Knowledge spaces from a topological point of view

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Abstract

In this paper we consider the operations of restriction, extension and gluing of pre-topologies, antimatroids, neighborhood systems and heritage choice functions. Pre-topologies are also known as knowledge spaces in knowledge space theory, and some of these operations were recently discussed in a paper by Falmagne [(2008). Projections and symmetric expansions of a learning space. arXiv:0803.0575]. We introduce the notion of a separator, which allows to decompose such objects into simpler parts, or conversely, to assemble them from simpler parts.

1 Introduction

This note was prompted by a recent paper by Falmagne (2008). In this paper Falmagne considers projections and symmetric expansions of learning space, which is a special case of knowledge space. The theory of knowledge spaces was introduced by Doignon and Falmagne (1999) as a mathematical framework for knowledge assessment and training. Technically, the notion of a knowledge space is a generalization of the notion of a topological space. The language of topology and its established generalizations are therefore useful for studying formal properties of knowledge spaces. For example, Falmagne's (2008) projections resemble the restriction operation for topologies. His notion of expansions turn out to be a particular case of gluing topological spaces.

The main aim of the present note is to describe the operations of restriction, extension, gluing, decomposition and assembly in the commonly accepted topological language, and in appropriate generality. In fact, we do not confine ourselves to studying pre-topologies (our term for knowledge spaces) only, but descend to the more primitive level of neighborhood systems. In addition to the problems of restriction, extension and gluing, we consider the problem of decomposing an object into finer parts, and the reverse problem of their synthesis from finer parts. Here the notion of a separator plays a key role. Roughly speaking, a separator is a part of the basic set on which the structure 'lives,' independently of the remaining part. The set of separators (which proves to be a topology) allows one to represent an object of interest as decomposed into independent parts. Such a decomposition enables structuring complicated objects and facilitates working with them.

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We begin with the pre-topologies and then present some applications to antimatroids (also known as learning spaces), a special class of pre-topologies. In the next step we generalize these notions and constructions to the neighborhood systems. Finally, we rewrite these matters in the cryptomorphic language of heritage choice functions.

2 Pre-topologies

In the following X denotes some basic set. Its elements are denoted in lower case letters such as x, y, z, a, b and called points. Its subsets are denoted in upper case letters such as A, B, Z. Calligraphic letters such as $\mathcal{T}$ or $\mathcal{N}$ will be used for subsets of the power set 2^X.

Definition. A *pre-topology* on the set X is a set $\mathcal{T} \subset 2^X$, which is stable with respect to arbitrary unions. In particular, $\emptyset \in \mathcal{T}$. The elements of $\mathcal{T}$ are called *open* sets of the pre-topology.

This notion is close to the classical notion of topology. Pre-topologies differ from topologies in that they do not require the set of open sets to be stable with respect to finite intersections. In particular, we do not assume that the whole set X is open. On the other hand, there exists the maximal open set, denoted as $\mathrm{supp}(\mathcal{T})$. Many common topological notions have meaning for pre-topologies as well.

The notion of a pre-topology is well-known; see, for example, Caspar and Monjardet (2003). If we focus on closed instead of open sets, we obtain the so-called *closure systems* (or *closure operators*). Doignon and Falmagne (1999) refer to pre-topologies by calling them *knowledge spaces* (open sets are termed *states*). We will, however, use the term 'pre-topology.'

Pre-topologies (as subsets of 2^X) can be compared with respect to set-inclusion. An intersection of pre-topologies is a pre-topology as well. Thus the poset (partially ordered set) of pre-topologies on the set X, $\mathbf{PTop}(X)$, is a (complete) lattice. The join $\vee_i \mathcal{T}_i$ of pre-topologies $\mathcal{T}_i$ (the minimal pre-topology containing all $\mathcal{T}_i$) consists of sets of the form $\cup_i U_i$, where $U_i \in \mathcal{T}_i$. The minimal (or *trivial*) pre-topology consists of the empty set only; the maximal (or *discrete*) pre-topology coincides with 2^X.

The lattice $\mathbf{PTop}(X)$ for the two-element basic set $X = \{a, b\}$ is drawn below (Figure 1). Here and henceforth we write ac instead of $\{a, c\}$, and so on.

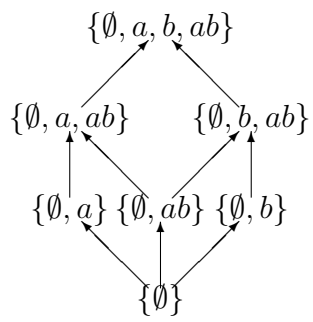

Figure 1. The lattice of pre-topologies on the set $\{a, b\}$.

