Dynamic consistency of expected utility under non-classical(quantum) uncertainty

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Abstract

Quantum cognition in decision-making is a recent and rapidly growing field. In this paper we develop an expected utility theory in a context of non-classical (quantum) uncertainty. We replace the classical state space with a Hilbert space which allows introducing the concept of quantum lottery. Within that framework we formulate axioms on preferences over quantum lotteries to establish a representation theorem. We show that demanding the consistency of choice behavior conditional on new information is equivalent to the von Neuman-Lüders postulate applied to beliefs. A dynamically consistent quantum-like agent may violate dynamic recursive consistency, however. This feature suggests interesting applications in behavioral economics as we illustrate in an example of persuasion.

1 Introduction

Alternatives available in decision problems can often be analyzed in terms of a variety of perspectives: a fur coat may be evaluated from an esthetical point of view or from the point of view of animal suffering. A military intervention in Syria can be evaluated from a geopolitical perspective or from a humanitarian one. Another type of example relates to the

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consumption of cigarettes: the immediate pleasure perspective contra the long term health perspective. In order to assess an alternative we need to build a representation of it, a "represented alternative" which is a mental construct[1]. Standard decision theory postulates that we always are able to combine any relevant perspectives into a synthetic and stable representation of the alternatives. However, it is also a common place for cognitive scientists that we face difficulties when building our representation of a complex alternative. We consider the alternative from different perspectives - one at a time. And most importantly we are not always able to synthesize information from various perspectives into one single coherent and stable representation of the alternative.

In this paper we are interested in decision-making under uncertainty and we want to capture the difficulties people show in combining all relevant information by analogy with incompatible properties in Quantum Mechanics. To many people it may appear unmotivated or artificial to turn to quantum mechanics (QM) when investigating human behavioral phenomena. However, the founders of QM, including Bohr and Heisenberg, were early to recognize an essential similarity between the two fields:[2] in both fields the object of investigation cannot (always) be separated from the process of investigation. QM and in particular its mathematical formalism was developed to respond to a general epistemological challenge: how can one study an object that is being modified by the measurement of its properties? It should therefore be viewed as truly legitimate to explore the value of the mathematical formalism of QM in the study of human behavioral phenomena - without reference to Physics.[3] Of particular interest in our context is that this formalism allows representing agents subject to the incapacity to simultaneously consider a choice alternative from all relevant perspectives. For instance when evaluating the "animal suffering" value of a fur coat, its esthetical (subjective) value, that was well-determined in our decision-maker's mind before considering animal suffering aspects, may become "blurred" i.e., uncertain.

The classical approach to decision-making under uncertainty e.g., in Savage (1972) and

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[1]Kahneman and Tversky (2000) write "the true objects of evaluation are neither objects in the real world nor verbal descriptions of those objects; they are mental representations" a conception which they further write is entirely natural for cognitive scientists (p. xiv).

[2]In particular Bohr was influenced by the psychology and philosophy of knowledge of Harald Höffding (see Bohr 1971 and the Introduction in Bitbol 2009 for an insightful discussion).

[3]The human mind behaves in a wide array of weird manners. But it is not the weirdness of quantum mechanics that makes it an attractive toolbox, but the fact that it is a most general paradigm for structural contextuality (i.e., non-separability between the object of and the operation of investigation).

Anscombe and Aumann (1963) builds on the notion of a state space S of states of nature. Very roughly a representation of the world (a belief) corresponds to a probability distribution on S. And the changes in belief following the acquisition of new information follow Bayes' rule which can be given a behavioral foundation as shown in Ghirardato (2002) : Bayesian updating secures dynamic consistency i.e., it secures that choices based on updated preferences are consistent with ex-ante preferences defined for the condition (event) that triggered updating. There exists however massive evidence of violations of Bayes rule. One source of violations is that measurements (in a broad sense) affect the object of measurement. Most clearly this happens in quantum physics and it is the reason why some properties may be incompatible. This is formally expressed in the non-commutativity of measurement operations which induces a non-Bayesian updating process. A related line of motivation appeals to the growing interest for applications of elements of the mathematical formalism of Quantum Mechanics to psychology, social sciences and in particular to decision-making (see e.g., Brandenburger and La Mura (2015), Khrennikov (2014) and Busemeyer and Bruza (2012) for an overview of the field). The approach has been successful in explaining a large variety of behavioral anomalies in decision-making ranging from cognitive dissonance, preference reversal, conjunction fallacy, disjunction effects to framing effects.

In a recent book Akerlof and Schiller (2015) labelled a new term "Phishing equilibrium" to express how markets systematically exploit the manipulability of real consumers with far reaching implications for the efficiency and welfare properties of free markets. In a simple economic example of a seller's attempt to persuade a potential buyer, we show that the quantum indeterminacy of beliefs implies a "manipulability" of economic agents much in line with Akerlof and Schiller's empirical evidence as well as with their understanding of the underlying psychological mechanism: "Just change people's focus and one changes the decisions they make" (p.173).

In this article, we substitute the Boolean lattice of events with a more general lattice of projectors in the Hilbert space as the suitable framework for modelling decision-making. The notions are introduced progressively and require no previous knowledge of Quantum Mechanics or Hilbert spaces. We show that a natural definition of a quantum lottery allows for the formulation of decision theoretical axioms similar to the classical ones with one exception. We need axiom A0 that secures the stability of preferences over lotteries defined

over different perspectives (resolutions of the state space). This axiom (that we labelled "no-framing") is trivially satisfied in the classical world (all lotteries can be expressed in a single finest partition(resolution) of the state space). We next show that the von Neumann-Lüders projection postulate of Quantum Mechanics used as an updating rule is both necessary and sufficient for dynamic consistency of preferences. In our context the von Neumann-Lüders postulate arises from purely behavioral considerations that is from a requirement of consistency applying to conditional (on new information) preference relations. Interestingly, the specificity of non-classical uncertainty (also referred to as "contextuality") is shown to imply a failure of the so-called "recursive dynamic consistency" (a dynamic version of the Savage's Sure Thing Principle). This feature is exploited in an example to illustrate the possible value of the results for behavioral economics.

There exists a few earlier works addressing quantum probabilities in the context of decision-making. These include Deutsch (1999), Pitowsky (2003), Lehrer and Shmaya (2006), Danilov and Lambert-Mogiliansky (2010) and Gyntelberg and Hansen (2012)). In particular Pitowsky writes about "betting on quantum measurements" but he is not working with preference relations. Interestingly, he formulates a rule saying that the probability for any specific outcome is independent of the specific measurement that yields it as one of its possible results. This rule is very much in line with our axiom A0. Lehrer and Shmaya propose a subjective approach to quantum probabilities but they do not work with quantum lotteries. Danilov and Lambert-Mogiliansky develop an expected utility theory in a general non-classical uncertainty context (ortho-modular lattices). A first distinction with the present work is that instead of assuming the existence of a certainty equivalent, we build on fundamentals which brings us closer to the approach of von Neumann and Morgenstein (1944) and Anscombe and Aumann (1963). We also adopt the structure of the Hilbert space which allows addressing more general type of lotteries.[4] These steps are necessary to develop the core contribution of the paper which is related to the dynamics of beliefs and choices in response to new information in a non-classical uncertainty environment. Gyltenberg and Hansen (2012) work with Hilbert space to develop an expected utility theory with subjective

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[4]In Danilov and Lambert-Mogiliansky 2010, only direct measurement (orthogonal resolution of the unit) were considered. In the present work we also address "fuzzy" measurement by means of POVM (positive operator valued measurements).

events. Their static setting shows similarities with ours. However their analysis appeals to a large number of axioms - 12 where we have 5 - and most importantly they do not address the issue of dynamic consistency.

The present work is a contribution to both decision theory and the foundations of quantum cognition. We extend previous works in two directions. First, we provide a complete characterization of expected utility theory under non-classical (quantum) uncertainty: a concise formulation of sufficient and necessary axioms in terms of preferences over quantum lotteries. Most importantly, this construction allows for a transparent characterization of dynamic consistency of choice behavior in such an environment. Finally, we discuss the value of the approach for economics and illustrate it with an example of "Phishing for Phools".

The paper proceeds as follows. First, we introduce the concept of quantum lottery which gives us the opportunity to define basic elements of the mathematical formalism. In section 3 we provide a straightforward construction and a complete characterization of preferences over quantum lotteries satisfying some standard properties. We formulate the corresponding axioms and derive our representation theorem. In section 4 we address the issue of information updating and formulate our central theorem of dynamic consistency. Thereafter we discuss the value of our results in economics and end with some concluding remarks.

2   Quantum lotteries

We are interested in a decision-maker's preferences over what we call quantum lotteries. In this section we define the notion of quantum or Q-lottery. As for any lottery, the prize that the DM obtains depends on the realization of some event which is the outcome of a measurement, it is an uncertain payoff. And the lotteries described below (roulette, horse and quantum lotteries) differ essentially in the type of measurement that is being performed. Therefore we first need to clarify the meaning of measurement and in particular of a quantum measurement. But we shall start by reminding basic facts about roulette lotteries and so called 'horse lotteries'.

Roulette lotteries

Hereafter we let X denote a set of prizes. A *roulette lottery* (with prizes in X) is defined

by a finite collection of prizes $x_1, ..., x_r$ together with the probabilities $p_1, ..., p_r$ (p_i are non-negative real numbers with $\sum_i p_i = 1$) for obtaining the corresponding prize. Such a lottery can be written as the string $l = (x_1, p_1; ...; x_r, p_r)$, but we prefer to write it as a formal sum $l = \sum_i x_i \otimes p_i$. We could think of it in the following way: a measurement in the form of a 'roulette' is performed and gives an outcome in the set $\{1, ..., r\}$. The probability of outcome i is p_i and, depending on the outcome of this 'measurement', a prize x_i is paid.

Such lotteries can be identified with (simple) probabilistic measures on the set X. We denote by $\Delta(X)$ the set of such measures (or lotteries). Under well-known conditions, von Neumann and Morgenstern obtained that the utility of a lottery $l = \sum_i x_i \otimes p_i$ for the decision-maker, henceforth DM, is given by a number $U(l) = \sum_i p_i u(x_i)$. Here $u : X \to \mathbb{R}$ is a 'utility function' defined on the set X of prizes.

