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Discrete convexity and equilibria in economies with indivisible goods and money

Vladimir Danilov[a,*], Gleb Koshevoy[a], Kazuo Murota[b]

[a]*Central Institute of Economics and Mathematics, Russian Academy of Sciences, Nahimovski prospect 47, Moscow 117418, Russia*

[b]*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan*

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Abstract

We consider a production economy with many indivisible goods and one perfectly divisible good. The aim of the paper is to provide some light on the reasons for which equilibrium exists for such an economy. It turns out, that a main reason for the existence is that supplies and demands of indivisible goods should be sets of a class of discrete convexity. The class of generalized polymatroids provides one of the most interesting classes of discrete convexity. © 2001 Published by Elsevier Science B.V.

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1. Introduction

We consider here an economy with production in which there is only one perfectly divisible good (numeraire or money) and all other goods are indivisible. In other words, the commodity space of the model takes the following form $\mathbb{Z}^K \times \mathbb{R}$, where K is a set of types of indivisible goods.

Several models of exchange economies with indivisible goods and money have been considered in the literature. In 1970, Henry (1970) proved the existence of a competitive equilibrium in an exchange economy with one indivisible good and money. He also gave an example of an economy with two indivisible goods in which no competitive

*Corresponding author. Tel.: +7-95-332-4606; fax: +7-95-718-9615.
E-mail address: danilov@cemi.rssi.ru (V. Danilov).

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equilibrium obtains. In 1982, Kaneko (1982) established existence of an equilibrium for a non-transferable extension of the Shapley–Shubik model. In 1984, Quinzii (1984), Gale (1984) and Svensson (1984) generalized his result for house markets. There has recently been renewed interest in exchange economies with indivisibilities and money. Van der Laan et al. (1997) consider a multi-product generalization of Gale's setup. Bevia et al. (1999) consider an economy with several types of indivisibles and specific transferable utilities. Danilov et al. (1995) and Bikhchandani and Mamer (1997) provided a necessary and sufficient condition of existence of equilibria for economies with transferable utilities.

However, these results do not provide general existence conditions akin to those prevailing for divisible goods economies. These are well known. These conditions boil down to continuity and convexity. One could hope that existence of equilibria in economies with indivisibles should be connected with an appropriate notion of convexity in the lattices of integer points.

The aim of the paper is to provide a general existence theorem for the Arrow–Debreu model with indivisibles and money. It turns out that the rationale for existence is that supplies and demands of indivisible goods be discrete-convex sets. In particular, our approach clarifies the afore-mentioned existence results.

There are two sets of conditions in the existence issue with indivisibles. The first ones consist in 'continuity' assumptions. They should warrant upper semi-continuity of demands and supplies with respect to prices and 'proper' behavior at the boundary of the price space. This *topological* part is common to both divisible and indivisible cases. The second set of conditions rules the shape of demands and supplies given prices. In economies with divisible goods, demand and supply should be convex subsets of an Euclidean space. In economies with indivisibles, demand and supply sets should belong to some *class of discrete convexity*.

Let us explain the essence of the matter. Consider an exchange economy with indivisibles and money (of course we have production in our model, but for the time being the reader might forget about it). Let $D_h(p) \subset \mathbb{Z}^K$ denote the 'discrete part' of the h-th consumer's demand, given a price $p \in \mathbb{Z}^K_+$, $h \in H$. Replace the 'discrete' demand sets $D_h(p)$ by their convex hulls $\mathrm{co}(D_h(p))$ and consider the convexified economy (with divisible goods) whose demands are $\mathrm{co}(D_h(p))$, $h \in H$. Assume that the sets $\mathrm{co}(D_h(p))$, $h \in H$, are 'well-behaved' with respect to prices. Then standard fixed-point arguments enable to assert existence of an equilibrium price p^* for the convexified economy. This means that the total initial endowment vector $\Sigma_{h\in H} W_h$ (an integral vector) belongs to the sum of the convexified demands $\Sigma_{h\in H}\ \mathrm{co}(D_h(p^*)) = \mathrm{co}(\Sigma_{h\in H}\ D_h\ (p^*))$. Generally speaking, this does not imply that $\Sigma_{h\in H} W_h$ belongs to the sum of the initial 'discrete' demands $\Sigma_{h\in H} D_h\ (p^*)$. Our requirement on discrete convexity of demands ensures that this implication holds. And therefore, the price p^* will be the equilibrium price in the discrete model.

Thus, to get an equilibrium in the discrete model, we need to ensure that the following equality holds:

$$\mathrm{co}\Big(\sum_{h\in H} (D_h(p^*)\Big) \cap \mathbb{Z}^K = \sum_{h\in H} D_h(p^*). \qquad (*)$$

Of course, we should require such an identity for each individual demand, i.e., $D_h(p) = \mathrm{co}(D_h(p)) \cap \mathbb{Z}^K$. We call *pseudoconvex* the subsets of $\mathbb{Z}^K$ having this property. But pseudoconvexity of individual demands is not enough to have (*). Indeed, the sum of pseudoconvex sets might, in general, fail to be a pseudoconvex set[1]. Nevertheless, this is not a hopeless case: there are subclasses of pseudoconvex sets, that are closed with respect to summation.

A class $\mathcal{D}$ of pseudoconvex subsets is said to be a *class of discrete convexity* if it is closed with respect to sums. Thus, if the sets $D_h(p^*)$, $h \in H$, belong to such a class $\mathcal{D}$, (*) holds, and, hence, the discrete model has an equilibrium.

Modulo the topological part of assumptions, our existence theorem (Theorem 1) states that equilibria in a discrete economy with money exist if the discrete parts of demands and supplies belong to some class of discrete convexity. Therefore, the discrete convexity of supplies and demands is, in our opinion, the true reason for the equilibrium existence in economies with indivisibilities.

However, finding both the rationale and the proper condition is only a half of the job. (Similar in spirit conditions appeared in Danilov et al. (1995) for a general case and in Bikhchandani and Mamer (1997) for a case of transferable utilities.) The second half consists in finding non-trivial interesting classes of discrete convexity. Fortunately, there are many classes of discrete convexity. The most interesting one with respect to the problem considered here is the class of integral generalized polymatroids (see Section 5). We also give a few examples of utility functions generating polymatroidal demands. In particular, in all afore-mentioned models (Henry, 1970; Kaneko, 1982; Gale, 1984; Quinzii, 1984; Svensson, 1984; van der Laan et al., 1997; Bevia et al., 1999) which dealt with existence of equilibria, demands turn out to be sets of the class discrete convexity associated with integral generalized polymatroids.

Remark. The authors became aware, after submitting their paper, of an article by Gul and Stacchetti (1999), in which some existence theorem for economies with indivisibles presented in single units and money is proven. Demands and supplies in this model also turn out to be sets belonging to the class of discrete convexity associated with integral generalized polymatroids. We will return to this question in another paper.

2. Model and definitions

We consider a production economy with a finite number of consumers and producers, a finite number of indivisible goods, and one perfectly divisible good. Indivisible commodities form only a part of the whole economy, the other part is represented by means of a perfectly divisible aggregate good, called money. We denote by K the set of indivisible goods. The commodity space of the model takes the form $\mathbb{Z}^K \times \mathbb{R}$.

There is a finite set L of producers. A producer $l \in L$ is described by its cost function $c_l : \mathbb{Z}^K_+ \to \mathbb{R} \cup \{+\infty\}$, $c_l(0) = 0$, c_l is a monotone function. This means that producers use

[1] Emmerson (1972) mistakenly assumed that the (*) always holds.

only this aggregate divisible good as productive input. The producer l produces a vector of discrete goods $Z \in \mathbb{Z}_+^K$ at the cost of using $c_l(Z)$ of money. If $c_l(Z) = +\infty$, then producer l cannot produce Z.

There is a finite set H of consumers. Consumer $h \in H$ has a preference $\preceq_h$ on $\mathbb{Z}_+^K \times \mathbb{R}_+$, which is a complete, closed, and transitive binary relation. We assume that preferences are increasing in indivisibles and strictly increasing in money. Each consumer $h \in H$ is endowed with a vector of initial endowment $(W_h, w_h) \in \mathbb{Z}_+^K \times \mathbb{R}_+$ and shares in the firms $\theta_{lh} \geq 0$, $l \in L$, $(\Sigma_{h\in H} \theta_{lh} = 1$ for any $l \in L)$.

Thus, a discrete economy $\mathscr{E}$ is a collection

$$\mathscr{E} = \{(\preceq_h, (W_h, w_h), (\theta_{lh}), l \in L), h \in H; (c_l), \ l \in L\}.$$

Markets and production are assumed to be perfectly competitive. We normalize prices in such a way that the price of money equals 1; and we define prices of indivisible goods as a positive linear functional on $\mathbb{Z}^K$, or, simply, a nonnegative vector $p \in \mathbb{R}_+^K$, $p(X) = \Sigma_{k\in K} p_k X^k$, where $p = (p_k)_{k\in K}$, $p_k \in \mathbb{R}_+$, $X = (X^k)_{k\in K}$, $X^k \in \mathbb{Z}$.