Often one is interested in pre-topologies with additional properties, such as topologies,

antimatroids, or dual matroids[1]. In Figure 1, all the pre-topologies shown are matroids except for $\{\emptyset, a, ab\}$ and $\{\emptyset, b, ab\}$, antimatroids except for $\{\emptyset, ab\}$, and topologies except for $\{\emptyset\}$, $\{\emptyset, a\}$ and $\{\emptyset, b\}$. In this section we consider the operations of restriction, extension and gluing for pre-topologies.

From now on, let Z denote a subset of X, and let $j : Z \to X$ be the canonical embedding of Z in X. We shall restrict pre-topologies from X to Z or extend them from Z to X.

Restriction

Let $\mathcal{T}$ be a pre-topology on X. Restricting $\mathcal{T}$ on Z means considering a pre-topology on Z which is naturally related to $\mathcal{T}$. There are at least two ways how this can be done. The first is called the proper *restriction* (or *strong inverse image*), or the *projection* in Falmagne's (2008) nomenclature, and it is denoted by $j^{\#}(\mathcal{T})$ or $\mathcal{T}|Z$.[2] By definition, $\mathcal{T}|Z$ consists of sets of the form $U \cap Z$, where $U \in \mathcal{T}$. In other words, open sets in Z are the traces on Z of open sets in X. It is clear that $\mathcal{T}|Z$ is a pre-topology. Thus, $j^{\#}$ provides a mapping $\mathbf{PTop}(X) \to \mathbf{PTop}(Z)$.

Lemma 1. *The mapping $j^{\#} : \mathbf{PTop}(X) \to \mathbf{PTop}(Z)$ commutes with $\vee$, that is $j^{\#}(\vee_i \mathcal{T}_i) = \vee_i (j^{\#}\mathcal{T}_i)$; in particular, it is isotone.* $\square$

The second way is through the *weak inverse image* j^*. By definition, $j^*(\mathcal{T})$ consists of subsets of Z which are open in X. Clearly, $j^*(\mathcal{T})$ is a pre-topology on Z and $j^*(\mathcal{T}) \subset j^{\#}(\mathcal{T})$.

Lemma 2. *The mapping $j^* : \mathbf{PTop}(X) \to \mathbf{PTop}(Z)$ commutes with $\vee$ and $\wedge$.*

Proof. The statement is obvious with respect to $\wedge$. Let us prove the statement for $\vee$. Suppose that $\mathcal{T}_i$ is a family of pre-topologies on X; we have to check that $j^*(\vee_i \mathcal{T}_i) = \vee_i j^*(\mathcal{T}_i)$. Let V be open in $j^*(\vee_i \mathcal{T}_i)$. This means that $V \subset Z$ and V is open in $\vee_i \mathcal{T}_i$. The latter means that $V = \cup_i V_i$, where $V_i \in \mathcal{T}_i$. But then all $V_i \subset Z$, so $V_i \in j^*(\mathcal{T}_i)$ and $V \in \vee_i j^*(\mathcal{T}_i)$. The reverse is proved analogously. $\square$

In the following we focus on the 'strong' restriction $j^{\#}$, for it contains more information about the initial pre-topology.

Extension

Let $\mathcal{P}$ be a pre-topology on Z. An *extension* of $\mathcal{P}$ to X is a pre-topology $\mathcal{T}$ on X such that $\mathcal{T}|Z = \mathcal{P}$. There exist many extensions in general, and there is the maximal $\mathcal{T} = j_{\#}(\mathcal{P})$ among these. By definition, $j_{\#}(\mathcal{P})$ consists of all sets $U \subset X$ such that $U \cap Z \in \mathcal{P}$. In other words, $\mathcal{T}$ consists of sets of the form $V \cup W$, where $V \in \mathcal{P}$ and $W \subset X - Z$.

This gives the mapping $j_{\#} : \mathbf{PTop}(Z) \to \mathbf{PTop}(X)$. It is clear that this mapping commutes with $\vee$ and $\wedge$.

One can also consider the minimal extension $j_*(\mathcal{P})$ consisting of all sets $V \in \mathcal{P}$. In other words, $j_*(\mathcal{P})$ coincides with $\mathcal{P}$ considered as a subset of 2^X.

[1] Antimatroids will be defined in Section 4. The notion of a matroidal pre-topology need not to be defined as it is not used elsewhere in this note; a definition can be found in Danilov and Koshevoy (2009).

[2] A related notion of a restriction or projection is defined in the framework of media theory; see Cavagnaro (2008).

Gluing or amalgam

Let us consider the following classical problem (in the case of topologies, see Bourbaki, 1940). Suppose X is covered by a family $(X_i,\ i \in I)$ of subsets, $X = \cup_i X_i$. Let $\mathcal{T}_i$ be pre-topologies on X_i. Is there a pre-topology $\mathcal{T}$ on X such that $\mathcal{T}|X_i = \mathcal{T}_i$? If such a pre-topology exists, then (due to Lemma 1) there exists the maximal pre-topology $\mathcal{T}$ with the property $\mathcal{T}|X_i = \mathcal{T}_i$.[3] This maximal pre-topology is called the *gluing* (or the *amalgam*) of the pre-topologies $\mathcal{T}_i$, and it is denoted by $\star_i \mathcal{T}_i$.