Horse lotteries

The next concept is that of a 'horse lottery' (in the terminology of Anscombe and Aumann) or 'act' (in Savage's terminology). A *horse lottery* is a mapping $f : S \to X$ from the set S of 'states of nature' to the set X of prizes. A measurement is performed in the form of a 'horse race' and, depending on the result of this measurement, the corresponding prize is paid.

Again under suitable conditions the utility of a horse lottery f can be written as $U(f) = \sum_s p_s u(f(s))$, where $u : X \to \mathbb{R}$ is again a utility function, and p is a (subjective) probability measure on the set S. A considerable simplification of the conditions was achieved by Anscombe and Aumann when taking roulette lotteries as prizes. They define a horse lottery as a function $L : S \to \Delta(X)$. A measurement defines the state s of nature, after that a drawing of the lottery $L(s)$ performs which gives a resulting prize.

In order to smoothly move over to quantum lotteries, it is convenient to present horse lotteries slightly differently. We denote by $l(s, x)$ the corresponding probabilities for realization of outcomes x in the lottery $L(s)$. Now we can form the functions $L_x : S \to \mathbb{R}$ by the rule $L_x(s) = l(s, x)$. And rewrite our horse lottery L as $\sum_x x \otimes L_x$. Here the function L_x can be understood as a *plausibility* (or as a *potentiality*) of getting prize x.

Generally, a finite family $(L_i, i \in I)$ of functions L_i on a set S is called a *positive decomposition of unit* if all these functions L_i are non-negative and their sum $\sum_i L_i$ is equal to

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the function 1_S identically equal to 1. One can understand such a family as a classical fuzzy measurement device with the set I of outcomes; in a state s of nature this measurement gives the outcome i with probability $L_i(s)$. If we associate a prize x_i to outcome i, we obtain a horse lottery $L = \sum_i x_i \otimes L_i$.

Quantum lotteries

A quantum lottery is also a bet on the outcome of a measurement, but now a quantum one. A measurement of some 'observable' is performed, and, depending on the result obtained, our DM receives some prize. To formalize the notion of quantum measurement we have to modify the notion of a state space. The set S is replaced by some Hilbert space H. The notion of function on S is replaced by the notion of Hermitian operator. Below we give precise definitions (a reminder of elementary notions about Hilbert spaces is provided in Appendix 1). For now we only say that the main difference with the classical state space model is that the Hilbert space model allows for measurements that cannot be performed simultaneously i.e., they are incompatible with each other. Therefore the performance of a measurement can modify the state of the system.

Quantum measurement

A quantum measurement device is modeled by a finite collection $(P_i,\ i \in I)$ of Hermitian operators such that

a) all P_i are nonnegative,[5] and

b) $\sum_i P_i = E$, **where E is the identity operator**.

In Physics such a collection is called POVM (positive operator valued measure); we prefer to speak about *positive decomposition of unit* (PDU). The elements of I are the possible outcomes of the device; the operators P_i express the potentiality for realization of the outcome i in a way similar to the functions L_x for horse lotteries (see above).

We shall distinguish between two classes of measurements. The first one consists of *von Neumann measurements* (they are known also as **projective measurements,** direct measurements, orthogonal measurements, first kind measurements, and reproducible measurements). They are defined by the requirement that the P_i are orthogonal to each other, that is $P_i P_j = 0$ for $i \neq j$. It is easy to see that in this case all operators P_i are projectors.

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[5] A definition of nonnegative operators is given in Appendix 1.

Conversely, it can be shown that if all P_i are projectors, they are orthogonal to each other.

The second and broader class of measurements includes $(Q_i, i \in I)$ such that Q_i commute with each other, that is $Q_i Q_j = Q_j Q_i$ for any $i, j \in I$. We call such measurements *internally consistent*. It is easy to see that orthogonal operators commute, so that von Neumann measurements are internally consistent.

Examples of such measurements are 'von Neumann measurements with a noise'. We make a von Neumann measurement $(P_i, i \in I)$ (with orthogonal P_i) and after having obtained an outcome i, we use a roulette lottery l_i with values in a set X to determine the final outcome x. Such a measurement device is modeled by the collection $(Q_x, \ x \in X)$, where $Q_x = \sum_i l_i(x) P_i$. Conversely, any internally consistent measurement can be represented as von Neumann measurement with a noise.[6]

An example of more general measurements is provided by the notion of a *compound measurement*. Suppose we have two von Neumann measurements devices, $\mathcal{P} = (P_i, \ i \in I$ and $\mathcal{Q} = (Q_j, \ j \in J)$. Then we can form the compound measurement $\mathcal{PQ}$ with the set of outcomes $I \times J$: we perform first measurement $\mathcal{P}$, then perform $\mathcal{Q}$ and write the obtained outcomes (i, j). If P_i commute with Q_j, $\mathcal{PQ}$ is von Neumann measurement as well. However in the general case, when $\mathcal{P}$ and $\mathcal{Q}$ are incompatible, the obtained measurement $\mathcal{PQ}$ is not a von Neumann measurement. This construction is one of the justifications for considering non-orthogonal measurements. Another line of justification relates to the possibility of defining mixtures (**see below**) and restrictions (**Section 4**).

Quantum Lottery

As we already wrote, a quantum lottery is a bet on the outcome of a quantum measurement. More precisely, a Q-lottery is a pair made of a quantum measurement device $\mathcal{P} = (P_i, \ i \in I)$ (the *base* of the lottery) and the prizes associated with the corresponding outcomes $(x_i, \ i \in I)$. We write such a Q-lottery as $\sum_i x_i \otimes P_i$. Intuitively the measurement $\mathcal{P}$ is performed and depending on the outcome i that obtains, the agent receives prizes x_i. The set of Q-lotteries is denoted as QL(H).

A constant Q-lottery is a lottery of the form $x \otimes E$; it gives the prize x with certainty.

Any Q-lottery $\sigma = \sum_i x_i \otimes P_i$ can be written in the *canonical form* $\sum_x x \otimes Q_x$, where

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[6]Hermitian operators that commute can be diagonalized in same orthonormal basis.

$Q_x = \sum_{i, x_i = x} P_i$.[7] Let $\mathbf{QL}_c(H, X)$ be the set of canonical Q-lotteries (or simply $\mathbf{QL}_c(H)$ because the specification of the set X does not play an essential role). Intuitively, the initial Q-lottery σ and its canonical form only differ in the way we write them and therefore can be considered as equivalent. Below we formulate this equivalence as our 'no-framing' axiom A0.

As it is the case for classical lotteries, we can define mixtures of Q-lotteries, but we restrict the mixture operation to canonical lotteries. Suppose that we have two Q-lotteries in the canonical form: $\sigma = \sum_x x \otimes P_x$ and $\tau = \sum_x x \otimes Q_x$. Then we can construct a new canonical Q-lottery as the *mixture* of the two (with weights α and $1 - \alpha$) $\alpha\sigma + (1 - \alpha)\tau :=$ $\sum_x x \otimes (\alpha P_x + (1 - \alpha)Q_x)$. The new lottery is interpreted as follows. First you use a "roulette device" to determine which one of σ or τ will be played and thereafter you play one of them.[8] (For instance, if $x^* \in X$ is an outcome never paid by τ, then $Q_{x^*} = 0$ and $\alpha\sigma + (1-\alpha)\tau$ pays x^* under the event αP_{x^*}.) The set $\mathbf{QL}_c(H, X)$ of canonical Q-lotteries is a convex space. We shall use this structure intensively in what follows.

3 Construction and characterization of 'nice' preferences

We are interested in preference relations over quantum lotteries that satisfy some 'natural' properties, familiar from von Neumann and Morgenstern, Savage, Aumann and others. We call such preference relations "nice". We start with a straightforward construction of nice preferences, thereafter we formulate their properties (axioms), and finally we show that these properties fully characterize nice preferences.

In order to construct preferences (in fact, to construct the utility of Q-lotteries), we should specify two things. First, a *utility function* $u : X \to \mathbb{R}$. Second, a linear '*belief*' *functional* $\beta : \mathbf{Herm}(H) \to \mathbb{R}$ which is

a) positive in the sense that $\beta(A) \geq 0$ for $A \geq 0$,

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[7]Although the set X can be infinite only a finite number of Q_x differ from 0.

[8]In the quantum case we do not in general have the equivalence - as in the classical case - with the alternative interpretation of the mixture: play both lotteries and use the roulette device to select which outcome determines the prize afterwards. This is because in the general case the two measurements implicit in the lotteries need not be compatible.

b) normalized in the sense that $\beta(E) = 1$.

With these two ingredients, we can define the '(u, β)-utility' $U(\sigma) = U_{u,\beta}(\sigma)$ of any Q-lottery $\sigma = \sum_i x_i \otimes P_i$ as

$$U(\sigma) = \sum_i u(x_i)\beta(P_i),$$

For Q-lotteries σ and τ we set $\sigma \preceq \tau$ if $U(\sigma) \leq U(\tau)$. Below we list some 'nice' properties A0-A4 that this preference relation $\preceq$ on $\mathbf{QL}(H)$ possesses.

No framing

A0. *For any* $\sigma, \tau \in \mathbf{QL}(H)$ *with respective canonical forms* σ' *and* τ', $\sigma \preceq \tau \Leftrightarrow \sigma' \preceq \tau'$.

Axiom A0 follows from the fact that the utility of a Q-lottery $\sigma = \sum_i x_i \otimes P_i$ is equal to the utility of its canonical form $\sum_x x \otimes (\sum_{i,x_i=x} P_i)$. Indeed, the utility of the latter is equal to $\sum_x u(x)\beta(\sum_{i,x_i=x} P_i) = \sum_x \sum_{i,x_i=x} u(x_i)\beta(P_i) = \sum_i u(x_i)\beta(P_i) = U(\sigma)$.

This axiom is implicit in the Savage and Anscombe-Aumann frameworks. However, in generalizations of these frameworks it must be imposed explicitly see e.g., Cohen and Jaffray (1980). They formulate an axiom of 'non influence of formalization' very similar to our axiom A0. There are also other works that reject that axiom in order to allow for framing effects see Ahn and Ergin (2010).

Weak order

A1. *The preference relation* $\preceq$ *is a weak order, that is, a complete and transitive binary relation.*

This follows from its representation via the utility U.

The next two properties assert that the preference relation is consistent with a mixture structure on the set $\mathbf{QL}(H)$.