Given a price p, each producer $l \in L$ maximizes the following program:

$$\max_{Y \in \mathbb{Z}_+^K} (p(Y) - c_l(Y)). \tag{1}$$

Denote $\pi_l(p) = \max(p(Y) - c_l(Y))$.

Each consumer $h \in H$ seeks a best element (X_h, m_h) with respect to the preference $\preceq_h$ in the budget set

$$B_h(p) = \{(X, m) \in \mathbb{Z}_+^K \times \mathbb{R}_+ \mid p(X) + m \leq \beta_h(p)\},$$

where the income, $\beta_h(p)$, is defined by

$$\beta_h(p) = p(W_h) + w_h + \sum_{l\in L} \theta_{lh} \pi_l(p).$$

Definition 1. A tuple $((X_h, m_h)_{h\in H}; (Y_l)_{l\in L}; p)$ is a *competitive equilibrium* in the economy $\mathscr{E}$ if

(a) Y_l is a solution to (1), $l \in L$,
(b) for each $h \in H$, (X_h, m_h) is a best element in the budget set $B_h(p)$ with respect to the preference $\preceq_h$,
(c) all markets clear:

$$\sum_{h\in H} X_h = \sum_{h\in H} W_h + \sum_{l\in L} Y_l, \tag{2}$$

$$\sum_{h\in H} m_h + \sum_{l\in L} c_l(Y_l) = \sum_{h\in H} w_h. \tag{3}$$

As preferences are strictly increasing with respect to money, the clearing of the indivisible goods markets implies that of the money market.

3. Discrete convexity and the existence issue

Here we present a general existence theorem for production economies with indivisibles and money. In order to emphasize the discrete aspect of the existence, i.e., a 'proper' notion of convexity in the lattices of integral vectors, we state it in a conditional form, omitting topological issues.

Given a price p, denote by

$$S_l(p) = \text{Argmax}_{Y \in \mathbb{Z}_+^K} (p(Y) - c_l(Y))$$

the supply of the producer $l \in L$. Each consumer h chooses a best element $(X_h, m_h) \in B_h(p)$ with respect to the preference $\preceq_h$. Monotonicity of preferences implies $p(X_h) + m_h = \beta_h(p)$. Denote by $D_h(p)$ the set of all $X_h \in \mathbb{Z}_+^K$ such that $(X_h, \beta_h(p) - p(X_h))$ is a best element in the budget set $B_h(p)$ with respect to the preference $\preceq_h$. An equilibrium exists whenever we find a price p^* such that the following inclusion

$$\sum_{h \in H} W_h \in \sum_{h \in H} D_h(p^*) - \sum_{l \in L} S_l(p^*) \tag{4}$$

holds, where the sum of sets is defined as the Minkowski sum, $A \pm B = \{a \pm b \mid a \in A, b \in B\}$.

Suppose the discrete economy $\mathscr{E}$ has been convexified. That is we have a new economy co($\mathscr{E}$) with divisible goods (whose commodity space is $\mathbb{R}^K \times \mathbb{R}$) in which individual demands and supplies are the convex hulls of demands and supplies of the initial economy $\mathscr{E}$.

Namely, in the convexified economy, for a price p, the demand of the h-th consumer is the set $\{(x, \beta_h(p) - p(x)),\ x \in \text{co}(D_h(p))\}$, where co($\cdot$) denotes the convex hull of a set. The supplies of co($\mathscr{E}$) are given by $\text{co}(S_l(p))$, $l \in L$. The profits of the producers in co($\mathscr{E}$) are equal to their profits in $\mathscr{E}$.

Assume now we have an equilibrium of the economy co($\mathscr{E}$)[2], i.e., there exists a price p^* such that

$$\sum_{h \in H} W_h \in \sum_{h \in H} \text{co}(D_h(p^*)) - \sum_{l \in L} \text{co}(S_l(p^*)). \tag{5}$$

When (5) implies (4), p^* is also an equilibrium price for the discrete economy $\mathscr{E}$. But, this implication does not hold for arbitrary discrete sets, and, thus, equilibria might fail to exist. We can see this in the following simple example.

Example 1. There are two agents and two types of goods, $H = K = \{1, 2\}$. The utilities of agent are $u_1((X^1, X^2), m) = 2\min(X^1, X^2) + m$, $u_2((X^1, X^2), m) = \min(2X^1 + 2X^2, 2) + m$. Since these utilities are quasi-linear in money, we do not specify the individual initial endowments and define only the aggregate endowment $\Omega = (1, 1)$ of indivisible goods. In the convexified economy, there exists a unique equilibrium price

[2]There are many results of existence of equilibria in economies with divisible goods, see, for example Arrow and Hahn (1971); McKenzie (1987); we also establish an existence result in the next section.

$p^* = (1, 1)$. At the price p^*, the agents' demands are respectively co({(1, 0), (0, 1)}) and co({(0, 0), (1, 1)}), and $x_1 = (1/2, 1/2)$, $x_2 = (1/2, 1/2)$ are the equilibrium allocations.

But this price p^* is not an equilibrium price of the initial discrete economy. In fact, at the price $p^* = (1, 1)$, the demands of agents are $D_1(p^*) = \{(1, 0), (0, 1)\}$ and $D_2(p^*) = \{(0, 0), (1, 1)\}$, respectively. The initial endowment (1, 1) belongs to the sum of convexified demands co({(1, 0), (0, 1)}) + co({(0, 0), (1, 1)}), but does not belong to the sum of their discrete counterparts, $\{(1, 0), (0, 1)\} + \{(0, 0), (1, 1)\} = \{(0, 1), (1, 0), (1, 2), (2, 1)\}$. For those sets, (5) does not imply (4). □

We introduce now our main notion.

Definition 2. A set $\mathscr{D}$ of subsets of $\mathbb{Z}^K$ which satisfies the following axioms DC1 and DC2 is said to be a class of discrete convexity.

DC1. For any $\mathbf{A} \in \mathscr{D}$, there holds

$$\mathbf{A} = (\mathrm{co}(\mathbf{A})) \cap \mathbb{Z}^K.$$

DC2. For any $\mathbf{A}$ and $\mathbf{B} \in \mathscr{D}$, we have $\mathbf{A} \pm \mathbf{B} \in \mathscr{D}$.

Given a class $\mathscr{D}$ of discrete convexity, suppose $D_h(p^*)$, $h \in H$, and $S_l(p^*)$, $l \in L$, are sets of $\mathscr{D}$. Then, the implication (5)⇒(4) holds. In fact, because of DC2, the set $A = \Sigma_{h\in H} D_h(p^*) - \Sigma_{l\in L} S_l(p^*)$ belongs to the class $\mathscr{D}$. Since $\Sigma_{h\in H} W_h \in \mathbb{Z}^K_+$ and $\Sigma_{h\in H} W_h \in \mathrm{co}A = \Sigma_{h\in H} \mathrm{co}D_h(p^*) - \Sigma_{l\in L} \mathrm{co}S_l(p^*)$, we conclude that DC1 implies $\Sigma_{h\in H} W_h \in A$.

Thus, we obtain the following conditional theorem.

Theorem 1. *Let $\mathscr{E}$ be a discrete economy. Assume* (1) *there exists an equilibrium price p^* for the convexified economy* co($\mathscr{E}$); (2) *there exists a class $\mathscr{D}$ of discrete convex sets such that, at the price p^*, the demands $D_h(p^*)$, $h \in H$, and supplies $S_l(p^*)$, $l \in L$, belong to $\mathscr{D}$. Then p^* is an equilibrium price for $\mathscr{E}$.*

Discrete convexity of demands and supplies is the main condition, which needs to be added to topological conditions, to ensure existence of equilibrium for discrete economy with money. We do not state that this condition is a necessary condition in a formal sense; it could occur that an equilibrium exists without discrete convexity. However, just as convexity is a 'necessary' condition for economies with divisible goods, the discrete convexity is a 'necessary' condition for economies with indivisibles and money. We can also say that our existence result does not rest upon any specifics of initial endowments and upon the number of agents.

In Section 4, we provide some topological conditions, ensuring that condition 1 of Theorem 1 is satisfied, and in Section 5, we consider an interesting class of discrete convexity associated with integral g-polymatroids.

Let us display the power of Theorem 1 in the following example. In this example we highlight the reason underlying the existence of equilibrium in Gale's model (Gale, 1984).

Example 2. Gale considers an exchange economy in which each trader owns one indivisible object (one may think of these objects as houses). Therefore, we can identify the set K of objects with the set of traders H. In other words, the initial endowments $W_h = \mathbf{1}_h$, $h \in H$, where, for $S \subseteqq K$, $\mathbf{1}_S$ denotes the vector in $\mathbb{R}^K$ with coordinates $(\mathbf{1}_S)_k = 1$, if $k \in S$ and 0 otherwise, $k \in K$.

The trader's preferences are such that no trader desires more than one object. Gale does not introduce explicitly trader's preferences in his model, instead he specifies trader's demand at each price.