There is an obvious necessary condition for the existence of the gluing: the pre-topologies $\mathcal{T}_i$ must be compatible on the intersections $X_{ii'} = X_i \cap X_{i'}$, in the sense that $\mathcal{T}_i|X_{ii'} = \mathcal{T}_{i'}|X_{ii'}$ for every $i, i' \in I$. Indeed, $\mathcal{T}_i|X_{ii'} = (\mathcal{T}|X_i)|X_{ii'} = \mathcal{T}|X_{ii'} = \mathcal{T}_{i'}|X_{ii'}$.

In the case of two elements this compatibility condition is also sufficient for the existence of the gluing. Suppose that $X = X_1 \cup X_2$, and that $\mathcal{T}_1$ and $\mathcal{T}_2$ are pre-topologies on X_1 and X_2 compatible on X_{12}. Then the gluing of $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by the formula:

$$\mathcal{T} = j_{1\#}(\mathcal{T}_1) \cap j_{2\#}(\mathcal{T}_2). \tag{1}$$

Here j_i is the canonical embedding of X_i in X. In other words, a set $U \subset X$ is open (with respect to $\mathcal{T}$) if and only if $U \cap X_i$ is open in X_i for $i = 1, 2$.

Proposition 1. *The pre-topology $\mathcal{T}$ given by (1) is the gluing of $\mathcal{T}_1$ and $\mathcal{T}_2$.*

Close assertions can be found in Theorems 1 and 2 of Stefanutti (2008).

Proof. Let us show that $\mathcal{T}|X_1 = \mathcal{T}_1$. The inclusion $\mathcal{T}|X_1 \subset \mathcal{T}_1$ is obvious, so it suffices to prove the inverse inclusion. Let $U_1 \in \mathcal{T}_1$; then $U_1 \cap X_{12} \in \mathcal{T}_1|X_{12} = \mathcal{T}_2|X_{12}$. Hence there exists $U_2 \in \mathcal{T}_2$ such that $U_2 \cap X_{12} = U_1 \cap X_{12}$. If we set $U = U_1 \cup U_2$, then $U \cap X_i$ is equal to U_i for $i = 1, 2$. Therefore U is open with respect to $\mathcal{T}$ and $U \cap X_1 = U_1$, that is, $U_1 \in \mathcal{T}|X_1$.

Let $\mathcal{T}'$ be a pre-topology on X such that $\mathcal{T}'|X_i = \mathcal{T}_i$ for $i = 1, 2$. Then $\mathcal{T}' \subset j_{1\#}(\mathcal{T}_1)$ and $\mathcal{T}' \subset j_{2\#}(\mathcal{T}_2)$, and hence $\mathcal{T}' \subset \mathcal{T} = j_{1\#}(\mathcal{T}_1) \cap j_{2\#}(\mathcal{T}_2)$. $\square$

In particular, if the subsets X_1 and X_2 do not intersect, the previous construction defines the *direct sum* of pre-topologies. As a remark: The (maximal) extension of the pre-topology $\mathcal{P}$ from Z to X can be considered the gluing (or the direct sum) of $\mathcal{P}$ on Z and the discrete topology on $X - Z$.

In the case of three or more elements the gluing may not exist.

Example 1. Let $X = \{a, b, c\}$ be covered by three subsets: $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$. Let the pre-topologies be specified as follows. The pre-topology (actually a topology) on $\{a, b\}$ consists of $\emptyset, a, ab$, the pre-topology on $\{b, c\}$ consists of $\emptyset, b, bc$, and the pre-topology on $\{c, a\}$ consists of $\emptyset, c, ca$. These pre-topologies are compatible on the intersections, however their gluing does not exist.

Indeed, assume $\mathcal{T}$ is their gluing. The subset $\{a\}$ is open in $\{a, b\}$, therefore $\{a\}$ or $\{a, c\}$ must belong to $\mathcal{T}$. If $\{a\} \in \mathcal{T}$, then $\{a\} \cap \{c, a\} = \{a\}$ is an open set in $\{c, a\}$; but this is not the case. If $\{a, c\} \in \mathcal{T}$, then $\{a, c\} \cap \{b, c\} = \{c\}$ is an open set in $\{b, c\}$; but this is not the case.

[3] Stefanutti (2008) considers *meshing* (as this classical problem is referred to in the knowledge space theory) in a slightly more general setup of knowledge structures. Meshability of knowledge structures is also discussed in Doignon and Falmagne (1999).

