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Hyper-relations, choice functions, and orderings of opportunity sets

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Abstract We prove a coincidence of the class of multi-preference hyper-relations and the class of *decent* hyper-relations (DHR), that is the class of binary relations on opportunity sets satisfying monotonicity, no-dummy, stability with respect to contraction and extension, and the union property. We study subclasses of DHR. In order to pursue our analysis, we establish a canonical bijection between DHR and the class of no-dummy heritage choice functions. From this we obtain that the no-dummy heritage choice functions have multi-criteria rationalizations with reflexive binary relations. We also prove that the restriction of this bijection to two subclasses of DHR, namely the transitive decent hyper-relations, and the *ample* hyper-relations, is a bijection between these subclasses and the classes of closed no-dummy choice functions and no-dummy path-independent choice functions (Plott functions), respectively.

1 Introduction

The ranking of sets of alternatives in terms of the degrees of freedom of choice that they offer to an agent has been the subject of considerable (axiomatic) analysis in recent years. The problem faced by a Decision Maker (DM) is to value an opportunity set in terms of the *flexibility* that it provides to the agent, when a DM is uncertain about

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her own *future preferences* at the time it would have to make her choice. Therefore, the greater the flexibility embodied by the opportunity set, the better it is for the agent. On the other hand, a DM can choose a subset of an opportunity set, and from this subset the future choice will be made.

At a first glimpse, these are two *different* problems: the first conceptualizes the problem of ranking opportunity sets, while the second 'reveals' the freedom-ranking of the sets of alternatives indirectly; the first one is concerned with the comparison of pairs of (opportunity) sets (A, B) with no restrictions (on A and B), while the second one deals with pairs of sets $(A, f(A))$, such that $f(A) \subseteq A$.

One of the main aim of the present work is to show that these two tasks become *equivalent* under the assumption that a DM is endowed with a '*binary rationality*'.

Specifically, we suppose a DM has a 'binary rationality' on the set of alternatives X whenever a DM can compare any pair of alternatives $x, y \in X$ by considering either one alternative preferred to another (denoted by $x < y$ or $y < x$) or both as equivalent (i.e. $x < y$ and $y < x$), or (further) as incomparable (namely $y \nless x$ and $x \nless y$), where $<$ denotes an *irreflexive* binary relation on the elements of X. We do not assume *transitivity* of $<$. (For example, a DM of the form of a committee of experts which compare projects based on the majority rule might produce a non-transitive preference on the set of projects.) We call an irreflexive binary relation $<$ on X a *preference*, while we use the term *hyper-relation* for indicating a binary relation $\preceq$ defined on the set of all subsets of X.

A DM endowed with $<$ can naturally compare the opportunity sets according to the following rule: for opportunity sets $A, B \in 2^X$, $A \, hyp(<) \, B$ if for any element $a \in A$, either $a \in B$, or there exists an element $b \in B$ such that $a < b$. We say that $hyp(<)$ is the hyper-relation generated by $<$.

The opportunity-sets ordering $hyp(<)$ satisfies the following properties: (i) every opportunity set is at least as good as every subset of itself (Monotonicity with respect to set-inclusion); (ii) a set providing more freedom of choice than another will provide definitely more than just a subset of the latter (Stability with respect to contraction); (iii) an opportunity set that contains another opportunity set that provides more freedom of choice than a third set will offer certainly more opportunities to choose than the latter (Stability with respect to extension); (iv) there are not (absolutely) undesirable opportunities (No-dummy), and (v) if an opportunity set A is at least as good as each of B and C, then A is at least as good as the union of B and C (Union) (see Sect. 2 for the formal definitions). Now, we take these axioms as a basis and call a hyper-relation $\preceq$ satisfying conditions $(i) - (v)$ *decent*. The class of decent hyper-relations and its subclasses will be the subject of an extensive analysis in the present work.

On the other hand, a DM, being equipped with the preference $<$, can choose, from each opportunity set A, the subset of non-dominated opportunities, namely:

$$Max(<)(A) := \{a \in A : \nexists\, b \in A \setminus a \text{ such that } a < b\}.$$

The choice function $Max(<)$ is called *rationalized* by $<$, and it satisfies No-dummy and Heritage properties: (1) the choice of an alternative is the alternative itself (No-dummy); and (2) if an alternative a is chosen from an opportunity set A, and a is also an element of a subset B of A, then a must be chosen from B, namely eliminating some of the not-chosen alternatives shouldn't affect the selection of a as the best option (Heritage property).

This simple example highlights the existence of a connection between the decent hyper-relations and the no-dummy heritage choice functions. It turns out to be a rule.[1]

Before formulating this rule as a precise statement, let us consider a DM with state-dependent preferences. Specifically, the pairwise comparisons of opportunities can depend on the 'states of nature'. That is, there is a set S of states of nature (a finite set for a finite set X), such that for each $s \in S$ and any $x, y \in X$, the comparison $x <_s y$ depends on s. Since a DM has a collection of preferences $<_s$ depending on $s \in S$, a hyper-relation generated by such a collection of preferences is called a *multi-preference hyper-relation*. In such a case, a DM ranks B at least as good as A, written $A\, hyp(<_s, s \in S)\, B$, if for any $a \in A$, either $a \in B$ or, for each $s \in S$, there exists an element $b(s)$ in B such that $a <_s b(s)$.

A multi-preference hyper-relation satisfies the same properties reviewed above.

The corresponding analogue in choice theory is a *multi-criteria* choice function. Namely, for a collection of criteria-preferences $<_s$, depending on $s \in S$, the choice set of a DM from a menu A is the union of non-dominated opportunities within A with respect to each of the criteria. Such a choice function still satisfies the heritage property and is no-dummy.

Thus, a DM endowed with a collection of preferences $<_s$, $s \in S$, can order opportunity sets by considering the intersection of hyper-relations generated by the $<_s$, with $s \in S$, or a DM can define a choice function that is the union of the choice functions *rationalized* by $<_s$, $s \in S$

In the present work, we will show that the two aforementioned approaches are isomorphic. Indeed, in what follows, we obtain a correspondence from the set of no-dummy heritage choice functions to the set of multi-preference hyper-relations. Such a correspondence is a mapping indeed, since it does not depend on a multi-criteria decomposition of a heritage choice function, and, moreover, is a bijection.

We prove that the class of multi-preference hyper-relations coincides with the class of decent hyper-relations (Theorem 1).

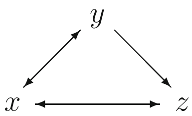
We illustrate Theorem 1 by the following example.

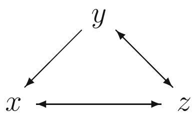
Example Let $X = \{x, y, z\}$ be the universal set of the alternatives. There are six experts which have the following orderings on the alternatives (1) $x > y > z$; (2) $z > x > y$; (3) $y > z > x$; (4) $y > x > z$; (5) $x > z > y$; and (6) $z > y > x$.