Independence

A2. *Let* $\sigma, \tau, \varphi \in \mathbf{QL}_c(H)$ *be canonical Q-lotteries, and* $\alpha \in [0, 1]$. *If* $\sigma \preceq \tau$ *then* $\alpha\sigma + (1-\alpha)\varphi \preceq \alpha\tau + (1-\alpha)\varphi$.

Property A2 follows from the linearity of β.

Continuity

A3. *Let $\sigma, \tau, \varphi \in \mathbf{QL}_c(H)$, and $\sigma \prec \varphi \prec \tau$. Then there exists α and β ($0 < \alpha, \beta < 1$) such that $\alpha\sigma + (1-\alpha)\tau \prec \varphi$ and $\varphi \prec \beta\sigma + (1-\beta)\tau$.*

Indeed, since $U(\sigma) < U(\varphi) < U(\tau)$, we have $U(\varphi) > \alpha U(\sigma) + (1-\alpha)U(\tau)$ for some $\alpha \in (0,1)$ and $U(\varphi) < \beta U(\sigma) + (1-\beta)U(\tau)$ for some $\beta \in (0,1)$. So $U(\varphi) > U(\alpha\sigma + (1-\alpha)\tau)$ and $U(\varphi) < U(\beta\sigma + (1-\beta)\tau)$.

Monotonicity

This property is the most subtle. It asserts, roughly speaking, that if we in a lottery $\sigma = \sum_i x_i \otimes P_i$ replace the prizes x_i with better ones then the new lottery will be preferred to the initial one. However, this formulation is too weak for our aims, and we formulate it in stronger form. To do that we first note that we can define the canonical form not only for 'prize valued Q-lotteries' but also for 'roulette valued Q-lotteries', that is for expression of the form $\sum_i l_i \otimes P_i$, where $l_i = \sum_x x \otimes l_i(x)$ are roulette lotteries. The canonical form of this lottery is $\sum_x x \otimes Q_x$, where $Q_x = \sum_i l_i(x)P_i$. Let $\mathbf{QL}(H, \Delta(X))$ denote the set of all roulette-valued Q-lotteries. We use this to extend the preference relation to $\mathbf{QL}(H, \Delta(X))$ as follows: for any $\sigma, \tau \in \mathbf{QL}(H, \Delta(X))$ with respective canonical form σ' and τ', $\sigma \preceq \tau$ if and only if $\sigma' \preceq \tau'$. We next have to consider the derived preference relation $\preceq_\Delta$ on the set of ordinary (roulette) lotteries $\Delta(X)$. For roulette lotteries l and m we set $l \preceq_\Delta m$ if $l \otimes E \preceq m \otimes E$. Here $l \otimes E$ denotes a constant Q-lottery, getting with certainty the prize l, and similarly for $m \otimes E$.

Note that the preference $\preceq_\Delta$ on $\Delta(X)$ is represented by the affine extension of u from X to $\Delta(X)$, that we still denote by u. Indeed, we have $U(l \otimes E) = U(\sum_x x \otimes l(x)E) = \sum_x u(x)\beta(l(x)E) = \sum_x u(x)l(x) = u(l)$, because $\beta(E) = 1$.

A4. *Let $\sigma = \sum_i l_i \otimes P_i$ and $\tau = \sum_i m_i \otimes P_i$ be elements of $\mathbf{QL}(H, \Delta(X))$ with the same base. If $l_i \preceq_\Delta m_i$ for any $i \in I$ then $\sigma \preceq \tau$.*

This property can be considered as a strong version of the sure-thing principle of Savage. It is a simple consequence of the formula $U(\sigma) = \sum_i u(l_i)\beta(P_i)$ and of the non-negativity of $\beta(P_i)$.

Definition 3. A preference relation $\preceq$ on the set $\mathbf{QL}(H)$ of Q-lotteries is *nice* if it has the properties $\mathbf{A0} - \mathbf{A4}$ (or satisfies the axioms $\mathbf{A0} - \mathbf{A4}$).

The discussion above shows that a preference relation $\preceq_{u,\beta}$ built with the help of a utility function u on X and a linear functional β on $\mathbf{Herm}(H)$ is nice. Our first result asserts that the reverse is also true.

Theorem 1. *A preference relation $\preceq$ on $\mathbf{QL}(H)$ is nice if and only if there exist a function u on X and a positive and normalized linear functional β on $\mathbf{Herm}(H)$ such that $\preceq = \preceq_{u,\mu}$.*

Moreover, if the preference $\preceq$ is not trivial (that is there exist Q-lotteries σ and τ such that $\sigma \prec \tau$) then β is unique and u is unique up to a positive affine transformation.

The proof is in Appendix 2. We here provide a brief sketch. First, fix a measurement device $\mathcal{P} = (P_i, \ i \in I)$. In the first stage of the proof, we use the primitive preference $\preceq$ on $\mathbf{QL}(H)$ to construct another preference $\preceq_{\mathcal{P}}$ on the set $\mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ of roulette-valued Q-lotteries with base $\mathcal{P}$. We view each such Q-lottery in $\mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ as an Anscombe-Aumann act from I to $\Delta(X)$. We next invoke the Anscombe-Aumann (1963) theorem and obtain a utility function $u_{\mathcal{P}}$ and a probability vector $\beta_{\mathcal{P}}$ on I which provide us with a Subjective Expected Utility (SEU) representation of $\preceq_{\mathcal{P}}$. The second stage of the proof involves showing that the various SEU representations are consistent with each other; that is, that the functions $u_{\mathcal{P}}$ are essentially independent of $\mathcal{P}$ and that the probability vectors $\beta_{\mathcal{P}}$ arise from a single belief functional β.

In a classical Anscombe-Aumann setting, it is straightforward to obtain the consistency of the various SEU representations across different partitions of the state space because acts are directly defined as functions from the state space to $\Delta(X)$. But suppose now that an act is given by $(E_1, l_1; \dots; E_n, l_n)$ where $(E_1, \dots, E_n)$ is a partition of the state space and $(l_1, \dots, l_n)$ is a corresponding collection of lotteries on X. Such an act induces a function $\sum_i \mathbb{1}_{E_i} l_i$. Under an additional axiom requiring two acts inducing the same function to be always indifferent, we again obtain the consistency of the various SEU representations across the various partitions of the state space. Coming back to our nonclassical setup, it takes a very similar axiom, namely A0, to obtain the consistency of the SEU representations $(u_{\mathcal{P}}, \beta_{\mathcal{P}})$, across different PDU. The full representation result stated in Theorem 1 easily follows. We next investigate some consequences of our result.

The shadow operator of a Q-lottery

Let $\preceq$ be a nice preference relation on Q-lotteries. We fix some function u from X to $\mathbb{R}$. We call the *shadow operator* of a lottery $\sigma = \sum_i x_i \otimes P_i$ the Hermitian operator $Sh(\sigma) = Sh^u(\sigma)$ defined as follows

$$Sh(\sigma) = \sum_i u(x_i)P_i.$$

We note that $\beta(Sh(\sigma)) = U(\sigma)$, and consequently the utility of Q-lottery σ depends only on its shadow operator. The notion of shadow operator is the equivalent of the notion of utility profile in the classical framework. And in the same way the expected utility of an act only depends on its utility profile. From here we could completely forget about Q-lotteries and discuss the utility of Hermitian operators expressed by the functional β.

Trace and belief operator

There exists a remarkably useful way of representing the belief functional β by means of a (Hermitian) operator of belief. For that we shall make extensive use of the concept of the trace of an operator (and precisely here the finite dimensionality of H becomes important). A reminder of the definition and properties of the trace is provided in Appendix 1. Of particular value in our context are two properties: commutativity $\mathbf{Tr}(AB) = \mathbf{Tr}(BA)$ and the fact that the trace of any Hermitian operator is a real number.

Definition 4. We call a *belief operator* (or a *cognitive state*) any nonnegative Hermitian operator with the trace equal to 1.

Given a belief operator B, we can define the functional β on $\mathbf{Herm}(H)$, setting $\beta(A) = \mathbf{Tr}(AB)$ for any $A \in \mathbf{Herm}(H)$.

Lemma 1. 1) *The functional β takes real values*;
2) $\beta(A) \geq 0$ *for* $A \geq 0$;
3) $\beta(E) = 1$.

Proof. It follows from the properties of the Trace operator see Appendix 1. $\square$

As a consequence, we obtain that the functional β built on the belief operator B is a belief functional. Moreover any belief functional has such a form (for a unique belief operator B). In fact the formula $(A, B) = \mathbf{Tr}(AB)$ gives a scalar product (and thereby also the structure of

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an Euclidean space) on the real vector space $\mathbf{Herm}\,(H)$. Relying on well-known description of linear functionals on Euclidean space, we have proved the following

Theorem 1′. *A preference relation $\preceq$ on $\mathbf{QL}(H)$ is nice if and only if there exist a utility function $u : X \to \mathbb{R}$ and a belief operator B such that $\preceq$ is represented by the function $\sigma \mapsto \mathbf{Tr}\,(Sh^u\,(\sigma)\,B)\,.$*

Moreover, if the preference $\preceq$ is non trivial (that is there exist Q-lotteries σ and τ such that $\sigma \prec \tau$) then B is unique and u is unique up to a positive affine transformation.

Remark. In quantum physics such belief operators are called 'states' or 'density operators'. We shall refer to them as 'belief-state' or 'cognitive state' because they allow constructing subjective probabilities in a most suitable way. Indeed, let B be a belief operator; then, for any event P, $\beta_B(P) = \mathbf{Tr}(PB)$ is the subjective probability for event P when the cognitive state is B.

Example 1. Assume that belief operator B is given as the projector $P = P_e$ on a one-dimensional subspace $\mathbb{C}e \subset H$, generated by a vector e of length 1 (that is $(e, e) = 1$; in Physics such operators are called *pure states*). In other words, $P(x) = (x, e)e$. It is easy to check that, for any Hermitian operator A (viewed as a Q-lottery), its utility $U(A)$ is equal to $\mathbf{Tr}(AP) = (Ae, e)$. The quadratic form (Ae, e) gives the utility of lottery A when the belief is represented by a pure state e. The value of the lottery is the expected utility when all but elementary event e have zero subjective probability.

Such a belief corresponds to a maximally precise (subjective) representation of the measured system. In the quantum context, a maximal information state does not correspond to complete information as in the classical context. In a maximal information state a measurement (incompatible with a one that has the current pure state as a possible outcome) generates new information and leads to the loss of some other previously known information: the state changes.