Even if the gluing of three elements does exist, the associativity law can fail.

Example 2. Again, let $X = \{a, b, c\}$ be covered by three subsets: $X_1 = \{a, b\}$, $X_2 = \{b, c\}$, and $X_3 = \{c, a\}$. The pre-topology $\mathcal{T}_1$ on X_1 consists of $\emptyset, b, ab$, the pre-topology $\mathcal{T}_2$ on X_2 consists of $\emptyset, bc$, and the pre-topology $\mathcal{T}_3$ on X_3 consists of $\emptyset, c, ac$. The gluing of these pre-topologies exists and is $\mathcal{T} = \{\emptyset, bc, abc\}$. The gluing of $\mathcal{T}_1$ and $\mathcal{T}_3$ exists and equals $\mathcal{T}' = \{\emptyset, b, c, bc, abc\}$. However, the pre-topologies $\mathcal{T}'$ and $\mathcal{T}_2$ are not compatible on the intersection $(X_1 \cup X_3) \cap X_2 = X_2$.

There is a simple case, though, when the gluing of three pre-topologies exists and the associativity law holds (see also Theorem 1 in the next section).

Proposition 2. *Let $X = X_1 \cup X_2 \cup X_3$, and let $\mathcal{T}_i$ be pre-topologies on X_i for $i = 1, 2, 3$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are compatible, and $\mathcal{T}_1 \star \mathcal{T}_2$ is compatible with $\mathcal{T}_3$, then $(\mathcal{T}_1 \star \mathcal{T}_2) \star \mathcal{T}_3 = \mathcal{T}_1 \star \mathcal{T}_2 \star \mathcal{T}_3$.*

Proof. Let $\mathcal{T} = (\mathcal{T}_1 \star \mathcal{T}_2) \star \mathcal{T}_3$. Clearly, the restrictions of $\mathcal{T}$ on X_i coincide with $\mathcal{T}_i$ for $i = 1, 2, 3$. We only have to prove that $\mathcal{T}$ is the maximal pre-topology with this property. Suppose that U is a set such that $U \cap X_i \in \mathcal{T}_i$ for $i = 1, 2, 3$. Then $U \cap (X_1 \cup X_2) \in \mathcal{T}_1 \star \mathcal{T}_2$ and $U \cap X_3 \in \mathcal{T}_3$, that is, U belongs to $\mathcal{T}$. $\square$

Corollary. *Let $X = X_1 \cup X_2 \cup X_3$, and let $\mathcal{T}_i$ be compatible pre-topologies on X_i for $i = 1, 2, 3$. Suppose $X_1 \cap X_3 = \emptyset$. Then $(\mathcal{T}_1 \star \mathcal{T}_2) \star \mathcal{T}_3 = \mathcal{T}_1 \star \mathcal{T}_2 \star \mathcal{T}_3 = \mathcal{T}_1 \star (\mathcal{T}_2 \star \mathcal{T}_3)$.*

Proof. We have to check that the pre-topologies $\mathcal{T}_1 \star \mathcal{T}_2$ and $\mathcal{T}_3$ are compatible on $(X_1 \cup X_2) \cap X_3 = X_2 \cap X_3$. Indeed, $(\mathcal{T}_1 \star \mathcal{T}_2)|(X_2 \cap X_3) = ((\mathcal{T}_1 \star \mathcal{T}_2)|X_2)|(X_2 \cap X_3) = \mathcal{T}_2|(X_2 \cap X_3) = \mathcal{T}_3|(X_2 \cap X_3)$. $\square$

The gluing of $\mathcal{T}_1$ and $\mathcal{T}_3$ can be incompatible with $\mathcal{T}_2$, though.

Example 3. Let $X = \{a, b\}$, $X_1 = \{a\}$, $X_2 = X$, and $X_3 = \{b\}$. Let $\mathcal{T} = \{\emptyset, ab\}$, and let $\mathcal{T}_i = \mathcal{T}|X_i$ for $i = 1, 2, 3$. The gluing of $\mathcal{T}_1$ and $\mathcal{T}_3$ is the discrete pre-topology on X, which strictly contains $\mathcal{T}_2$.

3 Separators

Let $\mathcal{T}$ be a pre-topology on X.

Definition. A subset $Z \subset X$ is a *separator* (for $\mathcal{T}$) if $U \cap Z$ is open for every open set U; that is, $U \in \mathcal{T}$ implies $U \cap Z \in \mathcal{T}$. An equivalent formulation of the condition is: $\mathcal{T}|Z \subset \mathcal{T}$.