There are two states of natures, summer (S) and winter (W). There are two committee of experts, the summer committee constituted from experts (1)–(4), and the winter committee constituted from experts (3)–(6). The preferences of the summer and winter committees are defined by the majority rule of the experts of the corresponding committees. These preferences are depicted below[2].

[1] It is worth noticing here that the idea to connect hyper-relations and choice functions goes back to Puppe (1996).

[2] Here an arrow $x \to z$ means $x > z$ and so on.

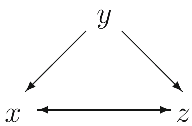




Specifically, the majority rule of the summer committee $<_S$ is presented on the left hand side picture and that of the winter committee, $<_W$, is on the right hand side. These preferences are not transitive.

In order to define a decent hyper-relation, one has to set 3×8 ($n \times 2^n$, n is the number of alternatives) relations of the form $a \preceq A$, $a \in X$, $A \subseteq X$ (see Lemma 1). However, in general a smaller set of relations specifies a decent hyper-relation[3]. One can check that a decent hyper-relation $\preceq$ is specified by $x \preceq y$, $y \not\preceq x$, $z \preceq y$, $y \not\preceq z$, $z \preceq x$, $x \preceq z$, and $y \preceq \{x, z\}$ is the intersection of $hyp(<_S)$ and $hyp(<_W)$. This hyper-relation is not transitive. Indeed, as a consequence of the Union property $\{x, z\} \sim x$ (as well $\{x, z\} \sim z$). Then, we have $y \sim \{x, z\} \sim x$ whereas $x \prec y$, hence $\preceq$ is non-transitive. Since $x \prec y$ and $z \prec y$ whereas $y \preceq \{x, z\}$, we can not find a single preference which generates $\preceq$.

The restriction of $\preceq$ to X is a transitive preference and coincides with the preference being the majority rule of experts (3) and (4):



The hyper-relation generated by this preference is transitive and differs from $\preceq$.

A subclass of transitive decent hyper-relations has been considered by Kreps (1979) in an intermediate result that he uses to characterize the complete transitive hyper-relations which satisfy Monotonicity and the following axiom of independence: if $A \sim A \cup B$, then for any C we have $A \cup C \sim A \cup C \cup B$. Moreover, the transitive decent hyper-relations have been studied in different set-ups [(see, for example, Armstrong (1974); Caspard and Monjardet (2003); Danilov and Koshevoy (2006); Domenach and Leclerc (2004)].

We show that the restriction of the bijection between the decent hyper-relations and the no-dummy heritage choice functions to this subclass is a bijection between the transitive decent hyper-relations and the class of choice functions called *closed* (see below for a formal definition). We prove that a multi-preference representation of a transitive decent hyper-relation can be chosen such that all state preferences are weak orders.

We will consider one more subclass of the decent hyper-relations, a class of hyper-relations that we call *ample* and such that, the Lattice equivalence property holds true, that is, for any collection A_i, $i \in I$, of equivalent menus, it holds that $\cup_i A_i \preceq \cap_i A_i$. We establish that the bijection restricted to this subclass is a bijection between the ample

[3] Here, the term 'specification' means that the list of comparisons $a \preceq A$ determines a decent hyper-relation.

hyper-relations and the Plott choice functions (or path-independent choice functions). For an ample hyper-relation, a multi-preference representation can be chosen such that all state preferences are linear orders.

The paper is organized as follows. In Sect. 2 we introduce the class of decent hyper-relations and provide a multi-preference model for them. A bijection between the decent hyper-relations and the no-dummy choice functions is established in Sect. 3. In Sect. 4 we consider a case of non-empty valued heritage choice functions. In Sect. 5, we study the class of transitive decent hyper-relations and their relations to the class of closed choice functions. In Sect. 6 we introduce the subclass of ample decent hyper-relations and provide a bijection between this subclass and the class of Plott choice functions.

2 Hyper-relations and the multi-preference model

Let X be a universal set of alternatives,[4] and 2^X be the set of all subsets of X. The sets $A, B, C, \ldots$, are the elements of 2^X to be interpreted as *opportunity sets* or *menus* from which a DM chooses. A *preference*, denoted by $<$, is an *irreflexive* binary relation on X. We call *hyper-relation*, denoted by $\preceq$, a binary relation on 2^X, and we identify its asymmetric part with $\prec$. **HR** is the set of all hyper-relations on 2^X.

Now, let $<$ be a preference of a DM on X. Then it is naturally to assume that a DM defines $a \preceq B$ if either $a \in B$ or $a < b$ for some $b \in B$. We say that a hyper-relation $\preceq$ is *generated by* $<$, $\preceq = hyp(<)$, if, for any $A, B \in 2^X$, $A\, hyp(<)\, B$ means that, for any $a \in A$, $a \preceq B$. The intended interpretation of $hyp(<)$ is that $A\, hyp(<)\, B$ if B entails at least as much opportunities as A.

Let us consider three examples of preferences and the corresponding hyper-relations.

Example 1 Let $<$ be the 'empty' preference, that is there are no elements $x \neq y$ of X such that $x < y$ or $y < x$ holds true. Then $hyp(<)$ is the set inclusion, $A\, hyp(<)\, B$ if and only if $A \subseteq B$.

Example 2 Let $<$ be the 'total' preference, for any $x \neq y \in X$, $x < y$. Then, for any non-empty A and $B \subseteq X$, $A\, hyp(<)\, B$.

Example 3 Pick up a pair (a, B), $a \in X$, $B \subseteq X$, $a \notin B$. For such a pair, we define the following preference $<_{(a,B)}$: for any different x and y such that $x \neq a$ and $y \notin B$, $x < y$ holds. (For $B = \emptyset$, we get the 'total' preference.) Then, for any non-empty menus A and C, $A\, hyp(<_{(a,B)})\, C$ does not hold if and only if $a \in A$ and $C \subseteq B$.

One can see that a hyper-relation $hyp(<)$ satisfies the following list of axioms. First, there are not (absolutely) undesirable alternatives, namely:

- **No Dummy (ND)**: For all $a \in X$, $\emptyset \prec a$.

The next property is

[4] The reader can assume for simplicity that X is a finite set, although the finiteness of X is almost nowhere used.

- **Monotonicity with respect to set inclusion (Mon)**: For all $A, B \in 2^X$, $A \subseteq B$ implies $A \preceq B$.

This assumption that every opportunity set is at least as good as every one of its subsets is one of the less controversial requirement in the freedom of choice literature.[5]

Another two axioms are

- **Stability with respect to contraction (Cont)**: For any $A, A', B \in 2^X$, $A' \subseteq A \preceq B$ implies $A' \preceq B$.
- **Stability with respect to extension (Ext)**: For any $A, B, B' \in 2^X$, $A \preceq B \subseteq B'$ implies $A \preceq B'$.