Given a price $p \in \mathbb{R}^H_+$, the demand of the trader h, $D_h(p)$, is a subset of $\{0\} \cup \{\mathbf{1}_h, h \in H\}$ (this holds because traders desire no more than one object). From the Gale's assumptions it follows that the demand correspondences $p \to D_h(p)$ are upper semicontinuous and have 'regular' behavior with respect to the boundary values of prices. In other words, the assumptions are 'topological' indeed. They warrant existence of an equilibrium in the convexification of his exchange economy, i.e., there exist $p^* \in \mathbb{R}^H_+$ and $x_h \in \text{co}(D_h(p^*))$, $h \in H$, such that

$$\sum_{h \in H} x_h = \mathbf{1}_H. \tag{6}$$

The equality (6) breaks down to the following list of equalities

$$\sum_{h \in H} x_{hh'} = 1 \text{ for any } h' \in H, \tag{7}$$

where $x_{hh'}$ denotes the h'-th coordinate of x_h. Moreover, every x_h being a convex combination of vectors of the set $\{0\} \cup \{\mathbf{1}_{h'}, h' \in H\}$, satisfies the inequality

$$\sum_{h' \in H} x_{hh'} \leq 1, \quad h \in H. \tag{8}$$

By (7), all inequalities (8) are tight. Hence, the matrix $(x_{hh'})$ is doubly stochastic. By Birkhoff's theorem, there exists a permutation $\pi{:}H \to H$ with $x_{h\pi(h)} > 0$, $h \in H$. This means that $\mathbf{1}_{\pi(h)} \in D_h(p^*)$. Therefore, the bundle $(p^*, (X_h = \mathbf{1}_{\pi(h)})_{h \in H})$ is an equilibrium of the discrete economy.

In the development above, there is a clear split between topologically based and discretely based arguments. The use of a fixed-point theorem is the topological argument; and the use of the Birkhoff theorem is the discrete one. Of course, the calling for the Birkhoff theorem is based on a particular feature of the agents' demands in his models: consumers need no more than one item, i.e., demands are subsets of $\{0\} \cup K$.

In our set-up, the topological part remains unchanged. But the discrete part, i.e. the going from 'divisible' equilibrium allocation to an indivisible allocation, differs. Namely, we pointed out that the sets $D_h(p)$ consist of integer points of some faces of the simplex $\Delta = \text{co}(0, \mathbf{1}_1, \ldots, \mathbf{1}_{|H|})$. The faces of Δ are the simplest integral g-polymatroids, and, as we will see later on in Section 5, the class of integral g-polymatroids is a class of discrete convexity. Therefore, the demands in Gale's model are discrete-convex sets, and this is the 'discrete' reason underlying the existence of equilibrium here. □

4. Existence of equilibria in the convexified economy

Here we describe a convexification procedure for a discrete economy and provide conditions which warrant existence of equilibria in the associated convexified economy.

4.1. Convexification

Suppose $\mathscr{E}$ is a discrete economy, as in Section 2. We shall convexify $\mathscr{E}$ by convexifying its cost functions and preferences.

Let $f:\mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ be a function. Denote by $\text{epi}(f) := \{(x, t) \in \mathbb{Z}^K \times \mathbb{R} \mid t \geq f(x)\}$ the epigraph of f. The *convexification* of f is a function $\text{co}(f):\mathbb{R}^K \to \mathbb{R} \cup \{+\infty\}$, whose epigraph is the closure of $\text{epi}(f)$, that is $\text{epi}(\text{co}(f)) = \overline{\text{co}}(\text{epi}(f))$. Equivalently, we might define $\text{co}(f)$ as the supremum of affine functions $h:\mathbb{R}^K \to \mathbb{R}$ such that $h(X) \leq f(X)$, for every $X \in \mathbb{Z}^K$. We implicitly assume here that such affine functions exist. Clearly, $\text{co}(f)$ is a closed convex function, i.e., its epigraph is a closed convex set.

Definition 3. A function $f:\mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be *pseudoconvex* if a) $\text{co}(\text{epi}(f))$ is a closed subset of $\mathbb{R}^K \times \mathbb{R}$, and b) for every $X \in \mathbb{Z}^K$, there holds $f(X) = \text{co}(f)(X)$. A function $f:\mathbb{Z}^K \to \mathbb{R} \cup \{-\infty\}$ is said to be *pseudoconcave* if $-f$ is pseudoconvex.

It follows from *a*) that, for any $x \in \mathbb{R}^K$, there exist points $X_0, X_1, \ldots, X_n$, $|K| = n$, in $\mathbb{Z}^K$ and weights $\alpha_0, \alpha_1, \ldots, \alpha_n$ ($\alpha_i \geq 0$, $\Sigma_{i=0}^n \alpha_i = 1$) such that $\text{co}(f)(x) = \Sigma_i \alpha_i f(X_i)$.

The convexification of the cost functions is straightforward from the definition. To define a convexification of preferences, we need to make some assumptions.

Assumption 1. For any agent $h \in H$ and for every bundle $Y \in \mathbb{Z}_+^K$ there holds

$$(W_h, w_h) \succ_h (Y, 0). \quad \square$$

This assumption is akin to 'abundance of money', and, in particular, it states that each consumer has a positive endowment of money. Assumption 1 is rather strong; it could be weakened, however not disposed of.

Remark. Since equilibrium allocations are individually rational allocations, the shape of the h-th agent preference on the set $\{(X, m) \in \mathbb{Z}_+^K \times \mathbb{R}_+ | (X, m) \prec_h (W_h, w_h)\}$ is irrelevant.

Assumption 2. For any agent $h \in H$ and any (X, m) with $m \geq 0$, there exists a certain amount of money $m' \in \mathbb{R}_+$ such that

$$(\mathbf{0}, m') \succeq_h (X, m). \quad \square$$

This assumption means that money can substitute any quantity of indivisible goods, and, in particular, implies that any bundle (X, m) possesses a money equivalent. In fact, if $t_h(X, m)$ denotes the infimum over all m' with $(\mathbf{0}, m') \succeq_h (X, m)$, then, it is easy to see that, $(\mathbf{0}, t_h(X, m)) \sim_h (X,m)$. Denote by m_h^0 the money equivalent of the initial endowment

(W_h, w_h). According to Assumption 1, for any $m \geq m_h^0$ and $X \in \mathbb{Z}_+^K$, there exists a unique number $q_h^m(X) \geq 0$ such that $(\mathbf{0}, m) \sim_h (X, q_h^m(X))$. The graph of $q_h^m(X)$ is the 'indifference curve' of $\preceq_h$ passing through the point $(\mathbf{0}, m)$. Thus, we can represent a preference $\preceq$ (satisfying Assumption 2) together with the money equivalent of a vector of initial endowments by the family of functions q^m on $\mathbb{Z}_+^K$, $m \geq m_0$.

Lemma 1. *This family of functions* $q^m(X)$, $m \geq m_0$, *has the following properties*: (1) $q^m(\mathbf{0}) = m$, (2) $q^m(X)$ *is decreasing in* X, (3) $q^m(X)$ *is* (*strictly*) *increasing in* m, (4) $q^m(X) \to +\infty$ *with* $m \to +\infty$, 5) $q^m(X)$ *is continuous in* m. *Conversely, a family of functions* $q^m(X)$ *with properties* (1)–(5) *defines a preference* $\preceq$ *which is continuous, increasing in* X, *strictly increasing in money, and which satisfies Assumption* 2. □

Proposition 1. *Suppose that functions* $q^m:\mathbb{Z}_+^K \to \mathbb{R}$ $(m \geq m_0)$ *have the properties* (1)–(5) *and are pseudoconvex. Then the functions* $co(q^m):\mathbb{R}_+^K \to \mathbb{R}$ *also satisfy properties* (1)–(5).

Proof. Properties (1)–(4) are almost obvious. Let us prove the property (5). Let $m \geq m_0$ and let m' be sufficiently close to m. Consider, to begin with, the case $m' \geq m$. Let $x \in \mathbb{R}_+^K$ and let X_i and α_i $(i = 0, \ldots, n)$ be such that $\mathrm{co}(q^m)(x) = \Sigma_i \; \alpha_i q^m(X_i)$. Since, for all i, there exists ϵ such that $\mathrm{co}(q^{m'})(X_i) \leq \mathrm{co}(q^m)(X_i) + \epsilon$, we have

$$\mathrm{co}(q^{m'})(x) \leq \sum_i \alpha_i \mathrm{co}(q^{m'})(X_i) \leq \sum_i \alpha_i \mathrm{co}(q^m)(X_i) + \epsilon.$$

Consider, now, the case $m' \leq m$ with m' tending to m. Let $x \in \mathbb{R}_+^K$ and let X_i' and α_i' be $(i = 0, \ldots, n)$ the corresponding points and weights for m' such that $x = \Sigma_i \; \alpha_i' X_i'$ and

$$\mathrm{co}(q^{m'})(x) = \sum_i \alpha_i' \mathrm{co}(q^{m'})(X_i').$$

We can suppose that each sequence α_i' converges to some (limit) weight α_i. Let $\alpha_0, \ldots, \alpha_s$ be positive and $\alpha_{s+1}, \ldots, \alpha_n$ be equal to 0. Then we can suppose that sequences $X_0', \ldots, X_s'$ also converge (and stabilize) to some (finite) points $X_0, \ldots, X_s$. In fact, if, say, X_0' is unbounded, then $\Sigma_i \alpha_i' X_i'$ moves away from the point x, but this cannot happen. Now,

$$\begin{aligned}\mathrm{co}(q^m)(x) - \mathrm{co}(q^{m'})(x) &\leq \sum_i \alpha_i' q^m(X_i') - \sum_i \alpha_i' q^{m'}(X_i') = \sum_i \alpha_i' (q^m(X_i') - q^{m'}(X_i')) \\ &= \sum_{i=0}^{s} \alpha_i' (q^m(X_i') - q^{m'}(X_i')) + \sum_{i=s+1}^{n} \alpha_i' (q^m(X_i') - q^{m'}(X_i')).\end{aligned}$$

When m' is sufficiently close to m, the first term is small because it is equal to $\Sigma_{i=0}^s \alpha_i' (q^m(X_i) - q^{m'}(X_i))$. The second term is also small, since $\alpha_{s+1}', \ldots, \alpha_n'$ are small and $q^m(X_i') - q^{m'}(X_i')$ are bounded (because $q^m(X_i') \leq q^{m'}(\mathbf{0})$ and $q^{m'}(X_i') \geq 0$). Q.E.D.