Example 2. Let us now consider the polar case, when our DM has a belief represented by the operator $B = E/\dim H$ which can be interpreted as '*uniform uncertainty*' i.e., the DM assigns equal probability to all states. The expected value of a lottery A under such a belief is $U(A) = \mathbf{Tr}(AE)/\dim(H) = \mathbf{Tr}(A)/\dim(H)$ which corresponds to the arithmetic average of the eigenvalues of operator A.

Note that as in the standard setting whatever the belief B, the expected value of a lottery-operator A equals to some convex combination of its eigenvalues.

4 Updating

Suppose that the belief of our decision-maker is given by belief operator B. That is, she believes that the state of the measured quantum system is B. Alternatively, her cognitive state is such that she assigns probabilities to events according to B. This means the following. We here consider events as subspaces of H, or as projectors. If P is a projector then the probability of P (with belief B) is equal to $\mathbf{Tr}(PB)$. Since P is a projector, we can write $prob(P) = \mathbf{Tr}(PBP)$ a formulation that we use extensively below. In the next section, we return to the interpretation of B. But for now suppose that she receives information of relevance for the lotteries. This information may concern the prizes or be relative to the system, the measurement of which determines the outcomes and the prizes. In the following, we restrict ourselves to the case when the value of the prizes does not change (and is given by a fixed utility function u) and all new information concerns the measured system. For instance, the DM (or someone else) performs some intermediate measurement and learns as a result of the measurement that some event P occurred. It is almost obvious that her belief and preferences on Q-lotteries should change, the question we ask in this section is how should her preferences on quantum lotteries change after receiving that information?

In Quantum Mechanics, it is simply postulated that the state (of a measured system) changes in accordance with the von Neumann-Lüders postulate. More precisely, a system that was in state B transits to the state $B' = PBP/\mathbf{Tr}(PBP)$ as a result of performing a measurement that yields event P. The operator PBP is Hermitian and non-negative $((PBPv, v) = (BPv, Pv) \geq 0$ by force of the nonnegativity of B). Thus, B' is indeed a state. Here, we need to clarify why $\mathbf{Tr}(PBP)$ is different from zero so we are allowed to divide by this number. We understand $\mathbf{Tr}(PBP)$ as the probability to discover event P in cognitive state B. Thus, by analogy with standard Bayesian updating, the von Neumann-Lüders postulate focuses on cases where the state assigns a positive probability to event P. If the trace $\mathbf{Tr}(PBP)$ were equal to 0, that would mean that something happened which had zero probability, i.e. an event that is considered impossible under belief B. That is, the

belief of our DM captured by the state B is simply incorrect and she has to update it in a more fundamental way.

We want to show that in quantum decision theory beliefs change in the same way. Clearly, we have to make some assumptions. In order to determine which assumptions we need, we return for a minute to the behavior of a classical decision-maker. She has preference over functions (acts) defined on the set S of states of nature; suppose that she learns in addition that the true state lies in some subset $T \subseteq S$. It is quite natural to assume that her new preference depends only on values of these functions on the subset T. That is, only on the restriction of the various functions to T.

I. Information as events

We want to proceed with Q-lotteries in a way analogous to the classical case. We shall assume that the new information comes from the performance of a projective measurement and that the obtained result informs that event P has occurred. Here P is a projector on subspace W. We have to define what we mean with the "restriction of a Q-lottery" to a subspace W.

As a subspace of the Hilbert space H, W is also a Hilbert space. Given an Hermitian operator A on H, one can consider the operator PA *as an operator on* W ($v \mapsto PA(v)$ for $v \in W$). To avoid confusion, we denote operator PA, constructed as an operator on W, by $A|W$ and call it the *restriction* of A on W. First we note that $A|W$ is an Hermitian operator (as an operator on W, not on H). Indeed, if $y, z \in W$ (and hence $Py = y$, $Pz = z$) then

$$(PAy, z) = (Ay, Pz) = (Ay, z) = (y, Az) = (Py, Az) = (y, PAz).$$

The same argument shows that $A|W$ is a nonnegative operator on W provided A is nonnegative. Note, finally, that $P|W$ is the identity operator on W.

The above indicates how to define a restriction to subspace W of any Q-lottery. If $\sigma = \sum_i x_i \otimes P_i$ is a Q-lottery on H, then $\sigma|W := \sum_i x_i \otimes P_i|W$ is a Q-lottery on W. We call $\sigma|W$ the *restriction* of lottery σ to W. Clearly, $Sh(\sigma|W) = Sh(\sigma)|W$.

Let us next turn to the problem of updating nice preferences. Suppose that $\preceq$ is a nice preference relation on the set $\mathbf{QL}(H)$; due to Theorem 1′ it is given by some belief operator

B. Suppose now that our DM receives information in the form of an event-subspace $W \subset H$ (or of an event-projector P). The updated preference relation is denoted by $\preceq_W$. Following Ghirardato (2002), we formulate two axioms connecting $\preceq_W$ and $\preceq$. The first one, A5, is 'consequentialism':

A5 *If σ and τ are Q-lotteries on H, and $\sigma|W = \tau|W$, then σ and τ are equivalent with respect to $\preceq_W$.*

To formulate the second axiom we need one more notion. We say that a projector P is compatible with a Q-lottery $\sigma = \sum_i x_i \otimes P_i$ if P commutes with every P_i. Now we can formulate axiom **A6** of compatible dynamic consistency:

A6 *Suppose that lotteries σ and τ are compatible with P, and $\sigma|W^\perp = \tau|W^\perp$, then $\sigma \preceq_W \tau$ if and only if $\sigma \preceq \tau$.*

Extending Ghirardato (2002) to the quantum context, we state the following

Theorem 2. *Let $\preceq$ be a non-trivial nice preference given by a belief operator B, and $Tr(PBP) > 0$. Then*

*a) The preference relation $\preceq_W$ given by the belief operator $B^{up} = PBP/Tr(PBP)$ satisfies axioms **A5** and **A6**.*

*b) Conversely, if a preference relation $\preceq_W$ satisfies Axioms **A1**, **A5** and **A6**, then it is nice and it is given by the 'updated' belief operator $B^{up} = PBP/Tr(PBP)$.*

The proof of Theorem 2 is in Appendix 3. It generalizes the classical equivalence result between Dynamic Consistency and Consequentialism on the one hand and Bayesian updating on the other hand to our nonclassical setting. Its proof is similar to the classical one. Given two classical Q-lotteries $\sigma, \tau \in \mathbf{QL}(H)$, we construct two other Q-lotteries $ad_W(\sigma), ad_W(\tau) \in \mathbf{QL}_c(H)$ with the following properties:

(1) $ad_W(\sigma)|W = \sigma|W$ and $ad_W(\tau)|W = \tau|W$,

(2) $ad_W(\sigma)|W^\perp = ad_W(\tau)|W^\perp$.

That is, $ad_W(\sigma)$ and $ad_W(\tau)$ agree on W with σ and τ respectively, while they agree with each other on $W^\perp$. Given Axioms A5 and A6, these two properties imply the equivalence between $\sigma \preceq_W \tau$ and $ad_W(\sigma) \preceq ad_W(\tau)$ for any Q-lotteries σ and τ. In a classical setting, this equivalence is essentially a form of Savage's (1954) Sure Thing Principle. Actually,

Savage postulates a preference satisfying the Sure Thing Principle and defines conditional preference through this equivalence. Moreover, we show that we can choose $ad_W(\sigma)$ and $ad_W(\tau)$ such that their shadow operators are respectively given by $PSh(\sigma)P$ and $PSh(\tau)P$. In classical terms, this means that the 'utility profiles' induced by $ad_W(\sigma)$ and $ad_W(\tau)$ are equal to those of σ and τ on event W and equal to 0 otherwise. From there, it takes a little algebra and the uniqueness part of Theorem 1' to conclude. We next discuss some corollaries of this theorem.

Remark 1. We above assumed that the probability of the event P is non-zero. In the opposite case the received information contradicts the initial belief. Consider now the case when the probability of the event P (equal, as we know, to $\mathbf{Tr}(PBP)$) is 1, that is our DM is sure that event P must occur. Receiving information that P occurred she does not learn anything and intuitively we expect her preferences to remain the same, which indeed obtains:

Proposition 1. *The distance between a prior B and the posterior B^{up} is $O((1 - \mathbf{Tr}(PBP))^{1/4})$.*

In particular, if $Tr(PBP) = 1$ then $B^{up} = B$. We prove Proposition 1 in Appendix 4.

Remark 2. Let us consider a situation when the initial (a priori) belief is maximally precise, that is, it is given (as in Example 1) by a one-dimensional projector or, to put it differently, B is a pure state. How does the DM update her belief as she receives new information? We expect the updated state to be pure as well. This is true in the classical context and this is true in the quantum case as well with a noticeable distinction that the new pure state is generally *not* the same as the initial one.

Proposition 2. *If a belief operator B is a one-dimensional projector then the updated operator B^{up} is a one-dimensional projector as well.*

Proof. The rank of operator PBP is less or equal to the rank of operator B, that is ≤ 1. □

The main difference with the classical situation is that the new state is generally different from the initial one. Indeed, let our belief operator B be the projection on the line $\mathbb{C}e$, where e is a normalized vector $((e, e) = 1)$. Such a projector B moves an arbitrary vector x to vector

$(x, e)e$. Then the operator PBP moves vector x to $P((Px, e)e) = (Px, e)P(e) = (x, Pe)Pe$, that is the projection (up to the multiplier (Pe, Pe)) on the line $\mathbb{C}Pe$. Thus, the pure state e is changed into another pure state $Pe/\sqrt{(Pe, Pe)}$. The reader can see here the projection postulate at work. We would like to add that the modification is smaller the closer the vector e to the subspace W associated with the projector P. That is the closer $\mathbf{Tr}(PBP) = (Pe, Pe)$ to 1.

Remark 3. Suppose that, after receiving information in the form of an event-projector P, the DM receives yet some new piece of information in the form of an event-projector Q (which does not contradict P). Then (up to a factor) the state changes as $B \mapsto PBP \mapsto QPBPQ$. However, in the case when events-projectors do not commute with each other, the updated belief depends on the order in which the updating is realized. Manipulating the order in which information is provided affects our DM preferences and thus her choice behavior. As we shall see later, together with Remark 2, Remark 3 paves the ground for interesting economic implications.