These axioms could be considered to weaken transitivity, i.e. if a transitive hyper-relation $\preceq$ satisfies **Mon**, then it is stable with respect to contraction (**Cont**) and to extension (**Ext**). These axioms reflect the following relationships between a hyper-relation and the set-inclusion: if a set provides more freedom of choice than another, then this *a fortiori* holds for any subset of the latter, and if a set offers more 'suitable alternatives' than another set, then the set containing the former as its subset will certainly provide more 'suitable alternatives' than the latter.

Finally,

- **Union (U):** Let $A_i \preceq B$ for some (possibly infinite) family of menus A_i $(i \in I)$ and $B \subseteq X$. Then $\cup_{i \in I} A_i \preceq B$.

Axiom **U** simply says that the union of worse (opportunity) sets is still the worst (opportunity) set. Moreover, for the case in which X is finite, it specializes to the so-called *Robustness* axiom [see Barberà et al. (2004)], saying that for all $A, A', B \in 2^X$, if $[A \preceq B$ and $A' \preceq B]$ then $A \cup A' \preceq B$.

Remark 1 Several scholars have used the Union axiom. In the present work, we systematically apply this property and indeed all our results heavily rely on it.

Kreps (1979, axiom (2.1), Ryan (2014, axiom **K**), and Lahiri (2003, Concordance axiom), consider the following condition:

- **K** For any $A, B \in 2^X$, $A \preceq B$ implies $A \cup B \preceq B$.

which, for the case of transitive monotonic hyper-relations, is equivalent to the axiom **U** (defined on finite families). Indeed, let $A \preceq C$ and $B \preceq C$. By **K**, $A \cup C \preceq C$ and $B \cup C \preceq C$. By **Mon**, $C \preceq B \cup C$, and by transitivity we have $A \cup C \preceq B \cup C$. Again, by **K** we have $A \cup B \cup C \preceq B \cup C \preceq C$. Using $A \cup B \subseteq A \cup B \cup C$ and transitivity, we obtain $A \cup B \preceq C$.

Example 4 For a finite set X, Pattanaik and Xu (1990) defined the cardinality-based ordering $\preceq_c$ of opportunity sets, as follows $A \preceq_c B$ if $|A| \leq |B|$. This hyper-relation is transitive and complete, satisfies **Mon**, but does not satisfy **U** and therefore it is not semi-decent. Moreover, it is worth noticing here that such a hyper-relation belongs to the class of complete hyper-relations characterized by Kreps (1979).

[5] This requirement is discussed in the works of several scholars [see, for example, Barberà et al. (2004)]. Kreps attributes this axiom to Koopmans [axiom (1.3) in Kreps (1979)].

A hyper-relation generated by a preference satisfies these axioms, and they are convenient for ordering opportunity sets. We take this list of axioms and introduce the class of hyper-relations that is the main object of the present work:

Definition 1 A hyper-relation $\preceq$ is *decent* if it satisfies **ND**, **Mon**, **Cont**, **Ext**, and **U**. (A hyper-relation $\preceq$ is *semi-decent* if it satisfies only **ND**, **Mon**, **Cont**, and **U**).

We denote by **DHR** and **SDHR** the class of decent and semi-decent hyper-relations respectively. An immediate consequence of the definitions is the following.

Lemma 1 *For a semi-decent hyper-relation $\preceq$ and for any $A, B \in 2^X$, $A \preceq B$ if and only if $a \preceq B$ for every $a \in A$.*

Thus, in order to define a semi-decent hyper-relation $\preceq$ it is sufficient to set a relation $a \preceq B$ between X and 2^X (i.e. such that $a \preceq B$ whenever $a \in A$).

We wonder if there is something more in the class of decent hyper-relations except those generated by preferences.

We use the 'operation of intersection' in order to define more general hyper-relations. In other words, if we denote by $<_s$ the state (or context) dependent preferences, where $s \in S$ and S is the set of all (future, possible) states of nature, then the intersection of hyper-relations generated by those preferences leads us to the construction of a so-called multi-preference hyper-relation. Thus, we state:

Definition 2 For a collection $\{<_s,\ s \in S\}$ of preferences, the hyper-relation $\preceq$ being the intersection of the hyper-relations $hyp(<_s)$, $s \in S$, that is $A \preceq B$ if and only if, for every $s \in S$, there holds $A\ hyp(<_s)\ B$, is called a *multi-preference hyper-relation*, and is denoted by $hyp(\{<_s, s \in S\})$.

Lemma 2 *If all hyper-relations $\preceq_i$, $i \in I$, are semi-decent (respectively: decent, transitive), then their intersection is also semi-decent (respectively: decent, transitive).*

From this lemma follows that any multi-preference hyper-relation belongs to the class of decent hyper-relations.

The example in Introduction shows a multi-preference hyper-relation with two states. Thus, the class of decent hyper-relations contains more hyper-relations than the class of hyper-relations generated by preferences.

We now state the main results of the this section:

Theorem 1 *The class of multi-preference hyper-relations coincides with the class of decent hyper-relations.*

This theorem means that every decent hyper-relation can be obtained as the intersection of hyper-relations generated by a family of state-dependent preferences. Thus, the class of multi-preference hyper-relations exhausts the class of decent hyper-relations.

Proof of Theorem 1 Let $\preceq$ be a decent hyper-relation. Consider the set $\mathcal{S}$ constituted from pairs (a, B) such that $a \in X$, $B \subseteq X$, and $a \npreceq B$. For each pair $(a, B) \in \mathcal{S}$, let us consider the preference $<_{(a,B)}$ (see Example 3) and the multi-preference hyper-relation $\preceq' := hyp(\{<_{(a,B)},\ (a, B) \in \mathcal{S}\})$. We claim that $\preceq$ coincides with $\preceq'$. According to

Lemma 1, it suffices to check that for any pairs (c, C), $c \in X$, $C \subseteq X$, and $c \notin C$, $c \not\preceq C$ and $c \not\preceq' C$ hold true simultaneously. Let $c \not\preceq C$, than $(c, C) \in \mathcal{S}$. Hence (see Example 3), for $\preceq'' := hyp(<_{(c,C)})$, there holds $c \not\preceq'' C$, that implies $c \not\preceq' C$.

Vice versa, let $c \not\preceq' C$, then, for some pair $(a, B) \in \mathcal{S}$, there holds $c \not\preceq'' C$, where $\preceq'' := hyp(<_{(a,B)})$. According to Example 3 this holds if and only if $a = c$ and $C \subseteq B$. Since $c = a \not\preceq B$, then, because of **Ext**, it holds that $c \not\preceq C$. □

For a hyper-relation $\preceq$, the *restriction* of $\preceq$ to X is a preference $< := res(\preceq)$ defined by $x < y$ if $x \preceq y$.

For a decent hyper-relation $\preceq$, we have the following inclusion $hyp(res(\preceq)) \subseteq \preceq$[6]. In fact, let $a\, hyp(res(\preceq))\, B$. Then either $a \in B$ and hence $a \preceq B$, or $a\, res(\preceq)\, b$ for some $b \in B$. The latter means that $a \preceq b$ and, hence by **Ext**, it holds that $a \preceq B$.