Assumption 3. The functions q_h^m $(h \in H, m \geq m_0^h)$ and the cost functions c_l $(l \in L)$ are pseudoconvex functions. □

By Lemma 1 and, for each h, the families of functions $\mathrm{co}(q_h^m)$ determine a preference on $\mathbb{R}_+^K \times \mathbb{R}_+$, which we call the convexification of $\preceq_h$, and denote by $\mathrm{co}(\preceq_h)$.

Thus, we have a convexified produced economy $\mathrm{co}(\mathscr{E})$, whose commodity space is $\mathbb{R}^K \times \mathbb{R}$, whose initial endowments are (W_h, w_h), $h \in H$, whose shares in the firms are θ_{lh}, $l \in L$, $h \in H$, whose preferences are $\mathrm{co}(\preceq_h)$, $h \in H$, and whose cost functions are $\mathrm{co}(c_l)$, $l \in L$.

We assert that, for any price $p \in \mathbb{R}_+^K$, there holds

$$\check{S}_l(p) = \mathrm{co}(S_l(p)),\ l \in L, \tag{9}$$

$$\check{D}_h(p) = \mathrm{co}(D_h(p)),\ h \in H, \tag{10}$$

where $\check{S}_l(p) = \mathrm{Argmax}_{y \in \mathbb{R}_+^K}\, p(y) - \mathrm{co}(c_l)(y)$, $l \in L$, $\check{D}_h(p)$ is the set of all $x \in \mathbb{R}_+^K$ such that $(x,\ \beta_h(p) - p(x))$ is a best element in the budget set $\overline{B}_h(p) = \{(x',\ m) \in \mathbb{R}_+^K \times \mathbb{R}_+ \mid p(x') + m \leq \beta_h(p)\}$ with respect to the preference $\mathrm{co}(\preceq_h)$, $h \in H$.

Let us prove Eqs. (9) (the equalities (10) are proven similarly). The inclusion $\subset$ is trivial. Let now $x \in \check{S}(p)$, that is

$$p(x) - \mathrm{co}(c)(x) \geq p(y) - \mathrm{co}(c)(y)$$

for any $y \in \mathbb{R}_+^K$. Let $x = \Sigma_i\ \alpha_i X_i$ be the convex representation such that $\mathrm{co}(c)(x) = \Sigma_i\ \alpha_i c(X_i)$. Then, for each i,

$$p(x) - \mathrm{co}(c)(x) \geq p(X_i) - \mathrm{co}(c)(X_i) = p(X_i) - c(X_i). \tag{11}$$

Multiplying (11) by α_i and summing up, we have

$$p(x) - \mathrm{co}(c)(x) \geq \sum_i \alpha_i (p(X_i) - c(X_i)) = p(x) - c(x).$$

We see from this that $p(x) - \mathrm{co}(c)(x) = p(X_i) - c(X_i)$ if $\alpha_i \neq 0$. Therefore

$$p(X_i) - c(X_i) \geq p(Y) - c(Y)$$

for every i and $Y \in \mathbb{Z}_+^K$, that is $X_i \in S(p)$ and $x \in \mathrm{co}(S(p))$.

Thus, the economy $\mathrm{co}(\mathscr{E})$ is the convexification of $\mathscr{E}$ in the sense of Section 3. We need the last assumption in order to ensure existence of equilibria in the convexified economy $\mathrm{co}(\mathscr{E})$.

Assumption 4. The total endowment of discrete goods is strictly positive: $\Sigma_{h \in H}\ W_h > 0$. The cost functions have the following property: $c_l(Y) \to +\infty$ with $\|Y\| \to +\infty$, $l \in L$. □

The second part of Assumption 4 is the ‘no-free-lunch’ condition. It holds, for example, if $c_l(Y) > 0$ for $Y \neq \mathbf{0}$.

Proposition 2. *Let* $\mathscr{E}$ *be a discrete economy and let Assumptions* 1–4 *hold. Then a competitive equilibrium exists in the convexified economy* $\mathrm{co}(\mathscr{E})$.

For a proof see Appendix A.

To combine this proposition with the conditional Theorem 1, we strengthen Assumption 3. Since we do not know which price would be an equilibrium price in the convexified economy, we need to assume supplies and demands be sets of some class of discrete convexity $\mathcal{D}$ for any price.

Definition 4. Let $\mathcal{D}$ be a class of discrete convexity in $\mathbb{Z}^K$. A pseudoconvex function $f{:}\mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be *$\mathcal{D}$-convex* if for any linear functional $p \in (\mathbb{R}^K)^*$ holds

$$\operatorname*{Argmax}_{X \in \mathbb{Z}^K} (p(X) - f(X)) \in \mathcal{D}.$$

Assumption 3′. There exists a class $\mathcal{D}$ of discrete convexity such that the functions q_h^m, $m \in \mathbb{R}_+$, $h \in H$, and the cost functions c_l, $l \in L$, are $\mathcal{D}$-convex functions. □

Thus, we have the following existence theorem.

Theorem 2. *Let $\mathcal{E}$ be a discrete economy and let Assumptions* 1, 2, 3′ *and* 4 *hold. Then a competitive equilibrium exists in $\mathcal{E}$.*

4.1.1. The case of transferable utilities

The special case of economies with transferable utility is interesting because the existence of equilibrium can be obtained as a solution to an auxiliary optimization problem. Moreover, we use this optimization problem in our proof of Proposition 2.

Recall, that a preference $\preceq$ on $\mathbb{R}_+^K \times \mathbb{R}$ is said to be *transferable* (quasi-linear in money) if it can be represented by a utility function of the form $U(X, m) = u(X) + m$ with some function u: $\mathbb{R}_+^K \to \mathbb{R}$.

Let $\hat{\mathcal{E}}$ be an economy with divisible goods and transferable utilities; consumers' utility functions are $u_h{:}\mathbb{R}_+^K \to \mathbb{R}$, $h \in H$, and producers' cost functions are c_l: $\mathbb{R}_+^K \to \mathbb{R} \cup \{+\infty\}$, $l \in L$.

Assume that functions u_h ($h \in H$) are closed concave, and that functions c_l ($l \in L$) are closed convex.

Given a price p, the demand $D_h(p)$ of the h-th consumer is defined as the set of solutions to the problem

$$\max_{x \in \mathbb{R}_+^K} (u_h(x) - p(x)). \tag{12}$$

The supply $S_l(p)$ of the l-th producer is the set of solutions to

$$\max_{x \in \mathbb{R}_+^K} (p(x) - c_l(x)). \tag{13}$$

A tuple $((x_h)_{h \in H}, (y_l)_{l \in L}, p)$ is said to be an *equilibrium of the economy* $\hat{\mathcal{E}} = \{(u_h, W_h), h \in H, c_l, l \in L\}$ *with transferable utilities* if, for every $h \in H$, $x_h \in D_h(p)$, for every $l \in L$, $y_l \in S_l(p)$ and the equality $\Sigma_{h \in H}\, x_h = \Sigma_{h \in H}\, W_h + \Sigma_{l \in L}\, y_l$ holds.

We remind that equilibria in economies with transferable utilities can be obtained with the help of solutions to an optimization task. In fact, consider the following aggregate utility and aggregate cost functions. The aggregate utility function U is given by the co-convolution

$$U(x) = \sup_{\sum_{h\in H} x_h = x,\ x_h \in \mathbb{R}^K_+} \left\{ \sum_{h\in H} u_h(x_h) \right\}, \quad x \in \mathbb{R}^K_+.$$

The aggregate cost function C is given by the convolution

$$C(y) = \inf_{\sum_{l\in L} y_l = y,\ y_l \in \mathbb{R}^K_+} \left\{ \sum_{l\in L} c_l(y_l) \right\}, \quad y \in \mathbb{R}^K_+.$$

Proposition 3. *The aggregate functions C and U have the following properties*:

$$S(p) = \sum_{l\in L} (S_l(p)), \tag{14}$$

and

$$D(p) = \sum_{h\in H} (D_h(p)), \tag{15}$$

where $S(p)$ *denotes* $\mathrm{Argmax}(p(y) - C(y))$, $D(p)$ *denotes* $\mathrm{Argmax}(U(x) - p(x))$, $S_l(p) := \mathrm{Argmax}(p(y) - c_l(y))$, *and* $D_h(p) := \mathrm{Argmax}(u_h(x) - p(x))$.