Remark 4. Above we assume that the information comes in the form of an event, that is a subspace W or a projector P. However we could consider the more general case when the information has the form a 'fuzzy-projector', that is an Hermitian operator P such that $0 \leq P \leq E$. In this case the updated belief takes the form $B^{up} = \sqrt{P}B\sqrt{P}/\mathbf{Tr}(PB)$. Proposition 1 generalizes to this setting; see Appendix 4.

II. Information as measurements

In the remaining of the paper we use the 'shadow operator' expression for lotteries: $A = Sh^u(\sigma)$. Due to Theorem 1', the utility of such a lottery under belief B is equal to $U(A) = \mathbf{Tr}(AB)$.

Usually, new information arises as the outcome of some measurement. Suppose we perform some von Neumann measurement, represented by an orthogonal decomposition of unit (ODU) $(P_i, \ i \in I)$. If, as the result of this measurement, we obtain outcome i, the belief-state B changes (is updated) into $B_i = P_i B P_i/\mathbf{Tr}(P_i B)$. We say that the state of the system *transits* into subspace $W = \mathrm{Im}(P_i)$. Note that the number $p_i = \mathbf{Tr}(P_i B)$ is precisely the probability (under the belief B) for the realization of outcome i when performing our

measurement. Excluding impossible results, we can assume that these numbers are strictly positive.

The utility of 'lottery' A, after the DM received information about the realization of outcome i, is now equal to $U_i(A) = \mathbf{Tr}(AB_i)$. It may clearly be either larger or smaller than the initial $U(A)$.

Further we discuss an interesting case when such a measurement has been performed (using ODU $\mathcal{P} = (P_i, \ i \in I)$), but its outcome is *not* known to the DM, i.e., she only knows that the measurement took place. This is a new type of information; in Physics one speaks about 'decoherence'. It is a fully non-classical phenomenon, because in the classical situation such an 'information' is useless. This is not the case in a quantum context.

We earlier established that when learning about the occurrence of an outcome i, the utility of the lottery-operator A is updated to $U_i(A) = \mathbf{Tr}(AB_i)$ (where B is the belief of our DM, and $B_i = P_i B P_i / \mathbf{Tr}(P_i B)$) that is equal to $\mathbf{Tr}(AP_i BP_i)/p_i$. Since the probability of outcome i is p_i, then the expected utility of our lottery $U'(A)$ (when learning about the performance of measurement $\mathcal{P}$) is

$$U'(A) = \sum_i p_i U_i(A) = \sum_i \mathbf{Tr}(AP_i BP_i).$$

(Of course, this means also that the state of belief B has changed into a new state $B' = \sum_i P_i B P_i$). And, as we shall see, the connection between these two utilities, i.e. *ex-ante* $U(A)$ and *ex-post* $U'(A)$, is not straightforward in general. Ex-post utility can be either larger or smaller than the initial (a priori) utility as the following simple example shows.

Example 3. Let H be a two dimensional Hilbert space with orthonormal basis (e_1, e_2). Let A be a projector on e_1, i.e., an operator of the form $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Consider A, a lottery that gives a utility equal to 1 in state e_1 and 0 otherwise. Consider another lottery $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ that gives utility 1 in state e_2 and 0 otherwise. Assume now that our DM's belief-state is given by $B = A$. Clearly the expected utility of lottery A is equal to 1 and the expected utility of C is equal to zero. So our DM strictly prefers A to C.

Assume now that we perform a measurement defined by the ODU (P_1, P_2), where $P_1 =$

$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$. If the outcome of the measurement is 1, the updated belief-state is given by operator $B_1 = P_1 B P_1 / \mathbf{Tr}(P_1 B P_1)$, which as can be seen easily is equal to P_1. The expected utility in the belief-state is $U_1(A) = \mathbf{Tr}(AB_1) = \mathbf{Tr}(AP_1) = 1/2$. And similarly if we obtain (the complementary) outcome 2, the belief-state is updated to $B_2 = P_2 B P_2 / \mathbf{Tr}(P_2 B P_2)$ and the corresponding expected utility is $U_2(A) = \mathbf{Tr}(AB_2) = \mathbf{Tr}(AP_2) = 1/2$. So we see that for any one of the two possible outcomes the value of the A lottery goes from 1 to $1/2$. With the same reasoning we obtain that $U_1(B) = \mathbf{Tr}(CB_1) = \mathbf{Tr}(CP_1) = 1/2 = U_1(C)$. So the two lotteries A and C yield the same expected utility. This violates "recursive dynamic consistency": lottery A is *ex-post* indifferent to C whatever the outcome of the measurement, yet *ex-ante* it is strictly preferred. Note that the intermediary measurement (P_1, P_2) is incompatible with either A or C and B.

It may seem odd that our dynamic consistency axiom A6 allows for such departures. But the appeal of recursive dynamic consistency is based upon the implicit assumption that "the act the agent performs has no effect on the resolution of uncertainty" [13]. However, the resolution of uncertainty is - in our setting - affected by the act that is selected and the measurements it entails (as well as by other measurements) performed to acquire new information. Once this is taken into account, recursive dynamic consistency loses much of its appeal. In the next section we discuss an economic example of the phenomenon exhibited above in connection with the behavior documented in the "Economics of Manipulation and Deception" [2].

Example 3 also allows illustrating 'information as measurements' a feature that lacks counter-part in the classical model. Imagine that we perform the measurement described above but our DM is not informed of the result. She only knows the measurement has been made. In the classical world such an information does not affect the DM's belief or the expected value of the lotteries. However in the quantum context the DM understands that for any of the two outcome (1 or 2) the expected value of lottery A has changed from 1 to $1/2$. Therefore, independently of her (lack of) knowledge about the outcome of the measurement, she will revise her belief-state. The new belief-state is $B' = p_1 B_1 + p_2 B_2 = E/2$ which corresponds to 'uniform ignorance'. And in this belief-state the expected utility of lottery A is equal to $\mathbf{Tr}(AB') = \mathbf{Tr}(AE)/2 = 1/2$. $\square$

There are two interesting cases when decoherence (that is a measurement with unknown outcome) does not change the utility of a lottery or the state of belief. These are cases when the measurement $\mathcal{P}$ is compatible either with the lottery A or with the belief B, i.e. when all the projectors P_i, $i \in I$, commute either with operator A or with operator B.

Proposition 3. *Assume that the intermediate measurement is compatible with either operator A or B. Then $U(A) = U'(A)$.*

Proof. $U'(A) = \sum_i \mathbf{Tr}(AP_iBP_i) = \sum_i \mathbf{Tr}(P_iAP_iB)$. Suppose that $P_iA = AP_i$. Then the second sum can be rewritten as

$$\sum \mathbf{Tr}(P_iP_iAB) = \sum \mathbf{Tr}(P_iAB) = \mathbf{Tr}((\sum P_i)AB) = \mathbf{Tr}(AB) = U(A).$$

Suppose now that $P_iB = BP_i$. Then the first sum can be rewritten as

$$\sum \mathbf{Tr}(AP_iP_iB) = \sum \mathbf{Tr}(AP_iB) = \mathbf{Tr}(A(\sum P_i)B) = \mathbf{Tr}(AB) = U(A). \blacksquare$$

Remark 5. Decoherence always changes state of belief toward a more dispersed one. One can give to this statement an exact sense, using notions from [3]. Here we would like to illustrate this by an example when initial state is pure, presented by a (normalized) vector e. As we see from the formula above (for updating the belief operator), the updated state is a probability mixture (with weights p_i) of pure states corresponding to the projections of e on the subspaces W_i. That is a pure (coherent) state disintegrates (decoheres) into a mixture of pure states.

# 5   Quantum Cognition in Economics

In this section we discuss the possible value of our results for behavioral economics. We start with a few words about quantum cognition. We know that in order to assess the world we build a representation of it, a "represented world" which is a mental construct. In classical standard theory, the represented world reflects our incomplete knowledge about the world expressed in our belief and this belief (should) evolves according to Bayes' rule in response to new information. Quantum cognition has been developed under the last decades as an

alternative approach to incorporate two observations: 1. People have difficulties to build a representation of a complex object. What people do is to consider a complex object from different perspectives - one at a time; 2. People may be unable to combine perspectives i.e., to synthesize all relevant information into one stable representation of the complex object. Quantum cognition models the incapacity to combine some pieces of information by analogy with incompatible properties (also called "Bohr complementary") in Quantum Mechanics: different properties may be incompatible in the sense that they cannot be given a determinate value simultaneously (cf speed and position) but they complement each other in the description of the system. Similarly different perspectives on an alternative may be incompatible in the mind in the sense that the individual cannot have a clear stand with respect to them simultaneously (i.e., combine them in a single coherent stable picture) but the different perspectives contribute in characterizing the alternative.

In the present paper, we have extended decision theory to a non-classical uncertainty environment. An interpretation of this move corresponds to proposing that the "represented world" used to evaluate lotteries exhibits quantum-like properties. The above mentioned cognitive limitations are then viewed as an expression of the quantum indeterminacy of beliefs. Our results with respect to dynamic consistency shed new light on observed behavioral anomalies in the spirit of Shiller (2000) and Akerlof and Shiller (2015) who write : "In our thoughts, as in our conversation, our minds may change. It is not just that we acquire new "information"; we change our point of view and we interpret information in a new way. Importantly these evolutions of our thoughts mean that our opinions, and the decisions that are based on them, may be quite inconsistent"(p.45). We next provide an example showing how the quantum indeterminacy of beliefs and in particular its dynamic properties illustrated in Example 3 above delivers the kind of manipulability at the core of Phishing for Phools ([2]).