Corollary 1 *Let $\preceq$ be a decent hyper-relation and $<$ be the restriction of $\preceq$ to X. Then, $\preceq$ is the intersection of hyper-relations $hyp(\{<_s\})$, generated by some set $\{<_s, s \in S\}$, where, for each $s \in S$, $< \subseteq <_s$.*

Proof According to Theorem 1, $\preceq$ is a multi-preference hyper-relation. Let $\{<_s, s \in S\}$ be the corresponding collection of preferences, $\preceq = hyp(\{<_s, s \in S\})$. Let $<$ be the restriction of $\preceq$ to X. Then, obviously, $a < b$ implies $a <_s b$, $s \in S$. □

We also have the following characterization of the complete decent hyper-relations.

Corollary 2 *A decent hyper-relation $\preceq$ is complete if and only if $res(\preceq)$ is complete.*

Proof Obviously, for a complete $\preceq$, the preference $res(\preceq)$ is also complete. Vice versa, let $res(\preceq)$ be complete. Then, for any pair $A, B \subseteq X$, either, for any $a \in A$, there exists some $b(a) \in B$ such that $a\, res(\preceq)\, b(a)$ and, hence, $A \preceq B$, or there exists $a \in A$ such that, for any $b \in B$, it holds that $b\, res(\preceq)\, a$, and hence $B \preceq A$. □

For a complete transitive decent hyper-relation $\preceq$, it holds that $\preceq = hyp(res(\preceq))$. Without the transitivity assumption, the latter equality might be not true, see the example in Introduction.

In what follows we consider two subclasses of the decent hyper-relations, the subclass of transitive and the subclass of ample hyper-relations, and we study the corresponding multi-preference representations. In order to proceed, we establish a connection between the decent hyper-relations and the choice functions.

3 Choice functions and hyper-relations

For a preference $<$, one can define a contracting operator $Max(<)$ by the rule:

$$Max(<)(A) := \{a \in A \;\; \nexists b \in A - a, \text{ such that } a < b\}.$$

Recall, that a contraction operator $f : 2^X \to 2^X$ sends a set to some of its subsets, that is $f(A) \subseteq A$, $A \subseteq X$.

[6] Let $\preceq$ and $\preceq'$ be two binary relations, we set $\preceq \subseteq \preceq'$ if, for any $A, B \subseteq X$, such that $A \preceq B$, it holds that $A \preceq' B$.

Contracting operators are also known as *choice functions*.

We interpret the choice function $Max(<)$ as the choice of the maximal alternatives from each subset. Such a choice function satisfies the following two properties

ND A choice function $f : 2^X \to 2^X$ is *no-dummy* if, for every $x \in X$ $f(x) \neq \emptyset$, that is $f(x) = x$.

H A choice function $f : 2^X \to 2^X$ is *heritage* if, for $A \subseteq B$, there holds $f(B) \cap A \subseteq f(A)$.

Let us warn that, for a no-dummy choice function f, it might be $f(A) = \emptyset$ for a non-singleton set A.

For a collection of preferences $<_s$, $s \in S$, the choice function $Max(\{<_s, s \in S\})$ being the union $\cup_{s \in S} Max(<_s)$ is a *multi-criteria choice function*.

It is easy to check that any multi-criteria choice function satisfies the axioms **ND** and **H**.

One can observe a similarity between the multi-criteria choice functions and the multi-preference hyper-relations. It turns out that this is a reflection of a deep relation between the choice functions and the semi-decent hyper-relations. Such a relation provides a bijection between the no-dummy heritage choice functions and the decent hyper-relations. As a consequence, we obtain the coincidence of the class of no-dummy heritage choice functions and the class of multi-criteria choice functions.

Heritage condition, the axiom **H**, has been first introduced by Chernoff (postulate 4 in Chernoff (1954)) and it means that if an opportunity a is chosen from an opportunity set B then it will be also chosen from the smaller opportunity set ($A \subseteq B$) including a.[7]

Denote by **CF** the class of no-dummy choice functions. Let us define a mapping κ : **CF** $\to$ **SDHR** by the following rule. Let f be a choice function, then we define a hyper-relation $\kappa(f)$ as follows, for any $A, B \in 2^X$,

$$A\,\kappa(f)\,B \text{ if, for every } a \in A,\ f(a \cup B) \subseteq B. \tag{3.1}$$

It is easy to check that $\kappa(f)$ is a semi-decent hyper-relation.

Let us define a mapping Φ : **SDHR** $\to$ **CF** by the rule:

$$\Phi(\preceq)(A) = \{a \in A : a \not\preceq A \backslash a\}.$$

In words, an alternative a is not chosen from A if $a \preceq A \backslash a$, i.e. adding a to $A \backslash a$ does not increase the attractiveness of A.

The mapping Φ is anti-monotone and it holds that $\Phi(\preceq \cap \preceq') = \Phi(\preceq) \cup \Phi(\preceq')$.

Proposition 1

- *κ is anti-monotone.*
- *For any $f \in$ **CF**, there holds $\Phi(\kappa(f)) = f$.*
- *For any $f \in$ **CF**, the hyper-relation $\kappa(f)$ satisfies* **ND, Mon, Cont** *and* **U**.

[7] There are some other reformulation of this axiom. For instance, in the literature on stable matchings (Roth and Sotomayor, 1990, Definiton 6.2 p. 173), the Heritage axiom is the Substitutability property: if a worker a is hired by a firm from a list B she will be also hired from any shorter list $A \subseteq B$.

Proof Item 1 is obvious. Item 2. Let, for some $a \in A$, we have $a \notin \Phi(\kappa(f))(A)$. By the definition of the mapping Φ, we get $a\,\kappa(f)\,A-a$. That means that $f(a \cup (A-a)) = f(A)$ is contained in $A - a$, that is nothing but $a \notin f(A)$.

Item 3. We only have to check that the no-dummy property is properly translates by κ, other properties of $\kappa(f)$ are obvious. Suppose that $\kappa(f)$ has a dummy, that is, for some $x \in X$, it holds that $x\,\kappa(f)\,\emptyset$. By definition this means that $f(x \cup \emptyset) \subseteq \emptyset$. This contradicts to that f is no-dummy. ◻

As a consequence of Proposition 1, we obtain a bijection between **CF** and **SDHR**. We illustrate mappings κ and Φ by the following examples:

Example 5 For the choice function **1** (i.e. $\mathbf{1}(A) = A$ for every $A \in 2^X$), the corresponding hyper-relation $\kappa(\mathbf{1})$ is nothing but the set-theoretical inclusion $\subseteq$.

Example 6 Let $<$ be a preference, and let $f = Max(<)$ be the choice function *rationalized* by the preference $<$. It is easy to check that the hyper-relation $\kappa(Max(<))$ is nothing but the hyper-relation $hyp(<)$.

Proposition 2 *A choice function $f \in CF$ is heritage if and only if the hyper-relation $\kappa(f)$ satisfies the axiom* **Ext**.