Examples. 1) The empty set and the whole set X are (trivial) separators.
2) An open singleton is a separator.
3) The maximal open set $\mathrm{supp}(\mathcal{T})$ in X is a separator.
4) If Z does not intersect $\mathrm{supp}(\mathcal{T})$, then Z is a separator.
5) Suppose $\mathcal{T} = j_\#(\mathcal{P})$ for a pre-topology $\mathcal{P}$ on Z. Then Z is a separator for $\mathcal{T}$.
6) If $\mathcal{T}$ is a topology, the set of separators for $\mathcal{T}$ coincides with $\mathcal{T}$.

Lemma 3. *Let $\mathcal{T}$ be a pre-topology on X.*
1) *Suppose that $Z \subset X$ is a separator for $\mathcal{T}$. Then, for any $Y \subset X$, the subset $Z \cap Y$ is a separator for $\mathcal{T}|Y$.*

2) *Suppose that $Z \subset X$ is a separator for $\mathcal{T}$ and $Y \subset Z$ is a separator for $\mathcal{T}|Z$. Then Y is a separator for $\mathcal{T}$.*

Proof. 1) Let V be an open set in Y (with respect to $\mathcal{T}|Y$). We have to check that $V \cap (Z \cap Y)$ is in $\mathcal{T}|Y$. By definition, $V = U \cap Y$ for some $U \in \mathcal{T}$. Therefore $V \cap (Z \cap Y) = (U \cap Z) \cap Y$. Since Z is a separator for $\mathcal{T}$, the set $U \cap Z$ is in $\mathcal{T}$. Hence $(U \cap Z) \cap Y$ is in $\mathcal{T}|Y$.

2) It holds that $\mathcal{T}|Y = (\mathcal{T}|Z)|Y \subset \mathcal{T}|Z \subset \mathcal{T}$. $\square$

The following proposition tells us that the set of separators always is a topology.

Proposition 3. a) *Intersection of a finite family of separators is a separator.*
b) *Union of any family of separators is a separator.*

Proof. a) It suffices to consider the case of two separators Z_1 and Z_2. Let $U \in \mathcal{T}$. Since Z_1 is a separator, $U \cap Z_1 \in \mathcal{T}$. Since Z_2 is a separator, $U \cap (Z_1 \cap Z_2) = (U \cap Z_1) \cap Z_2 \in \mathcal{T}$.

b) Let (Z_i) be a family of separators, and let $Z = \cup_i Z_i$. Let $U \in \mathcal{T}$. Since Z_i is a separator, $U \cap Z_i \in \mathcal{T}$. As the union of open sets, $U \cap (\cup_i Z_i) = \cup_i (U \cap Z_i)$ is open. $\square$

The set $\mathbf{Sep}(\mathcal{T})$ of separators for $\mathcal{T}$ thus is a topology on X. In general, this topology is incompatible with $\mathcal{T}$. (Though, if the whole set X is open, each separator is open and $\mathbf{Sep}(\mathcal{T}) \subset \mathcal{T}$.) Separators allow decomposing $\mathcal{T}$ into finer parts. An appreciation for the decomposition issue can be gained by considering the following simple case; later on we shall discuss a more general setting.

Decomposition: a simple case

Suppose that $\mathcal{T}$ is a pre-topology on X, $Z \subset X$, and $Y = X - Z$. In general, the restrictions $\mathcal{T}|Z$ and $\mathcal{T}|Y$ do not determine the pre-topology $\mathcal{T}$. To reconstruct $\mathcal{T}$ we consider, for every open set $V \in \mathcal{T}|Z$,

$$\mathcal{T}_V = \{U \subset Y,\ V \cup U \in \mathcal{T}\}$$

in 2^Y. (Note that $\mathcal{T}_V \subset \mathcal{T}|Y$.)

Lemma 4. *Let Z be a separator for $\mathcal{T}$. Then:*
a) *For every $V \in \mathcal{T}|Z$, $\mathcal{T}_V$ is a pre-topology on Y.*
b) *$\mathcal{T}_V$ is isotone in $V \in \mathcal{T}|Z$.*

Proof. a) It holds that $\emptyset \in \mathcal{T}_V$, that is, $V = V \cup \emptyset$ is open with respect to $\mathcal{T}$. We now use the fact that Z is a separator for $\mathcal{T}$; since $V \in \mathcal{T}|Z$, we have $V \in \mathcal{T}$. The proof for $\mathcal{T}_V$ being stable with respect to arbitrary unions is straightforward (and does not require Z to be a separator).

b) Suppose $U \in \mathcal{T}_V$ and $V \subset W \in \mathcal{T}|Z$. We have to show that $U \in \mathcal{T}_W$. Since $U \in \mathcal{T}_V$, $V \cup U$ is open in X. Since $W \in \mathcal{T}|Z$ and Z is a separator for $\mathcal{T}$, W is open in X as well. Hence, their union $(V \cup U) \cup W = (V \cup W) \cup U = W \cup U$ is open in X, and $U \in \mathcal{T}_W$. $\square$