Proof. Let us prove (14); (15) is proven similarly. By definition of the convolution, the epigraph of C is equal to the closure of the sum of the epigraphs of c_l. Since $c_l \geq 0$, its epigraph is a subset of the positive orthant $\mathbb{R}^K_+ \times \mathbb{R}_+$. Since the sum of closed convex subsets of $\mathbb{R}^K_+ \times \mathbb{R}_+$ is closed (see, for example, Rockafellar (1970)), we have

$$\mathrm{epi}(C) = \sum_{l} \mathrm{epi}(c_l). \tag{16}$$

(14) is now obtained from (16) by means of well-known properties of the summation operation. Q.E.D.

An equilibrium of the economy $\hat{\mathscr{E}}$ exists if and only if there exists a solution to the task

$$\max_{y\in\mathbb{R}^K_+} U(W + y) - C(y), \tag{17}$$

where $W = \sum_{h\in H} W_h$. Indeed, since equilibria are Pareto optimal, any equilibrium allocation gives a solution to (17). Conversely, let y^* be a solution to (17). Then we have

$$C(y) - C(y^*) \geq U(W + y) - U(W + y^*)$$

for any $y \in \mathbb{R}^K_+$. On the left hand side, the function is concave; on the right hand side, the function is convex. Therefore there exists a separating linear functional, and we can take any such functional to figure an equilibrium price. By (14) and (15), we can disaggregate both the vector $W + y^*$ into optimal consumers's solutions and the vector y^* into optimal producers' solutions. □

When, for example, the individual utility functions are bounded and $C(y) \to +\infty$ with

$\|y\| \to +\infty$, a solution to (17) exists. In such a case, the concave function $U(W+y) - C(y)$ goes to $-\infty$ with $\|y\| \to +\infty$. Existence of a solution is also warranted, when the cost functions have bounded effective domains[3]. Note that if we set $c(\mathbf{0}) = 0$ and $c(Y) = +\infty$ for $Y \neq 0$, this yields precisely a pure exchange economy.

Now, we consider utility functions of the form $U(X, m) = u(X) + m$ with some function $u : \mathbb{Z}_+^K \to \mathbb{R}$, $u(0) = 0$. In a discrete economy with transferable utilities, just as in the divisible case, the demand of the consumer h, at a price p, is defined as the set of solutions to the problem

$$\max_{X \in \mathbb{Z}_+^K} (u_h(X) - p(X)). \tag{18}$$

A tuple $((X_h)_{h \in H}, (Y_l)_{l \in L}, p)$ is said to be an *equilibrium of an economy* $\mathscr{E}^{tu} = \{(u_h, W_h), h \in H, c_l, l \in L\}$ *with transferable utilities* if, for every $h \in H$, X_h is a solution to (18), and if, for every $l \in L$, Y_l is a solution to (1) and the balance (2) holds.

An equilibrium of the convexified economy $\mathrm{co}(\mathscr{E}^{tu})$ exists if utility functions are pseudoconcave, and if cost functions are pseudoconvex, and if (17) has a solution for their convexified counterparts.

Thus, by our conditional Theorem 1, we have the following existence theorem for discrete economies with transferable utilities.

Theorem 3. *Let $\mathscr{E}^{tu}$ be a discrete economy with transferable utilities. Suppose, for some class $\mathscr{D}$ of discrete convexity, that the utility functions u_h $(h \in H)$ are $\mathscr{D}$-concave and the cost functions c_l $(l \in L)$ are $\mathscr{D}$-convex with bounded effective domains. Then there exists an equilibrium in $\mathscr{E}^{tu}$.*

Remark. Theorem 3 states existence of equilibrium in terms of individual utility functions and cost functions. Since the disaggregation of a solution to (17) yields an equilibrium in models with transferable utilities, a necessary and sufficient condition of existence equilibrium for $\mathscr{E}^{tu}$ can be formulated in aggregate terms (see, for example, Danilov et al. (1995) or Bikhchandani and Mamer (1997)). An equilibrium for the economy $\mathscr{E}^{tu}$ exists if and only if there exists a Pareto optimal allocation of indivisible goods in the economy $\mathscr{E}^{tu}$, which would be Pareto optimal in the convexified economy $\mathrm{co}(\mathscr{E}^{tu})$. However, this necessary and sufficient condition is of little interest, because of its aggregate formulation. □

Here, we have established existence of equilibrium for a discrete economy modulo a hypothetical class $\mathscr{D}$ of discrete-convex sets. However, our results would have no big value, if we could not provide interesting examples of such classes. In the next section, we show that there exists an interesting class of discrete convexity, namely, the class associated with integral generalized polymatroids. We also show that in all known models of exchange economies with indivisibilities and money, in which equilibria exist, the demand sets belong to such a class.

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[3]The effective domain of a function is the subset of the domain of definition where the function does not equal $+\infty$.

5. Discrete-convex sets and functions

Here we demonstrate how to construct some interesting classes of discrete convexity. We will construct them as integer points of integral polyhedra. A polyhedron is the intersection of a finite number of closed halfspaces. A polyhedron $P \subset \mathbb{R}^K$ is said to be an *integral polyhedron* if $P = \mathrm{co}(P(\mathbb{Z}))$, where $P(\mathbb{Z}) := P \cap \mathbb{Z}^K$.

Assume a class $\mathscr{P}$ of polyhedra with the following properties

DCP1. Any polyhedron $P \in \mathscr{P}$ is integral.

DCP2. For any polyhedra $P, Q \in \mathscr{P}$, we have $P \pm Q \in \mathscr{P}$ and

$$(P \pm Q)(\mathbb{Z}) = P(\mathbb{Z}) \pm Q(\mathbb{Z}). \tag{19}$$

A class of polyhedra $\mathscr{P}$ satisfying properties DCP1 and DCP2 is said to be a *polyhedral class of discrete convexity*. Given a class $\mathscr{P}$ of discrete convex polyhedra, the class $\mathscr{D}(\mathscr{P})$ of subsets of $\mathbb{Z}^K$, $\mathscr{D}(\mathscr{P}) = \{P(\mathbb{Z}), P \in \mathscr{P}\}$ satisfies DC1 and DC2, that is a class of discrete convexity. We say that the class $\mathscr{D}(\mathscr{P})$ is *associated* with the polyhedral class $\mathscr{P}$.

We may always assume that a polyhedral class of discrete convexity $\mathscr{P}$ contains all singleton sets of integer points. For such a class $\mathscr{P}$ of polyhedra, (22) is equivalent to the following property, which is more convenient to check out,

$$P \cap Q \text{ contains an integer point if non empty.} \tag{20}$$

It is easy to see that (20)⇔(19): Let $X \in \mathbb{Z}^K$ be some integer point of $P - Q$, that is $X = p - q$, $p \in P$, $q \in Q$. Then the intersection of P and $Q + X$ is non-empty and, because of (20), $P \cap (Q + X)$ is an integral polyhedron. Therefore, we can choose p and q in $\mathbb{Z}^K$, hence, the implication (20)⇒(19) is shown. In the other direction: if $P \cap Q \neq \emptyset$, then $0 \in P - Q$. According to (19), 0 is obtained as the difference of two integer points of P and Q, i.e. $P \cap Q$ contains an integer point.

In dimension one, the class of all integral polyhedra (which are segments with integral endpoints) is the polyhedral class of discrete convexity. This is, of course, not the case in higher dimensions (see, for example, Example 2). In higher dimensions, to get a class of discrete convexity, we need to narrow the class of pseudo-convex sets.

Example 3. *Hexagons*. Consider a class $\mathscr{H}$ of polyhedra in $\mathbb{R}^2$, which consists of polyhedra defined by the inequalities $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, $c \le x_1 + x_2 \le d$, where a_1, a_2, b_1, b_2, c and d are integers. It is easy to check that such hexagons (generally speaking they can degenerate to polyhedra with smaller number of edges) has integral vertices. Since the intersection of hexagons of $\mathscr{H}$ is a hexagon of $\mathscr{H}$, by (20), we conclude that $\mathscr{H}$ is a class of discrete convexity. □

Observe, that the edges of those hexagons are parallel to either e_1, e_2 or $e_1 - e_2$ (where we denote by e_1 and e_2 the standard basis of $\mathbb{Z}^2$). These vectors have the following property: if we take any two of them, then this pair will form a basis of the lattice $\mathbb{Z}^2$. As one can see from Example 1, if a class of integral polyhedra of $\mathbb{R}^2$ contains polyhedra, whose edges are parallel either $e_1 - e_2$ or $e_1 + e_2$, then it cannot be a class of discrete convexity. In fact, the pair of vectors $e_1 - e_2$ and $e_1 + e_2$ does not form a basis of $\mathbb{Z}^2$: for example, a point of the form $(2n+1)e_1$, $n \in \mathbb{Z}$, will never be obtained as a linear combination of the vectors $e_1 - e_2$ and $e_1 + e_2$ with integer coefficients.