Proof Let $f \in CF$ be heritage, $a\,\kappa(f)\,A$ and $A \subseteq B$. We have to check that $a\,\kappa(f)\,B$ holds true. In fact, $a\,\kappa(f)\,A$ means that $f(a \cup A) \subseteq A$. We have to check that $f(a \cup B) \subseteq B$. We may assume that $a \notin B$, otherwise the inclusion is obviously valid. On the contrary, suppose $f(a \cup B) \not\subseteq B$. Then, by **H**, $a \in f(a \cup B)$ implies $a \in f(a \cup A)$. Since $f(a \cup A) \subseteq A$, we have $a \in A$, that contradicts to $a \notin B$.

Vice versa, let $\kappa(f)$ satisfy **Ext** and the triple (a, A, B) be such that $A \subseteq B$ and $a \in f(B) \cap A$. Because of item 2 of Proposition 1, we have $f = \Phi(\kappa(f))$. The inclusion $a \in f(B)$ means that $a \not\preceq B - a$, where $\preceq := \kappa(f)$. Due to the axiom **Ext**, we have $a \not\preceq A - a$. The latter means $a \in f(A)$. ◻

A consequence of these Propositions and Theorem 1 is the following

Proposition 3 *The class of no-dummy heritage choice functions coincides with the class of multi-criteria choice functions.*

Remark 2 Puppe (1996) introduced the notion of *essential* elements. Namely, for a hyper-relation $\preceq$, an alternative $a \in A$ is *essential* in A if $A \backslash a \prec A$. In other words, removing such an alternative decreases the attractiveness of the opportunity set. We claim that, for a semi-decent hyper-relation $\preceq$, an alternative is essential in A if and only if it belongs to $\kappa^{-1}(\preceq)(A)$. This follows from:

Lemma 3 *For a semi-decent hyper-relation $\preceq$ the following two statements are equivalent:*

(1) $A \preceq A\backslash a$,
(2) $a \preceq A\backslash a$.

Proof Suppose that $A \preceq A\backslash a$, then, since $a \in A \preceq A\backslash a$, we obtain that $a \preceq A\backslash a$. Conversely, let $a \preceq A\backslash a$, then, since $A\backslash a \preceq A\backslash a$, we conclude by axiom **U** that $A = a \cup (A\backslash a) \preceq A\backslash a$. □

Remark 3 There are at least three other possible mappings from the set of choice functions to the set of hyper-relations:

(i) The first mapping is defined by associating a hyper-relation $\preceq'_f$ to a choice function f as follows:

$$A \preceq'_f B \Longleftrightarrow a \notin f(a \cup B) \quad \forall a \in f(A) - B.$$

that means that $a \notin f(a \cup B)$ holds only for the elements $a \in f(A)\backslash B$. Obviously, $A \preceq_f B$ implies $A \preceq'_f B$. This hyper-relation satisfies (only) the **Mon** axiom.

(ii) The second mapping was considered in Puppe (1996): for a choice function f, a hyper-relation $\preceq''_f$ is defined by the rule:

$$A \preceq''_f B \Longleftrightarrow f(A \cup B) \subseteq B.$$

This hyper-relation satisfies only **Mon**. We observe that if f is a heritage choice function, then $\preceq_f \subseteq \preceq''_f$, that is $A \preceq_f B \Rightarrow A \preceq''_f B$.[8]

(iii) The third mapping was considered in Puppe and Xu (2010) and Ryan (2014)

$$A \preceq^*_f B \Longleftrightarrow f(A \cup B) \cap B \neq \emptyset.$$

Such a hyper-relation satisfies only **Mon**.

Thus, we conclude that these three mappings do not have so many suitable properties as the mapping κ.

4 Non-emptiness

It is now well established that the preferences among the basic alternatives have a crucial role in capturing the desire for freedom of choice of a DM. Indeed, an available alternative can only contribute to the freedom of a decision maker if it is in some (weak) sense valuable to him/her. An undesirable option does not expand the personal freedom. In other words, not all available alternatives constitute an *essential* contribution to the freedom in a certain decision situation: for a menu A could exist an alternative $x \in A$ such that $A\backslash \{x\}$ is ranked not below A [see Puppe (1996)]. Those alternatives whose availability contributes to the agent's freedom -as said- are called *essential*.

[8] We provide a proof of the latter claim. Suppose that $A \not\preceq''_f B$, that is there exists $a \in f(A \cup B)$ such that $a \notin B$. Then, since f is a heritage choice function and $a \in A\backslash B$, we obtain that $a \in f(a \cup B)$ ($a \cup B \subseteq A \cup B$ and $a \in f(A \cup B)$). Hence, $A \not\preceq_f B$, since $A \preceq_f B$ implies $a \notin f(a \cup B)$).

Definition 3 A hyper-relation $\preceq$ is said to be *well* if every non-empty menu $A \in 2^X$ contains *essential* elements.

In terms of the corresponding choice function $\Phi(\preceq)$ this requirement means that $\Phi(\preceq)(A)$ is non-empty for any non-empty set $A \in 2^X$.

Let us characterize now the *well hyper-relations* (or non-empty-valued choice functions).

Definition 4 A preference $<$ is *well* if every increasing sequence $x_1 < x_2 < \ldots$ is finite.

Note that any well preference is irreflexive and acyclic. In the case of finite X, a preference $<$ is well if and only if it is acyclic. Then, we have:

Lemma 4 *The following two statements are equivalent:*

(1) *a preference* $<$ *is well;*
(2) *the hyper-relation* $hyp(<)$ *is well.*

Proof Suppose that $hyp(<)$ is not well. This means that some menu A has not a maximal element, that is, for any $a \in A$, there is a better element $a' \in A$ such that $a < a'$. Obviously, this leads to an infinite increasing chain.

Conversely, let $<$ be a non-well preference relation, and let $x_1 < x_2 < \ldots$ be an infinite increasing chain. It is obvious that the set $A = \{x_1, x_2, \ldots\}$ has not a maximal element. □

In terms of choice functions, we have:

Proposition 4 *Let f be a heritage choice function. The following three statements are equivalent:*

1. $f(A)$ *is non-empty for any non-empty menu* A*;*
2. *there exists a well preference* $<$ *such that* $Max(<) \subseteq f$*;*
3. *there exists a well linear order* $<$ *such that* $Max(<) \subseteq f$.

Let us note that applying this proposition to a well preference we obtain the following generalization of the Szpilrajn theorem: *any well preference can be extended to a well linear order.*

Proof Obviously 3) $\Rightarrow$ 2); 2) $\Rightarrow$ 1) by Lemma 4. It remains to prove that 1) $\Rightarrow$ 3). We (almost) explicitly construct such an order. To make the construction more clear, we consider from the beginning that X is finite. In this case, the desired linear order is constructed with the help the following simple peeling procedure.[9]

Since $f(X)$ is a non-empty set, we can take some element x_1 from it. Next, we take an element x_2 from the (non-empty) set $f(X \backslash x_1)$, and so on. In the case of a finite set

[9] The peeling order is a variant of the *peeling rank* in Statistics. Namely, for a finite set of points X in an Euclidean space, let us consider the following chain of sets $X_0 := X$, $X_1 := X_0 \setminus ex(X_0)$, where $ex(Y)$ denotes the set of points of Y which belong to the boundary of its convex hull, $X_2 := X_1 \setminus ex(X_1), \ldots$. Then, for a subset $A \subseteq X$ its rank r is defined as maximal number r such that $A \subseteq X_r$. The peeling rank is used in non-parametric rank tests [Eddy (1984)].