These data (the pre-topology $\mathcal{T}|Z$ on Z and the isotone family $(\mathcal{T}_V, V \in \mathcal{T}|Z)$ of pre-topologies on $X - Z$) allow us to restore the initial pre-topology $\mathcal{T}$. Namely,

$$\mathcal{T} = \{V \cup U,\ \text{where } V \in \mathcal{T}|Z \text{ and } U \in \mathcal{T}_V\}.$$

Indeed, $V \cup U$ belongs to $\mathcal{T}$ (by definition of $\mathcal{T}_V$). Conversely, if $W \in \mathcal{T}$, then $W = (W \cap Z) \cup (W - Z)$, $W \cap Z \in \mathcal{T}|Z$, and $W - Z \in \mathcal{T}_{W \cap Z}$.

Synthesis

Conversely, suppose we have the following data:

a) a pre-topology $\mathcal{T}_0$ on Z, and

b) for every $V \in \mathcal{T}_0$, a pre-topology $\mathcal{T}_V$ on $X - Z$ such that the correspondence $V \mapsto \mathcal{T}_V$ is isotone.

Given these data we define the following collection $\mathcal{T}$ of subsets of X,

$$\mathcal{T} = \{V \cup U, \text{ where } V \in \mathcal{T}_0 \text{ and } U \in \mathcal{T}_V\}. \tag{2}$$

Proposition 4. a) *$\mathcal{T}$ is a pre-topology on X.*

b) *It holds that $\mathcal{T}|Z = \mathcal{T}_0$.*

c) *Z is a separator for $\mathcal{T}$.*

d) *For every $V \in \mathcal{T}|Z$, $\mathcal{T}_V = \{U \subset X - Z, \text{ such that } V \cup U \in \mathcal{T}\}$.*

Proof. a) Let $(V_i \cup U_i)$ be a family of sets in $\mathcal{T}$. Since $\mathcal{T}_0$ is stable with respect to unions, $V = \cup_i V_i$ is in $\mathcal{T}_0$. Every U_i is in $\mathcal{T}_{V_i}$; since $V_i \subset V$, $\mathcal{T}_{V_i} \subset \mathcal{T}_V$ and $U_i \in \mathcal{T}_V$. As $\mathcal{T}_V$ is stable with respect to unions, $\cup_i U_i \in \mathcal{T}_V$, and consequently, $V \cup (\cup_i U_i) \in \mathcal{T}$.

b) The inclusion $\mathcal{T}|Z \subset \mathcal{T}_0$ is obvious; the inverse inclusion follows from $\emptyset \in \mathcal{T}_V$. This proves b), and the inclusion $\mathcal{T}_0 \subset \mathcal{T}$ implies c).

d) This statement is obvious. $\square$

This assembly construction thus is the reverse to the previously considered decomposition. One can say that $\mathcal{T}$ is the *semi-direct sum* of the pre-topology $\mathcal{T}_0$ and the family $(\mathcal{T}_V, V \in \mathcal{T}_0)$. If both Z and $Y = X - Z$ are separators, $\mathcal{T}$ is the direct sum of the pre-topologies $\mathcal{T}|Z$ and $\mathcal{T}|Y$.

Application to gluing

Consider the situation underlying the problem of gluing pre-topologies: $X = \cup_{i \in I} X_i$, $\mathcal{T}_i$ are pre-topologies on X_i, and $X_{ij} = X_i \cap X_j$.

Definition. The pre-topologies $\mathcal{T}_i$ are *strictly compatible* if, for any $i, j \in I$,

$$\mathcal{T}_i|X_{ij} \subset \mathcal{T}_j.$$

Lemma 5. *The pre-topologies $\mathcal{T}_i$ are strictly compatible if and only if $\mathcal{T}_i$ are compatible and, for any $i, j \in I$, the subsets X_{ij} are separators for $\mathcal{T}_i$ and $\mathcal{T}_j$.*

Proof. Suppose that $\mathcal{T}_i$ are strictly compatible. The inclusion $\mathcal{T}_i|X_{ij} \subset \mathcal{T}_j$ implies the inclusion $\mathcal{T}_i|X_{ij} \subset \mathcal{T}_j|X_{ij}$. Since the inverse inclusion $\mathcal{T}_j|X_{ij} \subset \mathcal{T}_i|X_{ij}$ is also true, the equality $\mathcal{T}_i|X_{ij} = \mathcal{T}_j|X_{ij}$ holds; that is, the pre-topologies $\mathcal{T}_i$ are compatible. Moreover, $\mathcal{T}_i|X_{ij} = \mathcal{T}_j|X_{ij} \subset \mathcal{T}_i$, that is, X_{ij} are separators for $\mathcal{T}_i$.