Consider a seller who wants to sell a smartphone at price 300$ and a customer considering buying one. Initially, the customer holds a belief about the quality of the smartphone that can be either Excellent (utility 600) or Standard (utility 100). Her initial cognitive state assigns subjective probabilities 0.25 and 0.75 respectively to the two possible events so the expected utility is $EU(S) = .25 \cdot 600 + .75 \cdot 100 = 225$. The alternative is to keep the money which has utility 300. So initially the customer does not want to buy the new smartphone since $225 < 300$. Now the seller engages in a conversation about the use of this brand

of smartphone among famous people. In so doing the customer is moved from a private user perspective on the smartphone to a "glamour" perspective (also with two outcomes: glamour, not glamour). Assume that the two perspectives are incompatible in the mind. Now what matters to our consumer is her idol Beyoncé, so she asks whether Beyoncé uses that smartphone. The seller answers truthfully either yes or no. After the conversation the consumer updates her belief, her cognitive state is modified. Consider for simplicity the case when private user and glamour perspectives are totally uncorrelated (the corresponding bases are 45° rotations of each other - see the figure below). As she turns back to the question whether or not to buy the smartphone, the lottery with the updated belief yields expected utility $EU(S) = .5 \cdot 600 + .5 \cdot 100 = 400 > 300$ whether she learned that the smartphone is glamour or not (and her cognitive state is projected onto the corresponding axis): "the phool has been phished!": the seller has exploited the quantum indeterminacy of the consumer's "represented smartphone" (that is her belief and her consistent updating rule when evaluating the lottery) to change her preferences so she chooses to purchase the smartphone. Quoting Akerlof and Shiller again "Just change people focus and you can change the decisions they make" ([2], p. 173).
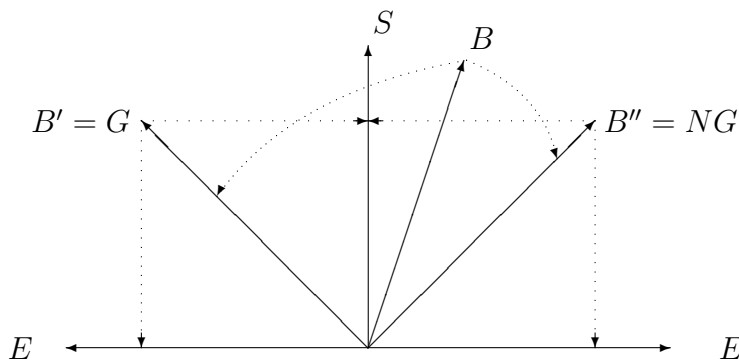
In this example, we have assumed that the user and glamour perspectives are incompatible in the mind of Receiver. An interpretation is that they appeal to arguments and emotions that cannot be processed simultaneously while still being related as aspects of the smartphone. The user perspective appeals to concrete functionalities of immediate practical significance for her. The glamour perspective appeal to her social persona, the pride and pleasure to be associated with a highly popular singer. The specific correlation (45°) can be interpreted as follows: when in the "glamour" perspective, she is (possibly uncon-

sciously) reminded of the general disconnect between what is temporarily "fashionable" and fundamental user values of commodities.

6   Concluding remarks

In this paper we provided a consistent choice theory that accounts for cognitive limitations affecting our capacity to build representations of choice alternatives. We found that the mathematical formalism of quantum mechanics offers a suitable framework for modeling such cognitive limitations. We characterize the rules for consistent choice behavior in a non-classical uncertainty environment. A concept of quantum lottery is introduced and sufficient and necessary conditions for choice behavior to be representable by an expected utility function are formulated. We next derive, from behavioral principles, an updating rule that secures the dynamic consistency of her preference relation as the decision-maker learns new information.

We found that most of the classical axioms of decision theory carry over to the context of quantum lotteries. This is because all but one axiom can be formulated in terms of a single orthogonal decomposition of the state space. Quantum lotteries operating in the Hilbert space are then equivalent to roulette lotteries in a classical state space. An additional axiom is required to secure that the probability for any specific event does not depend on the particular lottery that it belongs to. The necessity to impose that axiom stems from the fact that while it is trivially true in the classical world, it is not necessarily so in our general setting.

A most interesting result is that the von Neumann-Lüders postulate which is central to Quantum Mechanics and informs about the impact of a measurement on the state of a system can be derived from a consistency requirement on choice behavior. When the belief-state (cognitive state) is updated according to the postulate, the agent conditional preferences reflect a single preference order. Of particular interest for behavioral economics is that in contrast with classical subjective expected utility theory, dynamic consistency of preferences does not entail the so-called recursive dynamic consistency. This is an expression of the fundamental distinction between the two settings namely that the resolution of uncertainty depends on the operation(s) performed to resolve it. We suggest in an example that this

very feature allows for interesting applications in behavioral economics in particular related to the manipulability of agents.

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Appendix 1. Elementary facts about Hilbert spaces

Hilbert space

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and of complex numbers. For a complex number z, $\bar{z}$ denotes its complex conjugate.

Definition 1. Let H be a vector space over the field $\mathbb{C}$. An *Hermitian form* on H is a mapping $(.,.) : H \times H \to \mathbb{C}$ such that: a) it is linear in the first argument; b) $(v, w) = \overline{(w, v)}$ for any $v, w \in H$ (in particular, (v, v) is a real number); c) $(v, v) \geq 0$ for any $v \in H$, and $(v, v) = 0$ only for $v = 0$.

Vectors v and w are called *orthogonal* if $(v, w) = 0$; in this case $(w, v) = 0$ as well.

A *Hilbert space* is a vector space H endowed with an Hermitian form, which is complete relatively to the norm $|v| = \sqrt{(v, v)}$. In order to avoid unnecessary difficulties and subtleties we assume further that H has finite dimension; then H automatically is complete.

When discussing lotteries and measurements we shall not be dealing so much with vectors in H as with special operators (linear mappings from H to H) called Hermitian operators.

Hermitian operators

Definition 2. A linear operator $A : H \to H$ is called *Hermitian*, if $(Av, w) = (v, Aw)$ for any $v, w \in H$.

Clearly (Av, v) is a real number for any $v \in H$. Hermitian operator A is called *nonnegative* if $(Av, v) \geq 0$ for any v. For Hermitian operators A and B we write $A \geq B$ if $A - B$ is nonnegative. The identity operator E ($Ev = v$ for every $v \in H$) is Hermitian.

A most important, for the purpose of this paper, class of Hermitian operator consists of projectors. A projector is an idempotent Hermitian operator, that is $PP = P$. Since $(Pv, v) = (PPv, v) = (Pv, Pv) \geq 0$, any projector is nonnegative. Each projector P define a vector subspace $V = \text{Im} P \subset H$, consisting of vectors v such that with $Pv = v$. The kernel of the projector consists of vectors orthogonal to V, $\text{Ker} P = V^{\perp}$. The set of projectors can be identified with the set of (closed) subspaces of H.

Any linear combination of Hermitian operators with real coefficients is an Hermitian operator. In other words, the set **Herm**(H) of Hermitian operators is a real vector space. The crucial importance of projectors is underlined by the following central theorem.

Spectral theorem. *Let A be an Hermitian operator. Then there exists a family of projectors P_i and real numbers a_i such that: 1) $P_i P_j = 0$ for $i \neq j$, 2) $\sum_i P_i = E$, and 3) $A = \sum_i a_i P_i$.*

In other words, in some orthogonal basis the operator A can be represented by a diagonal matrix (with real coefficients). The coefficients a_i are eigenvalues of the operator A. The set of numbers a_i is called the *spectrum* of the operator A. Clearly A is nonnegative if and only if all coefficients a_i are nonnegative. An operator A is a projector if and only if its spectrum $SpecA$ consists of 0 or 1.

Each nonnegative operator A has a (unique) nonnegative square root $\sqrt{A}$ (or $A^{1/2}$, such that $(\sqrt{A})^2 = A$). If $A = \sum_i a_i P_i$ is a spectral representation of A then $\sqrt{A} = \sum_i \sqrt{a_i} P_i$.

Trace of operators

For arbitrary (not necessarily Hermitian) linear operator $A : H \to H$ it is possible to talk about its trace $\mathbf{Tr}\,(A)$. More precisely, for any quadratic matrix $A = (a_{ij})$, the trace $\mathbf{Tr}(A)$ is defined as $\sum_i a_{ii}$, the sum of its diagonal elements. A remarkable property of the trace is its 'commutativity': $\mathbf{Tr}(AB) = \mathbf{Tr}(BA)$ for any quadratic matrix A and B. This in particular implies that the trace of an operator is independent of the choice of basis, thereby allowing for an unambiguous definition of the trace of a linear operator.

For example, $\mathbf{Tr}(E) = \dim H$. More generally, if P is an (orthogonal) projector (on subspace $V = ImP$) then $\mathbf{Tr}(P) = \dim V$. Due to the spectral theorem, we obtain that the trace of Hermitian operator $A = \sum a_i P_i$ is equal to $\sum a_i \mathrm{rk}(P_i)$ and, in particular, it is a real number. The trace of nonnegative operator A is nonnegative and is strictly positive if $A \neq 0$.

For two Hermitian operators A and B define $(A, B)_{Herm} = \mathbf{Tr}(AB)$. We assert that this 'scalar product' gives a structure of Euclidean space on the real vector space $\mathbf{Herm}(H)$. This follows from the Lemma below.

Lemma. a) $(A, B)_{Herm}$ *is a real number*;

b) $(A, B)_{Herm} = (B, A)_{Herm}$;

c) $(A, A)_{Herm} \geq 0$ *and is equal to* 0 *only if* $A = 0$.

Proof. Due to the 'commutativity' of the trace, $2\mathbf{Tr}(AB) = \mathbf{Tr}(AB) + \mathbf{Tr}(BA) = \mathbf{Tr}(AB + BA)$. It is easy to understand that the operator $AB + BA$ is Hermitian, hence its

trace is real. This proves a).

b) follows from the 'commutativity' of the trace.

c) follows from the nonnegativity of the operator A^2. $\square$

Appendix 2. Proof of Theorem 1

Let $\preceq$ be a nice preference relation on the set $\mathbf{QL}(H)$. We shall be working with roulette-valued Q-lotteries, that is with expressions of the form $\sum_i l_i \otimes P_i$, where $(P_i, \ i \in I)$ is a PDU, and $(l_i, \ i \in I)$ is a collection of roulette lotteries on X. Let $\mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ denote the set of all roulette valued Q-lotteries with base $\mathcal{P}$. We first provide a result that shows that mixtures of canonical Q-lotteries in $\mathbf{QL}(H)$ and mixtures of Q-lotteries in $\mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ are compatible in a sense we make clear below.