X, we exhaust the whole X after a finite number of steps and we obtain a linear order $x_1 > x_2 > \cdots > x_n$. This order is compatible with f. Indeed, let A be a (non-empty) menu, and let x_k be a maximal element in A. We need to show that $x_k \in f(A)$. By definition, $A \subseteq X \backslash \{x_1, \ldots, x_{k-1}\}$ and contains x_k. By the Heredity property of f, $x_k \in f(A)$.

In the case in which X is infinite we can proceed as follows. Let s be a selector of f, that is a mapping $s : 2^X \backslash \{\emptyset\} \to X$ such that $s(A) \in f(A)$ for every non-empty menu A. Such a selector exists by Zermelo's axiom of choice. Applying Bourbaki (1957)[III, §2, Lemma 3], we obtain a well linear order $<$ on X such that $s(X \backslash [> x]) = x$ for any $x \in X$. Here $[> x]$ denote the set $\{y \in X,\ y > x\}$. We claim that $Max(<) \subseteq f$.

Indeed, let A be a non-empty set and m be the (unique) maximal element of A such that $a \leq m$ for any $a \in A$. Then $A \subseteq X \backslash [> m]$, and $m = s(X - [> m]) \in f(X - [> m])$. Since f is a heritage choice function, we have that $m \in f(A)$. □

Remark 4 Since the choice in $f(A)$ is arbitrary, the peeling procedure is not unique. However, we can construct a canonical weak order by picking $f(X)$ at the first step, $f(X \backslash f(X))$ at the second one, and so on. For the case in which X is infinite, this construction can be obtained by using transfinite induction.

5 Transitive decent hyper-relations and closed choice functions

Let $\preceq$ be a semi-decent hyper-relation. Then we define an extensive operator[10] $\mu(\preceq) : 2^X \to 2^X$ associated to $\preceq$ as follows: for any $A \in 2^X$, $\mu(\preceq)(A) = \{x \in X : x \preceq A\}$. Obviously, $A \subseteq \mu(\preceq)(A) \preceq A$. We denote the set of extensive no-dummy operators by **EO**. Thus, we have a mapping μ : **SDHR** $\to$ **EO**.

Let us define the inverse mapping from **EO** to **SDHR**: for an extensive operator μ, we set $A \preceq(\mu) B$ if $A \subseteq \mu(B)$. Obviously, $\preceq(\mu)$ is a semi-decent hyper-relation and it is easy to check that these mappings are one the inverse of the other.

An extensive operator μ is called *monotone* if $A \subseteq B$ implies that $\mu(A) \subseteq \mu(B)$. The set of all monotone no-dummy extensive operators is denoted as **MEO**. The following easy lemma holds true.

Lemma 5 *The extensive operator $\mu(\preceq)$ is monotone if and only if $\preceq$ is a decent hyper-relation.*

Thus, we have a bijection between **MEO** and **DHR**.

Kreps (1979) has obtained a decomposition-type result based on a relationship between the transitive decent hyper-relations and the closure operators, namely operators that are extensive, monotone and idempotent (i.e. $\mu(\mu(A)) = \mu(A)$).[11] Let

10 A mapping $\mu : 2^X \to 2^X$ is *extensive* if $A \subseteq \mu(A)$ for all $A \in 2^X$. An extensive mapping is no-dummy if $\mu(\emptyset) = \emptyset$.

11 Notice that the transitive decent hyper-relations have been studied in the literature on implication systems with the name of *complete implication systems, CIS*, see, for example Falmagne and Doignon (2011) and Domenach and Leclerc (2004). Specifically, CIS are the dual of transitive decent relations. In 1974, Armstrong (1974) has shown that CIS are in a one-to-one correspondence with the closure operators. Kreps proves the same result [Lemma 2, Kreps (1979)].

us recall this relationship by stating the following well-known result [see Armstrong (1974); Caspard and Monjardet (2003); Kreps (1979); Malishevskii (1996)]:

Proposition 5 *If* $\preceq$ *is a transitive decent hyper-relation then* $\mu = \mu(\preceq)$ *is a closure operator. Conversely, if* μ *is a closure operator then the hyper-relation* $\preceq(\mu)$ *is transitive and decent.*

In what follows, we present a characterization of the transitive decent hyper-relations in terms of the class of *closed choice functions*, namely heritage choice functions that also satisfy the following property:

- **W** Let $x \notin A$. If $x \in f(x \cup A)$ and $y \notin f(y \cup A)$ for every y from some set Y, then $x \in f(x \cup Y \cup A)$.

Danilov and Koshevoy (2009) have provided a particular formulation for **W** in the case in which the set Y is a singleton. When the universal set of alternatives X is finite both formulations of **W** are equivalent. Then, we have that:

Proposition 6 *For a decent hyper-relation* $\preceq$ *the following statements are equivalent*

(1) $\preceq$ *is transitive;*
(2) *The corresponding choice function* $f = \Phi(\preceq)$ *is no-dummy and closed.*

Proof 1) $\Rightarrow$ 2). Let x, y and A be as in **W**, and suppose that $x \in f(x \cup A)$, $y \notin f(y \cup A)$ for every $y \in Y$, but $x \notin f(x \cup Y \cup A)$. By definition, $y \notin f(y \cup A)$ means that $y \preceq A$ for every $y \in Y$. Hence, $Y \preceq A$ and (by **U**) $Y \cup A \preceq A$. Now, $x \notin f(x \cup Y \cup A)$ means $x \preceq A \cup Y$. Then, by transitivity of $\preceq$, we obtain $x \preceq A$, that is $x \notin f(x \cup A)$. Therefore, a contradiction.

2) $\Rightarrow$ 1). We have to check that, for a closed choice function f, the hyper-relation $\preceq := \preceq_f$ is transitive. That is, $a \preceq B$ and $B \preceq C$ imply $a \preceq C$. Obviously we can assume that $a \notin C \cup B$.