Conversely, $\mathcal{T}_i|X_{ij} = \mathcal{T}_j|X_{ij}$ due to the compatibility property, and $\mathcal{T}_j|X_{ij} \subset \mathcal{T}_j$ due to the separability property. Therefore $\mathcal{T}_i|X_{ij} \subset \mathcal{T}_j$. $\square$

Theorem 1. *Let the pre-topologies $\mathcal{T}_i$ on $X_i \subset X$, $i \in I$, be strictly compatible.*

a) *The gluing $\mathcal{T} = \star_i \mathcal{T}_i$ of $\mathcal{T}_i$ exists and is equal to $\cap_i j_{i\#}(\mathcal{T}_i)$.*

b) *Suppose that* $I = I' \cup I''$, *and let* $X' = \cup_{i' \in I'} X_{i'}$, $X'' = \cup_{i'' \in I''} X_{i''}$. *Define the pre-topology* $\mathcal{T}'$ *on* X' *as the gluing of all* $\mathcal{T}_{i'}$, $i' \in I'$, *and define similarly* $\mathcal{T}''$ *on* X''. *Then,* $\mathcal{T}'$ *and* $\mathcal{T}''$ *are strictly compatible, and* $\mathcal{T} = \mathcal{T}' \star \mathcal{T}''$.

Proof. a) Let $\mathcal{T} = \cap_i j_{i\#}(\mathcal{T}_i)$. We have to check that $\mathcal{T}|X_i = \mathcal{T}_i$. The inclusion $\mathcal{T}|X_i \subset \mathcal{T}_i$ is trivial. To prove the reverse inclusion, let U be an open set in X_i. Due to the strict compatibility property, all $U \cap X_j$ are open with respect to $\mathcal{T}_j$; hence U is open with respect to $\mathcal{T}$. This proves a).

b) Let U' be open with respect to $\mathcal{T}'$. We have to prove that $U' \cap (X' \cap X'')$ is open with respect to $\mathcal{T}''$. The set U' being open with respect to $\mathcal{T}'$ means that

$$U' \cap X_{i'} \in \mathcal{T}_{i'} \text{ for every } i' \in I'. \tag{3}$$

The set $U' \cap (X' \cap X'')$ being open with respect to $\mathcal{T}''$ means that $U' \cap (X' \cap X'') \cap X_{i''} \in \mathcal{T}_{i''}$ for every $i'' \in I''$. Now $U' \cap (X' \cap X'') \cap X_{i''} = U' \cap X' \cap X_{i''} = \cup_{i' \in I'}(U' \cap X_{i'} \cap X_{i''})$. Therefore it suffices to prove that $U' \cap X_{i'i''}$ is in $\mathcal{T}_{i''}$ for every $i' \in I'$. Due to (3), $U' \cap X_{i'}$ is in $\mathcal{T}_{i'}$. Since $\mathcal{T}_{i'}$ and $\mathcal{T}_{i''}$ are strictly compatible, $(U' \cap X_{i'}) \cap (X_{i'} \cap X_{i''}) = U' \cap X_{i'i''}$ is in $\mathcal{T}_{i''}$. This proves the strict compatibility of $\mathcal{T}'$ and $\mathcal{T}''$.

The last assertion is now obvious, because $[U \in \mathcal{T}' \star \mathcal{T}''] \Leftrightarrow [U \cap X' \in \mathcal{T}'$ and $U \cap X'' \in \mathcal{T}''] \Leftrightarrow [U \cap X_{i'} \in \mathcal{T}_{i'}$ for every $i' \in I'$ and $U \cap X_{i''} \in \mathcal{T}_{i''}$ for every $i'' \in I''] \Leftrightarrow [U \cap X_i \in \mathcal{T}_i$ for every $i \in I] \Leftrightarrow [U \in \mathcal{T}]$. $\square$

4 Application to antimatroids

In the case of topologies the previously obtained results and constructions are well-known; see Bourbaki (1940). The notion of matroids is not taken on here as it deserves a separate investigation; let us only mention an analogy of separators and modular flats and Brylawski's theorem (Brylawski, 1975) about amalgams of matroids. In this note we focus on the case of antimatroids. Throughout this section, X is assumed to be a finite set.

Antimatroids are special pre-topologies. In terms of closed sets (i.e., complements of open sets) they are known as the so-called *convex geometries*; see Theorem 1.3 in Chapter 3 of Korte, Lovasz, and Schrader (1991). The following definition is more appropriate for the purposes of this note as it directly refers to open sets.

Definition. An *antimatroid* on a set X is a pre-topology $\mathcal{T}$ on X possessing the following property: for every non-empty open set $U \in \mathcal{T}$, a point $x \in U$ exists such that the set $U - \{x\}$ is open as well.

In the following we will write $U - x$ instead of $U - \{x\}$.