Consider a lottery in the canonical form $\sigma = \sum_x x \otimes P_x$, where $x \in X$, P_x are nonnegative Hermitian operators which add up to E, $\sum_x P_x = E$. The mixture $\alpha\sigma + (1 - \alpha)\tau$, where $\tau = \sum_x x \otimes Q_x$ and $\alpha \in [0, 1]$, is given as $\sum_x x \otimes (\alpha P_x + (1 - \alpha) Q_x)$. On the other hand a roulette valued Q-lottery writes $\sigma = \sum_i l_i \otimes P_i$, where $\mathcal{P} = (P_i, i \in I)$ is a measurement device (the base of the lottery) and $l_i \in \Delta(X)$. A mixture of such lotteries is defined by the following formula: $\alpha\sigma + (1 - \alpha)\tau = \sum_i (\alpha l_i + (1 - \alpha) r_i) P_i$, where $\tau = \sum_i r_i \otimes P_i$ is another Q-lottery in $\mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ and $\alpha \in [0, 1]$.

We next define the canonization mapping $can : \mathbf{QL}_{\mathcal{P}} \to \mathbf{QL}_c$, which maps lottery $\sum_i l_i \otimes P_i$ (where the roulette lottery l_i has the form $\sum_x x \otimes l_i(x)$ i.e., l_i gives value x with probability $l_i(x)$) into the canonical lottery $\sum_x x \otimes (\sum_i l_i(x) P_i)$.

Lemma 2. *The mapping 'can' preserves the operation of mixture; that is, for any* $\sigma, \tau \in \mathbf{QL}_{\mathcal{P}}(H, \Delta(X))$ *and* $\alpha \in [0, 1]$, $can(\alpha\sigma + (1 - \alpha)\tau) = \alpha can(\sigma) + (1 - \alpha) can(\tau)$.

Proof. Assume we have two $\mathcal{P}$−based lotteries $\sigma = \sum_i l_i \otimes P_i$ and $\tau = \sum_i r_i \otimes P_i$ and some $\alpha \in [0, 1]$. We want to show that $can(\alpha\sigma + (1 - \alpha)\tau) = \alpha can(\sigma) + (1 - \alpha) can(\tau)$.

The left hand side is equal to $can\left(\sum_i (\alpha l_i + (1 - \alpha) r_i) \otimes P_i\right) =$

$\sum_x x \otimes \sum_i (\alpha l_i + (1 - \alpha) r_i)(x) P_i = \sum_x x \otimes \sum_i (\alpha l_i(x) + (1 - \alpha) r_i(x)) P_i =$

$= \sum_x x \otimes \sum_i (\alpha \sum_i l_i(x) + (1 - \alpha) \sum_i r_i(x)) P_i$.

The right hand side is $\alpha \sum_x x \otimes (\sum_i l_i(x) P_i) + (1 - \alpha)(\sum_x x \otimes \sum_i r_i(x) P_i)$

$= \alpha can(\sigma) + (1 - \alpha) can(\tau)$. $\square$

Returning to the proof of Theorem 1, let $\preceq_\Delta$ denote the derived preference relation on the set $\Delta(X)$ of roulette lotteries. The assertion of the theorem is true if the preference $\preceq$ is trivial. Indeed, we can take u to be a constant and take an arbitrary functional β. So, from now on, we assume that the preference $\preceq$ is nontrivial. That is $\tau \prec \sigma$ for some Q-lotteries $\sigma = \sum_i l_i \otimes P_i$ and $\tau = \sum_j m_j \otimes Q_j$.

Claim 0. *Let l^* be the best lottery among (l_i) and m^* be the worst among (m_j) with respect to the derived weak order $\preceq_\Delta$. Then $\sigma \preceq l^* \otimes E$ and $m_* \otimes E \preceq \tau$.*

Proof. Consider the 'constant' lottery $\sigma^* = \sum_i l^* \otimes P_i$. Due to A4, $\sigma \preceq \sigma^*$, due to A0, $\sigma^* \approx l^* \otimes E$ and due to the transitivity of $\preceq$ (see A1), we conclude that $\sigma \preceq l^* \otimes E$. The proof $m_* \otimes E \preceq \tau$ is analoguous. $\square$

From Claim 0 it follows that the derived preference $\preceq_\Delta$ is non-trivial, that is $l_* \prec_\Delta l^*$ for some ordinary lotteries l_* and l^*. We fix such lotteries $l_* \prec_\Delta l^*$; a function $u : \Delta(X) \to \mathbb{R}$ is said to be *normalized* if $u(l_*) = 0$ and $u(l^*) = 1$.

Fix now some measurement device $\mathcal{P} = (P_i, \ i \in I)$, and let $\mathbf{QL}_\mathcal{P}(H, \Delta(X))$ denote the set of all roulette valued Q-lotteries with base $\mathcal{P}$. We first extend $\preceq$ into a preference relation $\preceq_\mathcal{P}$ defined on $\mathbf{QL}_\mathcal{P}(H, \Delta(X))$ by setting $\sigma \preceq \tau$ if and only if $can(\sigma) \preceq can(\tau)$ for any $\sigma, \tau \in \mathbf{QL}_\mathcal{P}(H, \Delta(X))$.

Claim 1. *There exists a normalized affine function $u_\mathcal{P}$ on $\Delta(X)$ and a function $\beta_\mathcal{P}$ on the set of outcomes I ($\beta_\mathcal{P}(i) \geq 0$ and $\sum_i \beta_\mathcal{P}(i) = 1$) such that the preference $\preceq_\mathcal{P}$ is represented by the function $U_\mathcal{P}$, $U_\mathcal{P}(\sum_i l_i \otimes P_i) = \sum_i u_\mathcal{P}(l_i)\beta_\mathcal{P}(i)$. Moreover, both $u_\mathcal{P}$ and $\beta_\mathcal{P}$ are unique.*

Proof. Each Q-lottery $\sigma = \sum_i l_i \otimes P_i$ can be considered as a 'horse' lottery $f : I \to \Delta(X)$, where $f(i) = l_i$. Moreover, due to axioms A1-A4 and Lemma 2, the relation $\preceq_\mathcal{P}$ satisfies all the Anscombe-Aumann axioms. Therefore, by theorem 13.2 in Fishburn (1970), we obtain an affine utility function $u_\mathcal{P}$ on $\Delta(X)$ and a probability measure $\beta_\mathcal{P} \in \Delta(I)$ that achieve the representation stated in Claim 1. The uniqueness of $\beta_\mathcal{P}$ is also given by this theorem. The uniqueness of $u_\mathcal{P}$ follows from normalization of $u_\mathcal{P}$. $\square$

Claim 2. *The functions $u_\mathcal{P}$ are independent of $\mathcal{P}$ (and we denote them as u).*

Proof. Due to A0, each of the functions $u_\mathcal{P}$ represents the derived preference $\preceq_\Delta$ on

$\Delta(X)$. Therefore they are positive affine transformations of each other. Normalization gives that they are in fact equal to each other. $\square$

Claim 3. *For any Q-lottery σ, there exists an ordinary lottery $l \in \Delta(X)$ such that σ is equivalent to l, that is $\sigma \approx l \otimes E$.*

Proof. Let $\sigma = \sum_i l_i \otimes P_i$, and let l^b (l^w) be a best (a worst) lotteries among $(l_i, i \in I)$. Due to A4, we have $\sum_i l^w \otimes P_i \preceq \sigma \preceq \sum_i l^b \otimes P_i$, and all these Q-lotteries have the same base $\mathcal{P} = (P_i, i \in I)$. Therefore we can apply Claim 1, which gives inequalities

$$u(l^w) \leq \sum_i u(l_i)\beta_{\mathcal{P}}(i) \leq u(l^b).$$

Hence $\sum_i u(l_i)\beta_{\mathcal{P}}(i) = \alpha u(l^w) + (1-\alpha)u(l^b) = u(l)$ for some $\alpha \in [0,1]$, where $l = \alpha l^w + (1-\alpha)l^b$. By Claim 1, we have $\sigma \approx \sum_i l \otimes P_i \approx l \otimes E$. $\square$

Due to Claim 1, the function $U_{\mathcal{P}}$ allows comparing Q-lotteries with base $\mathcal{P}$. But we assert that it allows comparing Q-lotteries with different bases as well.

Claim 4. *Let $\sigma = \sum_i l_i \otimes P_i$ be a Q-lottery with a base $\mathcal{P} = (P_i, i \in I)$, and let $\tau = \sum_j m_j \otimes Q_j$ be a Q-lottery with a base $\mathcal{Q} = (Q_j, j \in J)$. Then $\sigma \preceq \tau$ if and only if $U_{\mathcal{P}}(\sigma) \leq U_{\mathcal{Q}}(\tau)$.*

Proof. Due to Claim 3, lottery σ is equivalent to some lottery $l \otimes E$, or to the lottery $\sum_i l \otimes P_i$. Therefore, $U_{\mathcal{P}}(\sigma) = U_{\mathcal{P}}(\sum_i l \otimes P_i) = u(l)$. Similarly, τ is equivalent to $m \otimes E$, and $U_{\mathcal{Q}}(\tau) = u(m)$. Now

$$\sigma \preceq \tau \Leftrightarrow l \otimes E \preceq m \otimes E \Leftrightarrow U_{\mathcal{P}}(\sigma) = u(l) \leq u(m) = U_{\mathcal{Q}}(\tau). \quad \square$$

Let us return now to the functions $\beta_{\mathcal{P}}$. We assert that $\beta_{\mathcal{P}}(i)$ depends only on the operator P_i, not of $\mathcal{P}$ and i.