Set $Y = B \backslash C$. Since for every $y \in Y$, $y \preceq C$ and $y \notin C$, we have $y \notin f(y \cup C)$ for every $y \in Y$. Suppose that $a \npreceq C$. Then, $a \in f(a \cup C)$, and by **W** we obtain $a \in f(a \cup Y \cup C) = f(a \cup B)$. The latter means that $a \npreceq B$, that contradicts the assumption $a \preceq B$. □

Suppose that a preference $<$ is the irreflexive part of a preorder (or quasi-ordering, that is a reflexive and transitive binary relation) $\leq$ on X. It is obvious that the hyper-relation generated by $<$ is transitive and no-dummy. The following result is well-known:

Theorem 2 [Kreps (1979)]. *Let* $\preceq$ *be a decent hyper-relation. The following property of* $\preceq$ *are equivalent:*

(1) $\preceq$ *is transitive;*
(2) $\preceq$ *has a multi-preference representation whose state preferences are the irreflexive parts of preorders;*
(3) $\preceq$ *has a multi-preference representation whose state preferences are the irreflexive parts of weak orders;*

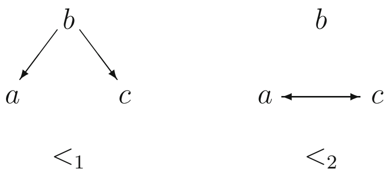
(4) $\preceq$ *has a multi-preference representation whose state preferences are the irreflexive parts of dichotomous weak orders.*

Let us recall briefly the argument of the proof. As we have mentioned above, a transitive decent hyper-relation defines a closure operator μ, or a closure space $\mathcal{F}$ of 'closed' sets (a set F is closed if $\mu(F) = F$). Now, for every closed set F, we can consider the closure subspace $\mathcal{F}_F$ (in $\mathcal{F}$) consisting of three closed sets $\{\emptyset, F, X\}$. Since $\mathcal{F} = \cup_F \mathcal{F}_F$, the initial hyper-relation $\preceq$ is represented as the intersection of hyper-relations corresponding to the closure systems $\mathcal{F}_F$. It is straightforward to check, that the corresponding hyper-relations are generated by the irreflexive parts of dichotomous preorders $\leq_F$, defined by the rule: $x \leq_F y$ if $x \in F$ or if $y \in X \backslash F$.

We illustrate the foregoing result by:

Example 7 Let $\preceq$ be a hyper-relation corresponding to the following closure operator μ: $\mu(x) = x$ for every $x \in X$ and $\mu(A) = X$ if $|A| \geq 2$. This hyper-relation is generated by the preference $\neq$, that is the irreflexive part of the total preference relation $\leq$ ($x \leq y$ for any $x, y \in X$).

Example 8 Let $X = \{a, b, c\}$ and let $\preceq$ be a hyper-relation corresponding to the closure operator μ defined as follows: $\mu(x) = x$ for every $x \in X$, $\mu(ac) = ac$, and $\mu(A) = X$ if $b \in A$. Therefore $\preceq$ is a decent transitive hyper-relation. It has a multi-preference representation $(<_1, <_2)$, where:



Note that the preference $<_2$ is not transitive, but it is the irreflexive part of the preorder $\leq$, where $a \leq c \leq a$.

6 Ample hyper-relations and Plott functions

We now consider a subclass of the decent hyper-relations such that in the multi-preference representations the state preferences can be chosen as complete, asymmetric, and transitive preferences (orders).

In the case in which the universal set of alternatives X is finite, it is well-known that this subclass is closely related to so-called *path-independent* or Plott choice functions [see, for example, Barberà et al. (2004); Nehring and Puppe (1999); Danilov and Koshevoy (2006)].

In order to proceed with our analysis, we introduce two notions.

Let us say that a hyper-relation $\preceq$ satisfies the property of *Lattice Equivalence* if, for any collection of equivalent menus $A_i, i \in I$, that is, for any $i, j \in I$, it holds that $A_i \preceq A_j$, the following holds true

- **LE** $\cup_{i\in I} A_i \preceq \cap_{i\in I} A_i$.

Definition 5 A decent hyper-relation $\preceq$ is *ample* if it satisfies Lattice Equivalence property **LE**.

Let $\preceq$ be a decent hyper-relation, and let $f := \Phi(\preceq)$ be the corresponding choice function. Then, we say that $\preceq$ is *framed* if, for every menu $A \in 2^X$, the following property holds:

- **F** $A \preceq f(A)$.

Property **F** means that any menu has sufficiently many essential alternatives. In Puppe (1996), this axiom is called '*Independence of Non-essential Alternatives*'. It also reminds the so-called Krein-Milman property of the convex polytopes, saying that every point of a convex polytope is a convex combination of the extreme points of the polytope. We will show that axioms **LE** and **F** are also closely related to the following well-known condition:

- **O** For any $A, B \in 2^X$ and a choice function f, if $f(A) \subseteq B \subseteq A$ then $f(B) = f(A)$.

The property **O** was introduced by Chernoff [postulate 5* in Chernoff (1954)] and is called *Outcast* property. It says that removing from an opportunity set the alternatives that are not chosen by a DM does not affect the freedom of choice provided by the set itself. In the language of essential elements, this property takes the following form: if, for a collection A' of elements of A, it holds that $A \preceq A - a, a \in A'$, then $A \preceq A \setminus A'$.

It is easy to see that, for the class of decent hyper-relations, the latter property follows from **LE**.

Choice functions that satisfy both conditions **H** and **O** constitute an important class of contracting operators called *path-independent* because they also satisfies the following suitable property:

- **PI** A choice function f is *path-independent* if, for any $A, B \in 2^X$,

$$f(A \cup B) = f(f(A) \cup B)$$

More precisely (see e.g. Aizerman and Malishevskii (1981); Moulin (1985); Danilov and Koshevoy (2005)), a choice function is *path-independent* if and only if it satisfies the axiom **H** and **O**. These choice functions are also called *Plott functions* in Danilov and Koshevoy (2005) after the celebrated work of Plott (1993).

Lemma 6 *Let $\preceq$ be a decent hyper-relation and let f be the corresponding choice function. Suppose $A \preceq B \subseteq A$. Then $f(A) \subseteq B$.*

Proof Indeed, if B does not contain some $a \in f(A)$ then $B \subseteq A \backslash a$. Since $A \preceq B$, by **Ext** we have that $A \preceq A \backslash a$. But this contradicts the fact that a is essential in A. □

We then state the following:

Theorem 3 *Let f be a choice function and $\preceq := \kappa_f$ be the corresponding hyper-relation. The following statements are equivalent:*

1. $\preceq$ *is an ample hyper-relation;*
2. f *is a no-dummy Plott function;*
3. $\preceq$ *is a framed hyper-relation.*

Proof 3) $\Rightarrow$ 2). By Theorem 1, we have that f satisfies **H**. Let us check that f also satisfies **O**. Suppose that $f(A) \subseteq B \subseteq A$. Then, by **F**, $B \subseteq A \preceq f(A)$, therefore, we have $B \preceq f(A)$. Applying Lemma 6 to $B \preceq f(A) \subseteq B$ we obtain $f(B) \subseteq f(A)$. The reverse inclusion follows by the heredity property of f.

2) $\Rightarrow$ 3). By Theorem 1, $\preceq$ is a decent hyper-relation. We have to show that $A \preceq f(A)$ for any menu A, i.e. that $f(a \cup f(A)) \subseteq f(A)$ holds for every $a \in A$. This follows by applying **O** to the inclusions $f(A) \subseteq a \cup f(A) \subseteq A$.