Moreover, the property that 'for a given collection of integral polytopes, any linear independent subset of $|K|$ primitive vectors, which are parallel to edges of these polytopes form a basis of the abelian group $\mathbb{Z}^K$' is crucial for the collection to be a polyhedral class of discrete convexity.

Let us introduce the following notion: A collection $\mathcal{R}$ of vectors of $\mathbb{R}^K$ is said to be a *unimodular system* if, for any subset $R \subset \mathcal{R}$, the abelian group $\mathbb{Z}(R) = \{\Sigma_i\ a_i r_i \mid r_i \in R, a_i \in \mathbb{Z}\}$ coincides with the abelian subgroup $\mathbb{R}(R) \cap \mathbb{Z}^K$, $\mathbb{R}(R) = \{\Sigma_i\ a_i r_i \mid r_i \in R, a_i \in \mathbb{R}\}$. We now give a precise statement (for proof see Danilov and Koshevoy, 1998).

Theorem 4. *Let $\mathcal{P}$ be a collection of integral pointed polyhedra of $\mathbb{R}^K$ closed under taking faces. Let $\mathcal{R}(\mathcal{P})$ denote the set of primitive vectors in $\mathbb{Z}^K$, which are parallel to edges of polyhedra of $\mathcal{P}$*[4]*. Then $\mathcal{P}$ is a class of discrete convexity if and only if $\mathcal{R}(\mathcal{P})$ is a unimodular system.*

Recall that a polyhedron is said to be *pointed* if it has at least one vertex. Of course, a polytope is a pointed polyhedron.

Remark. Since, in our model, convex hulls of demands and supplies are subsets of $\mathbb{R}^K_+$, they are pointed polyhedra. Therefore, the only relevant classes of interest for economic applications are the classes of discrete convexity associated to pointed polyhedra. We can also assume that the set $\mathcal{R}(\mathcal{P})$ contains the standard basis $\{e_1, \ldots, e_{|K|}\}$ of $\mathbb{Z}^K$. In such a case, $\mathcal{R}(\mathcal{P})$ is a unimodular system if and only if every $|K|$ linear independent vectors $r_1, \ldots, r_{|K|}$ of $\mathcal{R}(\mathcal{P})$ is a basis of the abelian group $\mathbb{Z}^K$ (see, for example, Schrijver (1987), Chapter 19). □

We thus have the following recipe to construct polyhedral classes of discrete convexity: Take a unimodular system $\mathcal{R}$ and consider all the integral polytopes whose edges are parallel to vectors of $\mathcal{R}$. Denote by $\mathcal{P}t(\mathcal{R},\mathbb{Z})$ such a class of polytopes.

We now give an example of the interesting and famous unimodular system.

Example 4. The set $\mathbb{A}_K := \{\pm e_i,\ e_i - e_j,\ i,\ j \in K\}$ of vectors of $\mathbb{Z}^K$ is a unimodular system. Since $\mathbb{A}_K$ contains the standard basis, we need to show that any subset of $|K|$ linear independent vectors of $\mathbb{A}_K$ form a basis of $\mathbb{Z}^K$ (see the previous Remark). Let $B \subset \mathbb{A}_K$ be a basis of $\mathbb{R}^K$. We check that B is a basis of $\mathbb{Z}^K$. One of the $\pm e_i$, $i \in K$, belongs to B, otherwise B would be a subset of the hyperplane $\Sigma_{i \in K} x_i = 0$, and, hence, B could not be a basis of $\mathbb{R}^K$. Let $e_1 \in B$. If none of the vectors $\pm(e_i - e_1)$ belongs to B, then the set $B \backslash \{e_1\}$ is contained in the subspace $x_1 = 0$, where $x_1, \ldots, x_n$ denotes the dual

[4] A vector r belongs to $\mathcal{R}(\mathcal{P})$ if and only if there is a polyhedron $P \in \mathcal{P}$ which has an edge of the form $[x, x + ar]$ or $x + \mathbb{R}_+ r$ with some $x \in \mathbb{Z}^K$ and $a \in \mathbb{N}$.

basis to $e_1, \ldots, e_{|K|}$. By induction, $B \backslash \{e_1\}$ forms a basis of $\mathbb{Z}^{K\backslash\{1\}}$. Hence B is a basis of $\mathbb{Z}^K$. If $e_j - e_1$ belongs to B with some j, then, substituting $e_j - e_1$ by $e_j = e_1 + (e_j - e_1)$, we receive a new basis B'. Obviously, both B and B' are either bases or are not bases of $\mathbb{Z}^K$. Repeating, we can assume that none of vectors $\pm(e_i - e_1)$ belongs to B'. Therefore, B' is a basis of $\mathbb{Z}^K$, and, hence, so is B. □

Denote by $\mathscr{P}t(\mathbb{A}_K)$ the class of pointed polyhedra of $\mathbb{R}^K$, whose edges are parallel to vectors of $\mathbb{A}_K$, and denote by $\mathscr{IGP}$ the subclass of integer polyhedra. The class $\mathscr{P}t(\mathbb{A}_K)$ coincides with the class of polyhedra, known in discrete mathematics as *generalized polymatroids* (g-polymatroids). Generalized polymatroids were introduced by Frank (1984) as a generalization of polymatroids of Edmonds (1970). This class of polyhedra, moreover, is equivalent to the class of cores of convex cooperative games, explored by Shapley (1971). We mention that in discrete mathematics generalized polymatroids were defined as polyhedra given by systems of specific linear inequalities (see Frank and Tardos, 1988; Fujishige, 1991). However, this viewpoint is not used in the sequel.

A subset of $\mathbb{Z}^K$ is said to be a *PM-set* if it belongs to the class of discrete convexity associated with integral g-polymatroids.

We can construct new g-polymatroids by summing up of already known; $\mathscr{IGP}$ contains segments $[0, r]$, $r \in \mathbb{A}_K$, the sums of such segments, the simplex Δ and all its faces, the sums of these faces. We can also construct products; if P and Q are g-polymatroid in $\mathbb{R}^N$ and $\mathbb{R}^M$, correspondingly, then $P \times Q$ is a g-polymatroid in $\mathbb{R}^N \times \mathbb{R}^M$. We can use projections along a set of coordinates and, more generally, homomorphisms of lattices of integers of special types (Danilov and Koshevoy, 1998) to construct new g-polymatroids who will be the images of already known under such homomorphisms.

We now describe a few classes of easily recognizable g-polymatroids given by specific systems of inequalities. Recall, that a family $\mathscr{T}$ of subsets of a set K is called *laminar* if, for any $A, B \in \mathscr{T}$, there holds either $A \subseteqq B$, or $B \subseteqq A$, or $A \cap B = \emptyset$. Observe, that if $\mathscr{T}$ is a laminar family, then we can always assume that $\mathscr{T}$ contains all singletons $\{k\}$, $k \in K$, and the whole set K.

Example 5. Let $\mathscr{T}$ be the collection of singletons and a chain $\mathscr{C} = \{K \supset C_1 \supset C_2 \supset \cdots \supset C_m\}$, $m \leq |K|$. $\mathscr{T}$ is a laminar family. □

Proposition 4. *Let $\mathscr{T}$ be a laminar family. A polyhedron defined by the inequalities*

$$a_A \leq x(A) \leq b_A, \quad A \in \mathscr{T},$$

is a generalized polymatroid. If, for every $A \in \mathscr{T}$, a_A, $b_A \in \mathbb{Z}$, the above polyhedron is an integral g-polymatroid.

Proof. We check that the edges of a polyhedron defined by the inequalities $a_A \leq x(A) \leq b_A$, $A \in \mathscr{T}$, are parallel to vectors of $\mathbb{A}_K$. For this, consider $(n-1)$ linearly independent functionals $\mathbf{1}_{S_i}$, $S_i \in \mathscr{T}$, $i = 1, \ldots, n-1$, $n := |K|$ $(\mathbf{1}_{S_i}(x) = \Sigma_{l \in S_i} x^i)$.

There are two different cases: $K \neq \cup_{i=1}^{n-1} S_i$ and $K = \cup_{i=1}^{n-1} S_i$. In the first case, there exists exactly one element outside $\cup S_i$, say $k \in K$, and, then, the common kernel of $\mathbf{1}_{S_i}$, $i = 1, \ldots, n-1$, is generated by e_k. In the second case, since $\mathcal{T}$ is laminar and the kernel is one-dimensional, one can show that there exists a unique 'nonseparable pair' k, $l \in K$, i.e., there is a unique pair of elements k, $l \in K$ such that there is no set in the collection S_i, $i = 1, \ldots, n-1$, which separates k and l ($k \in S_i$, $l2 \in S_i$, or vice versa). Therefore, the kernel is proportional to $e_k - e_l$.

