set in $\widetilde{X}$. Suppose that a super-point $\widetilde{y} = (y, V)$ lies in $\widetilde{U} - \widetilde{x}$; we have to show that the super-point $\widetilde{x} = (x, U) \notin \widetilde{V}(\widetilde{y})$. Suppose the contrary, that $\widetilde{x} \in \widetilde{V}(\widetilde{y})$. Then $U \subseteq V$ from where $U = V$, and therefore $y \neq x$. Since V is a minimal neighborhood of y, the axiom **K'** implies that $U - y$ is a neighborhood of x, what contradicts to the assumption that U is a minimal neighborhood of x. This contradiction shows that $\widetilde{U} - \widetilde{x}$ is open, and the topology $\widetilde{\mathcal{P}}$ is Kolmogorov. $\square$

As a corollary we obtain the following result from [6]: *Any Plott choice function is the direct image of a choice function rationalized by a partial order.*

Example 3. Let us provide an illustration to Theorem 5. Suppose that $X = \{a, b, c\}$, and consider the following closure operator σ on X: all subsets in X ate closed except the set $\{a, c\}$, $\sigma(\{a, c\}) = X$. The lattice of closed sets is drawn below.



In the following picture we draw (in the form of minimal neighborhoods) the corresponding pre-topology (which is Kolmogorov) and the topological development of this pre-topology. The development consists of four points a, b', b'', c; $\psi(a) = a$, $\psi(c) = c$, $\psi(b') = \psi(b'') = b$.



This example is a particular case of the canonical pre-topology on the set $X = 2^I$, see Example 2 from Section 11. In the general case, the topological development is the direct sum

$$\tilde{X} = \coprod_{i \in I} 2^{I-i}$$

of posets 2^{I-i}. The mapping ψ transforms x from summand 2^{I-i} of $\widetilde{X}$ to $x \cup \{i\} \in X$. Open sets of the topology on 2^{I-i} are upper contours of the natural order on 2^{I-i}.

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