Claim 5. *Let $\mathcal{P} = (P_1, ..., P_n)$ and $\mathcal{Q} = (Q_1, ...Q_k)$ be two measurement devices (bases), and $P_1 = Q_1 = R$. Then $\beta_{\mathcal{P}}(1) = \beta_{\mathcal{Q}}(1)$.*

Proof. Consider Q-lottery $\sigma = l^* \otimes P_1 + \sum_{i=2}^n l_* \otimes P_i$ with base $\mathcal{P}$. Its $\mathcal{P}$-utility $U_{\mathcal{P}}(\sigma)$ is equal to $\beta_{\mathcal{P}}(1)$. Now let us form the auxiliary base $\mathcal{R} = (R, E - R)$ and the following Q-lottery $\rho = l^* \otimes R + l_*(E - R)$. Since $E - R = P_2 + ... + P_n$, the lottery ρ is equivalent

to σ (see Axiom A0). Therefore $\mathcal{R}$-utility $U_\mathcal{R}(\rho)$ (which equals $\beta_\mathcal{R}(1)$) is, by Claim 4, equal to $\beta_\mathcal{P}(1)$. The same applies to $\mathcal{Q}$ and gives the equality $\beta_\mathcal{R}(1) = \beta_\mathcal{Q}(1)$. Together with the equality $\beta_\mathcal{R}(1) = \beta_\mathcal{P}(1)$ we obtain the equality $\beta_\mathcal{P}(1) = \beta_\mathcal{Q}(1)$. $\square$

As a consequence, we can speak about the number $\beta(P)$ for any 'event' P, that is for any Hermitian operator P, $0 \le P \le E$. $\beta(P)$ is $\mathcal{R}$-utility $U_\mathcal{R}$ of the following Q-lottery $l^* \otimes P + l_* \otimes (E - P)$. Correspondingly, the utility of an arbitrary Q-lottery $\sigma = \sum_i l_i \otimes P_i$ can be rewritten as

$$U(\sigma) = \sum_i u(l_i)\beta(P_i).$$

Obviously, $\beta(0) = 0$ and $\beta(E) = 1$. Moreover, $\beta(P) \ge 0$ for any 'event' P, and $\sum_i \beta(P_i) = 1$ provided $\sum_i P_i = E$.

Claim 6. *If $0 \le P$, $0 \le Q$, and $P + Q \le E$, then $\beta(P + Q) = \beta(P) + \beta(Q)$.*

Proof. Indeed, consider the Q-lottery $l^* \otimes P + l^* \otimes Q + l_* \otimes (E - P - Q)$. Its utility is $\beta(P) + \beta(Q)$. On the other hand, due to A0, this lottery is equivalent to the lottery $l^* \otimes (P + Q) + l_* \otimes (E - P - Q)$, whose utility is $\beta(P + Q)$. $\square$

Claim 6 implies that β can be extended to a (unique) linear functional β on the vector space $\texttt{Herm}(H)$. Obviously, $\beta(A) \ge 0$ for $A \ge 0$, and $\beta(E) = 1$. That is β is a belief functional. This completes the proof of Theorem 1.

Appendix 3. Proof of Theorem 2

Proof of Assertion a). Here we can work with lotteries in the form of Hermitian operators. The utility $U(A)$ of such an operator A is equal to $\mathbf{Tr}(AB)$ and the utility under the condition P is equal to $U(A|P) = \mathbf{Tr}(APBP)$ (up to the factor $\mathbf{Tr}(PBP)$).

Proof of property **A5**. We assume that $A|W = 0$ and we have to show that $U(A|P) = 0$. Note that $A|P = 0$ is equivalent to $PAP = 0$. Now $U(A|P) = \mathbf{Tr}(APBP) = \mathbf{Tr}(PAPB) = \mathbf{Tr}(0B) = 0$.

Proof of property **A6**. Here we assume that A commute with P and that $(E - P)A = 0$ (that is $A = PA = AP$). We have to show that $U(A|P)$ is equal to $U(A)$. But $U(A|P) = \mathbf{Tr}(APBP) = \mathbf{Tr}(PAPB) = \mathbf{Tr}(AB) = U(A)$, because $PAP = A$.

Proof of Assertion b). We define (for an arbitrary Q-lottery $\sigma = \sum_i x_i \otimes P_i$ and an event W given by a projector P) some special (*adapted to P*) Q-lottery that we denote $ad_W(\sigma)$. Roughly speaking, $ad_W(\sigma) = \sum_i x_i \otimes PP_iP$. However, the sum $\sum_i PP_iP$ is equal to P, not to E. Therefore we add to this sum a tail-end $x_* \otimes (E - P)$. Here x_* is a prize with zero utility. The final formula yields

$$ad_W(\sigma) = \sum_i x_i \otimes PP_iP + x_* \otimes (E - P).$$

Lemma 3. a) $\sigma|W = ad_W(\sigma)|W$;

b) *the adapted lottery $ad_W(\sigma)$ is compatible with P;*

c) *for any Q-lottery σ we have $ad_W(\sigma)|W^\perp = x_* \otimes (E - P)$.*

Proof. a) It is obvious that $P_i|W = PP_iP|W$. Moreover, $(E - P)|W = P(E - P) = 0$.

b) For any i we have $PPP_iP = PP_iPP$, since $PP = P$. Moreover, $P(E - P) = (E - P)P = 0$.

c) It is clear that $(E - P)PP_iP = 0$ for any i. $\square$

Proposition 4. *Suppose that a preference relation $\preceq_W$ on the set $\mathbf{QL}(H)$ is a weak order and satisfies the axioms A5 and A6. Then it is given by the following explicit formula (where σ and τ are Q-lotteries on H):*

$$\sigma \preceq_W \tau \text{ if and only if } ad_W(\sigma) \preceq ad_W(\tau). \tag{1}$$

Proof. Indeed, due to the axiom A5 and Lemma 3, we have $\sigma \approx_W ad_W(\sigma)$ and $\tau \approx_W ad_W(\tau)$. Applying axiom A6 to the lotteries $ad_W(\sigma)$ and $ad_W(\tau)$ (which is possible due to points 2) and 3) of Lemma 3), we obtain that $ad_W(\sigma) \preceq ad_W(\tau)$ if and only if $ad_W(\sigma) \preceq_W ad_W(\tau)$. The rest follows from the transitivity of $\preceq_W$. $\square$

We recall that the (unconditional) utility of lottery $ad_W(\sigma)$ is equal to $\mathbf{Tr}(Sh(ad_W(\sigma))B)$. If $A = Sh(\sigma)$ then $Sh(ad_W(\sigma)) = PAP + 0(E - P) = PAP$. Therefore $U(ad_W(\sigma)) = \mathbf{Tr}(PAPB) = \mathbf{Tr}(APBP) = \mathbf{Tr}(PBP)\mathbf{Tr}(AB^{up})$, which is (up to the factor $\mathbf{Tr}(PBP)$) the utility of σ with respect to 'updated' belief operator $B^{up} = PBP/\mathbf{Tr}(PBP)$. This completes the proof of Theorem 2.

Appendix 4. Proof of Proposition 1.

We here prove an assertion that generalizes Proposition 1 when we substitute projector P with an arbitrary 'fuzzy-projector' F, that is an operator $0 \leq F \leq E$ (see Remark 4 of Section 4). The posterior B^{up} is given as $\sqrt{F}B\sqrt{F}/\mathbf{Tr}(\sqrt{F}B\sqrt{F}) = \sqrt{F}B\sqrt{F}/\mathbf{Tr}(FB)$. We shall denote it as $B|F$.

Proposition 1′. *The distance between a prior B and the posterior $B|F$ is $O((1 - \mathbf{Tr}(FB))^{1/4})$.*

In other words, if $\varepsilon = 1 - \mathbf{Tr}(FB)$ then the distance between B and $B|F$ is $O(\varepsilon^{1/4})$.

Proof. Choose an orthonormal basis of H in which the operator F (as well as $\sqrt{F}$) has diagonal form

$$F = \texttt{diag}(f_1, ..., f_n).$$

In this basis operator B is represented by Hermitian matrix (b_{ij}), where i and j run over 1 to $n = \dim H$. The matrix $\sqrt{F}B\sqrt{F}$ has coefficients $\sqrt{f_i}b_{ij}\sqrt{f_j}$. The matrix $\sqrt{F}B\sqrt{F}$ differs of $B|F = \sqrt{F}B\sqrt{F}/Tr(FB)$ by less than $O(\varepsilon)$. Thus we need to compare B and $\sqrt{F}B\sqrt{F}$ and to show that the distance between them is $O(\varepsilon^{1/4})$. Or equivalently we need to show that (for any i and j) distance between b_{ij} and $\sqrt{f_i}b_{ij}\sqrt{f_j}$ is $O(\varepsilon^{1/4})$.

Let us divide the set of indices i from $\{1, ..., n\}$ into two groups. Say that an index i is *non-essential*, if $b_{ii} \leq \sqrt{\varepsilon}$, and is *essential* in the opposite case.

Lemma 4. *If i is non-essential then $|b_{ij}| \leq \varepsilon^{1/4}$.*

Proof. A sub-matrix of the matrix B, formed by the rows and columns from the set $\{i, j\}$, is Hermitian and therefore has non-negative determinant $b_{ii}b_{jj} - |b_{ij}|^2$. That is $\sqrt{\varepsilon} \geq b_{ii}b_{ij} \geq |b_{ij}|^2$. $\square$

Due to Lemma 4, if i or j is non-essential then the coefficient b_{ij} of the matrix B as well as the corresponding coefficient $b_{ij}\sqrt{f_i}\sqrt{f_j}$ of the matrix $\sqrt{F}B\sqrt{F}$ is $\leq \varepsilon^{1/4}$. Therefore the distance between them is $\leq 2\varepsilon^{1/4}$.

Hence we can suppose that the both indexes i and j are essential.

Lemma 5. *If an index i is essential then $f_i = 1 + O(\varepsilon^{1/2})$.*

Proof. We have $\mathbf{Tr}(FB) = 1 - \varepsilon$, that is $\sum_i f_i b_{ii} = 1 - \varepsilon$. Moreover the trace of B is equal to 1, that is $\sum_i b_{ii} = 1$. Substracting we obtain that $\sum_i b_{ii}(1 - f_i) = \varepsilon$, therefore each term of this sum is less or equal to ε, $b_{ii}(1 - f_i) \leq \varepsilon$. . In particular, if i is essential then $b_{ii} > \varepsilon^{1/2}$ and $1 - f_i < \varepsilon^{1/2}$, that is $f_i = 1 + O(\varepsilon^{1/2})$. $\square$

As a corollary we obtain (for an essential i) that $\sqrt{f_i} = 1 + O(\varepsilon^{1/2})$.

Let us return to evaluation of the distance between b_{ij} and $b_{ij}\sqrt{f_i}\sqrt{f_j}$ in the case when i and j are essential indexes. It is clear that $b_{ij} - b_{ij}\sqrt{f_i}\sqrt{f_j} = b_{ij}(1 - (1 + O(\varepsilon^{1/2}))(1 + O(\varepsilon^{1/2})))$. The module of this number is no more than $|b_{ij}|$ (which is ≤ 1) multiplied by $Q(\varepsilon^{1/2})$. Therefore the distance is no more than $\leq O(\varepsilon^{1/2}) \leq O(\varepsilon^{1/4})$ which proves Proposition 1'. $\square$