Since the edges of the polyhedron defined by $a_A \leq x(A) \leq b_A$, $A \in \mathcal{T}$, are parallel to kernels of $(n-1)$ linearly independent functionals $\mathbf{1}_{S_i}$, $S_i \in \mathcal{T}$, $i = 1, \ldots, n-1$, and, since these kernels are generated by vectors of $\mathbb{A}_K$, this polyhedron is a g-polymatroid. Similarly, one can show that if a_A and b_A, $A \in \mathcal{T}$, are integers, then any vertex of this polyhedron is an integer vector. Hence it is an integral g-polymatroid (for details, see Danilov and Koshevoy (1998), or see Frank and Tardos (1988) for an alternative proof). Q.E.D.

Given a laminar family $\mathcal{T}$, denote by $\mathcal{P}(\mathcal{T})$ the class of polyhedra of the form

$$a_A \leq x(A) \leq b_A, \quad A \in \mathcal{T},$$

a_A, b_A, $A \in \mathcal{T}$, are integers. Obviously, the intersection of polyhedra of $\mathcal{P}(\mathcal{T})$ remains in the class. The sum of polyhedra of $\mathcal{P}(\mathcal{T})$ can turn out to be a polyhedron outside of $\mathcal{P}(\mathcal{T})$, but it is, of course, always an integral g-polymatroid.

Remark. Demand sets in Gale's model (see Example 2) are *PM*-sets. In fact, the polytope given by inequalities $x(K) \leq 1$, $x^k \geq 0$, $k \in K$, is the simplex $\Delta_K = \text{co}\{0, \mathbf{1}_k, k \in K\}$. Δ_K belongs to $\mathcal{P}(\mathcal{T})$, where $\mathcal{T}$ is the trivial laminar set consisting of the collection of all singletons and the whole set K. All faces of Δ_K are defined by similar systems of inequalities. Thus, demands, as integer points of the faces of the simplex Δ_K, are *PM*-sets. □

5.1. $\mathcal{IGP}$-convex functions

A function $f:\mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be *$\mathcal{IGP}$-convex* if it fits Definition 4 for the class of *PM*-sets.

Murota and Shioura (1999) considered such a class of functions and called these functions M$^\natural$-convex, see, also Murota (1996, 1998).

How does one construct $\mathcal{IGP}$-convex functions? Well, since $\mathcal{IGP}$ is a class of discrete convexity, the convolution of a few $\mathcal{IGP}$-convex functions, defined on $\mathbb{R}_+^K$, is $\mathcal{IGP}$-convex.

The sum of $\mathcal{IGP}$-convex functions is not $\mathcal{IGP}$-convex in general, since the class of integral g-polymatroids fails to be closed with respect to intersections. However, we have the following

Proposition 5. *Let $\mathcal{T}$ be a laminar family of subsets of K. For every $A \in \mathcal{T}$, let $f_A:\mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ be a pseudoconvex function. Then the function $f:\mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(X)=\sum_{A\in\mathcal{T}} f_A(X(A)),\quad X\in\mathbb{Z}^K, \tag{21}$$

is $\mathcal{IGP}$-convex.

Proof. Recall that $f^*:(\mathbb{R}^K)^*\to\mathbb{R}$, $f^*(p):=\sup_{x\in\mathbb{R}^K}(p(x)-f(x))$, is called the conjugate function or the Legendre–Young–Fenchel transform of f.

Let f and g be closed convex functions, such that $\mathrm{epi}(f^*)+\mathrm{epi}(g^*)$ is a closed set. Then, it is not difficult to check (see the proof of Proposition 3) that $\partial^*(f+g)(p)=\partial(f^*g^*)(x)$ holds with $x\in\partial^*(f^*g^*)(p)$, where $\partial f(x)=\{p\in(\mathbb{R}^K)^*|\,p(y)-p(x)\le f(y)-f(x)\ \forall y\in\mathbb{R}^K\}$ denotes the subdifferential of f at the point x. This implies that $\partial^*(f+g)(p))=\partial^* f(p_1)\cap\partial^* g(p_2)$ for some p_1 and p_2 such that $p=p_1+p_2$, $p_1,p_2\in(\mathbb{R}^K)^*$.

The effective domain of the conjugate function $(f_A)^*$, $A\subset N$, is located on the line $\mathbb{R}(\mathbf{1}_A)$. Therefore, the sum of the epigraphs $\mathrm{epi}(f_A^*)+\mathrm{epi}(f_B^*)$ is closed for any $A\neq B$. Hence, areas of affinity of the function $\Sigma_{A\in\mathcal{T}} f_A(X(A))$ are intersections of strips of the form $a_A\le x(A)\le b_A$, $A\in\mathcal{T}$. These areas of affinity are polyhedra of $\mathcal{P}(\mathcal{T})$. The latter polyhedra are integral g-polymatroids. Hence f is an $\mathcal{IGP}$-convex function. Q.E.D.

By Proposition 5 and Example 5, we have the following.

Corollary 1. *Let $\mathcal{T}$ be the collection of singletons and elements of a chain $\mathcal{C}=\{K=C_1\supset C_2\supset\cdots\supset C_m\}$. Let $f_k:\mathbb{Z}\to\mathbb{R}\cup\{+\infty\}$, $k\in K$, and $g_i:\mathbb{Z}\to\mathbb{R}\cup\{+\infty\}$, $i=1,\ldots,m$, be pseudoconvex functions. Then the function*

$$f(X)=\sum_{k\in K} f_k(X^k)+\sum_{i=1}^{m} g_i\Big(\sum_{k\in C_i} X^k\Big),\quad X\in\mathbb{Z}^K, \tag{22}$$

is $\mathcal{IGP}$-convex.

We now propose several examples of $\mathcal{IGP}$-convex functions to illustrate Corollary 1.

Example 6. *Take separable convex functions, i.e., functions of the form*

$$f(X)=\sum_{k\in K} f_k(X^k),$$

where f_k $(k\in K)$ are convex functions on $\mathbb{R}$. They fit (22) for $\mathcal{T}$ consisting only of singletons. These functions are $\mathcal{IGP}$-convex, and Theorem 2 along with this class of functions is precisely a multidimensional generalization of Henry's result (Henry, 1970). □

Example 7. Take now quasi-separable convex functions, that is functions defined as follows,

$$f(X)=\sum_{k\in K} f_k(X^k)+f_0\Big(\sum_{k\in K} X^k\Big),\ X\in\mathbb{Z}^K,$$

where f_0 and f_k, $k\in K$, are convex functions on $\mathbb{R}$. They also fit (22) for $\mathcal{T}$ consisting of singletons and the one element chain $\mathcal{C}=\{K\}$. □

Theorem 2 along with this type of $\mathcal{IGP}$-convex functions extends the existence result in Bevia et al. (1999).

The indifference levels of the utility functions appearing in Quinzii (1984) and Svensson (1984) also fit (22). In fact, the essential feature of Quinzii's model is that consumers never have use for more than one item of any of the goods. Formally, the consumer utility function takes the form $u(Z, m) = \max_{k \in \mathrm{supp}(Z)} u(k, m)$, for any $Z \in \{0, 1\}^K$, $m > 0$. The associated indifference levels, such a function, are of the form $q^m(Z) = \min_{k \in \mathrm{supp}(Z)} q^m(k)$. In the following Example 8, we show that these indifference levels are $\mathcal{IGP}$-convex, and, moreover, also fit (22). This is a rationale underlying Quinzii's (1984) existence result.

Example 8. Consider the function $U{:}2^K \to \mathbb{R}$ of the form $U(A) = \min_{k \in A} u(k)$, with some function $u{:}K \to \mathbb{R}$. We show that it satisfies Corollary 1. For this, we extend U on $\mathbb{Z}_+^K$ as follows

$$U(X) = \min_{k \in \mathrm{supp}(X)} u(k), \quad X \in \mathbb{Z}_+^K, \tag{23}$$

where $U(0) = \max_{k \in K} u(k)$ by convention, and $\mathrm{supp}(X) = \{k \in K | X^k > 0\}$.

Now, rank elements of the set K in decreasing order with respect to the values of u: $u(1) \geq \ldots \geq u(n) \geq 0$, $n = |K|$. Set $d_1 = u_1$, $d_k = u(k) - u(k-1)$, $i = 2, \ldots, n$. And consider the following pseudoconcave function $\theta{:}\mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$, defined by

$$\theta(t) = \begin{cases} -\infty, & t < 0; \\ 0, & t = 0; \\ 1, & t \geq 1. \end{cases}$$

It is easy to see, that

$$U(X) = d_1 + d_2\theta(X_2 + \ldots + X_n) + \cdots + d_n\theta(X_n). \tag{24}$$

In fact, let k be a maximal element in supp$X$. Then, the left hand side of (24) yields $u(k)$, while the right hand side sums up to $d_1 + \ldots + d_k = u(k)$. Since $d_i \leq 0$ (for $i > 1$), the functions $d_i\theta$, $i = 1, \ldots, n$, are pseudoconvex. Therefore, the function $d_1 + d_2\theta(X_2 + \ldots + X_n) + d_n\theta(X_n)$ fits (23) for $\mathcal{T}$ consisting of the chain $\mathcal{C} = \{\{1, \ldots, n\} \supset \{2, \ldots, n\} \supset \ldots \supset \{n\}\}$. Thus, functions of the form (23) are $\mathcal{IGP}$-convex. □

Example 9. Let K be partitioned, $K = \cup_{s \in S} K_s$, $K_s \cap K_{s'} = \emptyset$ with $s \neq s'$. For each $s \in S$, pick a class of discrete convexity $\mathcal{K}_s$ in $\mathbb{Z}^{K_s}$ and a $\mathcal{K}_s$-convex function f_s. Then, since the cartesian product of classes of discrete convexity is a class of discrete convexity, the function defined by

$$f(X) = \sum_{s \in S} f_s(X|_{K_s}),$$

where $X|_{K_s} \in \mathbb{Z}^{K_s}$ is the projection of X on K_s, is a $\prod_{s \in S} \mathcal{K}_s$-convex function. □

Example 9 and Theorem 2 together ensure that an exchange economy, in which consumers have preferences with indifference levels of the form $q^m(X) = \Sigma_{s\in S} \min_{k\in \mathrm{supp}(X)\cap K_s} q^m(k)$, will have equilibria.