For the implication 1) $\Rightarrow$ 3), recall that, for $a \in A$, $a \notin f(A)$ holds if and only if $a \preceq A - a$ holds. Because of the union axiom **U**, $a \preceq A - a$ implies $A \preceq A - a$, or equivalently, A is equivalent to $A - a$. Thus, $f(A)$ is equal to the intersection of all $A - a$ which are equivalent to A. This and **LE** imply $A \preceq f(A)$.

For the implication 3) $\Rightarrow$ 1): By the union axiom **U**, we have, for every $j \in I$, $\cup_i A_i \preceq A_j$; since $f(\cup_i A_i) \subseteq \cup_i A_i$, due to **Cont**, it holds that $f(\cup_i A_i) \preceq A_j$. Thus, we have $f(\cup_i A_i) \preceq A_j \subseteq \cup_i A_i$, and we are in the framework of Lemma 6, hence, it holds that $f(\cup_i A_i) \subseteq A_j$ for any $j \in I$. Thus, $f(\cup_i A_i) \subseteq \cap_i A_i$. By the property **F**, we have $\cap_i A_i \preceq f(\cap_i A_i)$ and, hence, it holds that $f(\cup_i A_i) \subseteq \cap_i A_i \preceq f(\cap_i A_i)$. Thus by **Ext**, we conclude that **LE** holds true. □

Lemma 7 *Let f be a Plott function, and let B and C be some menus. The following three statements are equivalent:*

(1) $B\,\kappa(f)\,C$;
(2) $f(B \cup C) \subseteq C$;
(3) $f(B \cup C) = f(C)$.

Proof 1) $\Rightarrow$ 2). Let $B\kappa(f)C$, then, by **U**, $B \cup C\kappa(f)C$, hence, by **Cont** ($f(B \cup C) \subseteq B \cup C$), we have $f(B \cup C)\kappa(f)C$. Thus, it holds that $f(B \cup C)\kappa(f)C \subseteq B \cup C$. By Lemma 6, we have $f(B \cup C) \subseteq C$, that is 2).

2) $\Rightarrow$ 3). This follows by applying **O** to the chain of inclusions $f(B \cup C) \subseteq C \subseteq (B \cup C)$.

3) $\Rightarrow$ 1). By **F**, we have $B \cup C\,\kappa(f)\,f(B \cup C) = f(C) \subseteq C$. Therefore, due to **Ext**, it holds that $B \cup C\,\kappa(f)\,C$ and, by **Cont**, we have $B\,\kappa(f)\,C$. □

From this lemma it follows that, for a Plott function f, the hyper-relation $\kappa(f)$, $\preceq'_f$ and $\preceq''_f$ in Remark 2 coincide.

Corollary *An ample hyper-relation is transitive.*

Proof Let $f = \Phi(\preceq)$. Consider a triple $A \preceq B \preceq C$. Then, by Theorem 3 (equivalence 1) and 3)), we have $A \cup C \subseteq A \cup B \cup C \preceq f(A \cup B \cup C)$. Due to **Cont**, it holds that $A \cup C \preceq f(A \cup B \cup C)$. Then, by Theorem 3 (equivalence 1) and 2)), we have $f(A \cup B \cup C) = f(f(A \cup B) \cup C)$. Due to Lemma 7, it holds that $f(A \cup B) = f(B)$ and $f(B \cup C) = f(C)$. Hence, it holds that $f(f(A \cup B) \cup C) = f(C)$. Thus, $A \cup C \preceq f(C)$, and by Lemma 7, it holds that $f(A \cup C \cup f(C)) = f(C)$, that is nothing but $f(A \cup C) = f(C)$. Therefore, by Lemma 7, we have $A \preceq C$. □

Let us observe that a no-dummy Plott function f is non-empty-valued. In fact, suppose $f(A) = \emptyset$ for a non-empty set A, and let $a \in A$. From $f(A) = \emptyset \subseteq \{a\} \subseteq A$ we obtain $f(a) = f(A) = \emptyset$, that is a contradiction.

It is well known (see e.g. Aizerman and Malishevskii (1981); Moulin (1985); Danilov and Koshevoy (2005) that, for the case in which X is finite, any no-dummy Plott function can be represented as the union of choice functions rationalized by linear orders. We show that this result can be generalized to an arbitrary set X.

In order to do that, we start with the rationalized choice functions. Let $<$ be a preference and $Max(<)$ be the choice function rationalized by $<$.

Lemma 8 *$Max(<)$ is a no-dummy Plott function if and only if $<$ is a well order.*

Proof Suppose that $<$ is a well order on the set X. We know that $f := Max(<)$ is no-dummy and heritage. We need to check the axiom **O**. Suppose that $f(A) \subseteq B \subseteq A$; we have to show that $f(B) \subseteq f(A)$. Let $b \in f(B)$, that is b is a maximal element of B. We assert that b is maximal in A. In the opposite case b is dominated by some element $a \in A$ ($b < a$), which we can assume to be maximal in A. Then $a \in f(A) \subseteq B$, and $b < a$. But this contradicts the maximality of b in B.

Vice versa, suppose that $Max(<)$ is a no-dummy Plott function. By Lemma 4 and Theorem 3, $<$ is a well relation. From Corollary 3, $<$ is transitive, i.e. it is an order. □

We now establish

Theorem 4 *Let f be a heritage choice function. The following statements are equivalent:*

(1) *f is a no-dummy Plott function;*
(2) *f is the union of choice functions rationalized by well linear orders;*
(3) *f is the union of choice functions rationalized by well orders.*

Proof 1) ⇒ 2). Following the proof of Proposition 4, every selection s of f defines a well linear order $<_s$ on X such that $f_s := Max(<_s) \subseteq f$. Let us show that the union of these 'linear' choice functions f_s is equal to f.

Let $a \in f(A)$, then there exists a selection s of f such that $f_s(A) = a$. Indeed, consider a set B containing A, i.e. $A \subseteq B$. There are two cases:

1. $f(B) \nsubseteq A$, and
2. $f(B) \subseteq A$. Notice that in this case $a \in f(B)$ by Heredity of f.

Let us pick a selection s of f which satisfies two additional properties: in case 1. $s(B) \notin A$; in case 2. we require $s(B) = a$. Such a selector exists. Now let $\leq_s$ be the well linear order constructed in Proposition 5. We claim that $f_s(A) = a$. Indeed, let $m \in A$ be the maximal element in A with respect to the order $\leq_s$ (so that $f_s(A) = m$), and let $B = X - [> m]$. Obviously, $A \subseteq B$ and $m = s(B)$. Therefore, only case 2 is possible and $m = a$.

By Lemma 8, we have that 2) ⇒ 3) and 3) ⇒ 1). □

Corollary 3 *Every ample hyper-relation has a multi-preference representation, where the state dependent preferences are well orders.*

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