For example, the generalization of Gale's model by van der Laan et al. (1997) is a case in which the demands are products of *PM*-sets. They are obtained as products of integer points of faces of different standard unit simplexes, $\prod_s \Delta_{K_s}$. Hence, existence of equilibrium, in this model, can be obtained as a consequence of Theorem 1 (see Example 2).

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Appendix A. Proof of Proposition 2.

Denote $W = \Sigma_{h\in H} W_h$. Let $Z^* \in \mathbb{Z}^K_+$ be such that, for any $Z \geq Z^*$, the aggregate cost of producing Z is larger than the total amount of money in the economy, i.e. $C(Z) > \Sigma_{h\in H} w_h$ for any $Z \geq Z^*$, where $C(Z) = \min_{Z=\Sigma_{l\in L} Z_l,\, Z_l\in \mathbb{Z}^K} \Sigma_{l\in L} c_l(Z_l)$. Z^* exists by Assumption 4.

We slightly modify the preferences of consumers. Denote by T_h the amount of money such that $(0,T_h) \sim_h (W + Z^*, \Sigma_{h\in H} w_h)$, $h\in H$. Then the function $q_h^{T_h}$ defines the indifference level of the preference $\preceq_h$ which passes through the point $(W + Z^*, \Sigma_{h\in H} w_h)$, or, equivalently, through the point $(0,T_h)$. Define the modified preference $\tilde{\precsim}_h$ by setting its indifference levels as follows: For any $m_h^0 \leq m \leq T_h$ the indifference level of $\tilde{\precsim}_h$ passing through the point $(0, m)$ coincides with the indifference level of $\preceq_h$ passing through the same point. For any $m > T_h$ set $\tilde{q}_h^m$ to be a parallel translation of $q_h^{T_h}$, i.e.,

$$\tilde{q}_h^m(X) = q_h^{T_h}(X) + m - T_h, \quad X \in \mathbb{Z}^K. \tag{A.1}$$

Let $\tilde{\mathscr{E}} = \{(\tilde{\precsim}_h, (W_h, w_h)_{h\in H}; (c_l)_{l\in L}\}$ be the modified economy. We assert that equilibria of $\mathscr{E}$ and $\tilde{\mathscr{E}}$ coincide. In fact by Assumption 1, every consumer in $\mathscr{E}$ has a positive amount of money, inferior to $\Sigma_{h\in H} w_h$ (see (3)), at an equilibrium. Now according to (2) and (3), at any equilibrium of $\mathscr{E}$, each consumer has a vector of indivisibles bounded by $W + Z^* = \Sigma_{h\in H} W_h + Z^*$. However, for any $X \leq W + Z^*$ and $m \leq \Sigma_{h\in H} w_h$, the indifference levels of the preferences $\preceq_h$ and $\tilde{\precsim}_h$ passing through (X, m) coincide. Therefore, the sets of equilibria of $\mathscr{E}$ and $\tilde{\mathscr{E}}$ coincide. Thus, we may assume that (A.1) is already satisfied in the initial economy $\mathscr{E}$.

We prove existence of equilibrium for the convexified economy co($\mathscr{E}$).

Set $Q = \Sigma_{h\in H} T_h$. Take the price cube

$$\mathbf{Q} = \{p \in \mathbb{R}^K_+ \mid 0 \leq p_k \leq Q,\ \forall k \in K\},$$

and define the price correspondence $\mathbf{P}:\mathbf{Q}\Rightarrow\mathbf{Q}$ through the following auxiliary construction:

For every $h\in H$, set

$$m_h(p) := \inf\{m : q_h^m(X)\geq \beta_h(p)-p(X), X\in\mathbb{Z}_+^K, X\leq W+Z^*\}. \tag{A.2}$$

(That is $m_h(p)$ is such that the indifference level $q_h^{m_h(p)}(\cdot)$ 'touches' the budget set $B_h(p)$.)

Define now the indirect utility functions

$$u_h^p(X)=m_h(p)-q_h^{m_h(p)}(X), X\in\mathbb{Z}_+^K.$$

According to Assumption 3, these functions are pseudoconcave and, because of (A.1), the functions u_h^p are bounded by T_h for any $p\in\mathbf{Q}$.

Let the aggregate utility function U^p be given by the co-convolution

$$U^p(x)=\min_{\sum_{h\in H} x_h=x}\left\{\sum_{h\in H}\hat{u}_h^p(x_h)\right\},\quad x\in\mathbb{R}_+^K,$$

and let the aggregate cost function C be given by the convolution

$$C(y)=\min_{\sum_{l\in L} y_l=y}\left\{\sum_{l\in L}\mathrm{co}(c_l)(y_l)\right\},\quad y\in\mathbb{R}_+^K.$$

Recall that $\hat{u}$ denotes the concavification of a function u, $\hat{u}=-\mathrm{co}(-u)$, where $\mathrm{co}(c)$ denotes the convexification of a function c.

Let y^p be a solution to

$$\max_{y\in\mathbb{R}_+^K} U^p(W+y)-C(y).$$

Such a solution exists, since $C(y)\to+\infty$ with $\|y\|\to\infty$ and U^p is bounded by $Q=\Sigma_h T_h$ (since each u_h^p is bounded by T_h).

Denote $M=\max_{y\in\mathbb{R}_+^K} U^p(W+y)-C(y)$. Then, U^p is a concave function; C is a convex function and, for any $x\in\mathbb{R}_+^K$, $C(x)+M\geq U^p(W+x)$ holds. Therefore, there exists a separating affine function of the form $M+p'$ with some linear function p', i.e., we have

$$C(x)+M\geq p'(x)+M\geq U^p(W+x),\qquad x\in\mathbb{R}_+^K. \tag{A.3}$$

For $p\in\mathbf{Q}$, let $\mathbf{P}(p)$ be the set of separating linear functionals p' satisfying (A.3).

We claim that $\mathbf{P}(p)\subset\mathbf{Q}$, i.e. $\mathbf{P}:\mathbf{Q}\Rightarrow\mathbf{Q}$. In fact, monotonicity of C and U^p implies that $p'\geq 0$, and hence $\mathbf{P}(p)\subset(\mathbb{R}^K)^*{}_+$. For any $p'\in\mathbf{P}(p)$, there holds $U^p(W+y^p)-p'(W+y^p)\geq U^p(W+y^p)-C(y^p)\geq 0$. Therefore, we have $Q\geq U^p(W+y^p)\geq p'(W+y^p)$. Since W is an integral vector with positive coordinates, we have $W\geq\mathbf{1}$, which implies that $p'_k<Q$ for all $k\in K$.

The set $\mathbf{P}(p)$ is convex and compact, as the set of separating linear functionals. Since U^p is continuous with respect to p (every u_h^p is continuous with respect to p), the correspondence $\mathbf{P}$ is closed. By the Kakutani theorem, $\mathbf{P}$ has a fixed point, say $p^*\in\mathbf{P}(p^*)$. Let us check that p^* is an equilibrium price for $\mathrm{co}(\mathscr{E})$.

Obviously, $y^{p^*} < Z^*$. Since $q_h^{m_h(p^*)}$ touches the budget set $B_h(p^*)$, the following equality is satisfied

$$\text{Argmax}_{x \in \mathbb{R}_+^K} (u_h^{p^*}(x) - p^*(x)) = \text{co}(D_h(p^*)).$$

(Recall that $D_h(p)$ denotes the set of all $X_h \in \mathbb{Z}_+^K$ such that $(X_h, \beta_h(p) - p(X_h))$ is a best element in the budget set $B_h(p)$ with respect to the preference $\preceq_h$.)

We have $\text{Argmax}(p^*(y) - C(y)) = \Sigma_{l \in L}\ \text{co}(S_l(p^*))$, because this holds for all prices (see (14).

Thus at price p^*, because of (15) and (14) (U^{p^*} and C are aggregate functions, the co-convolution and the convolution, respectively), we have

$$W \in \sum_{h \in H} \text{co}(D_h(p^*)) - \sum_{l \in L} \text{co}(S_l(p^*)). \tag{A.4}$$

And p^* is an equilibrium price for the convexified economy $\text{co}(\tilde{\mathscr{E}})$. Q.E.D.

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