In terms of choice functions (see Section 6), antimatroids are equivalent to the so-called *Plott choice functions*; see Danilov and Koshevoy (2005). In the theory of knowledge spaces, antimatroids are known as *learning spaces* or *well-graded knowledge spaces*.

Antimatroids possess a property formally stronger than the property given in the definition.

Lemma 6. *Let* $\mathcal{T}$ *be an antimatroid on* X, *and let* $V \subsetneqq U$ *be two distinct open sets. Then a point* $x \in U - V$ *exists such that* $U - x$ *is open.*

Proof. We shall use an induction on the size of the set $U - V$. If $U - V$ consists of a single point, the assertion of the lemma is obviously true. For the general case, let x be an arbitrary point of $U - V$. Let $U_x \subset U$ be a minimal open set containing the point x.

Note that the set $U_x - x$ is open. This is obvious if $U_x = \{x\}$. Otherwise, there exists $y \in U_x$ such that $U_x - y$ is open. If $y \neq x$, then $U_x - y$ is a proper subset of U_x containing the point x. This contradicts the minimality of U_x. Hence $y = x$.

Consider now two cases. If $U_x \cup V = U$, then $U - x = (U_x - x) \cup V$, as the union of the open sets $U_x - x$ and V, is open. If $V' = U_x \cup V$ is strictly smaller than U, we obtain the pair of open sets $V' \subset U$ with strictly smaller difference $U - V'$. By the induction assumption, there exists a point $x' \in U - V'$ such that $U - x'$ is open. □

Proposition 5. *If $(\mathcal{T}_i,\ i \in I)$ is a family of antimatroids on a set X, then $\vee_i \mathcal{T}_i$ is an antimatroid as well.*

(In the context of the Plott choice functions this proposition is known as Blair's theorem.)

Proof. It suffices to show the assertion for two antimatroids $\mathcal{T}_1$ and $\mathcal{T}_2$. An open set U of the pre-topology $\mathcal{T}_1 \vee \mathcal{T}_2$ has the form $U_1 \cup U_2$, where $U_i \in \mathcal{T}_i$. Since $\mathcal{T}_1$ is an antimatroid, there exists a point $x \in U_1$ such that $U_1 - x$ is open with respect to $\mathcal{T}_1$. If $x \notin U_2$, then $U - x = (U_1 - x) \cup U_2$ is open with respect to $\mathcal{T}$, and the proof is completed. If $x \in U_2$, then $U = (U_1 - x) \cup U_2$, and U_1 is reduced. Continuing this process (and making use of the finiteness of U_1), we can assume that U_1 is empty. But then $U = U_2$, and we are done because $\mathcal{T}_2$ is an antimatroid. □

An intersection of antimatroids is not generally an antimatroid. However, since the trivial pre-topology is an antimatroid, the poset $\mathbf{AMat}(X)$ of antimatroids is a (non-distributive) lattice; see Danilov and Koshevoy (2005).

After these preliminaries we return to the discussion of restrictions, extensions and gluings.

Restriction

Let Z be a subset of X.

Proposition 6. *If $\mathcal{T}$ is an antimatroid on X, the pre-topologies $\mathcal{T}|Z$ and $j^*(\mathcal{T})$ are antimatroids on Z.*

The assertion about $\mathcal{T}|Z$ is the Projection Theorem of Falmagne (2008).

Proof. The assertion about j^* is simple and is a particular case of a more general result by Danilov and Koshevoy (2005) or Danilov and Koshevoy (2009). Therefore we only show that $\mathcal{T}|Z$ is an antimatroid. Let V be a (non-empty) open set with respect to $\mathcal{T}|Z$. This means that $V = U \cap Z$, where $U \in \mathcal{T}$. At this point we can assume that U is a minimal open set with the property $V = U \cap Z$. Since $\mathcal{T}$ is an antimatroid, $U - x$ is open for some point $x \in U$. Then $(U - x) \cap Z$ is strictly smaller than $V = U \cap Z$, hence $x \in V$ and $V - x = (U - x) \cap Z$ is open with respect to $\mathcal{T}|Z$. □

Extension

Let $\mathcal{P}$ be an antimatroid on Z. Then $\mathcal{T} = j_\#(\mathcal{P})$ is an antimatroid on X. Indeed, suppose $U = V \cup W$ (with $V \in \mathcal{P}$ and $W \subset X - Z$) is open with respect to $\mathcal{T}$. If W is

not empty, we can remove any element from W. If W is empty, then $U = V$ and we can remove some element from V.

Gluing

Let $X = X_1 \cup X_2$, let $\mathcal{T}_i$ be antimatroids on X_i compatible on the intersection X_{12}, and let $\mathcal{T} = \mathcal{T}_1 \star \mathcal{T}_2$ be the gluing of $\mathcal{T}_1$ and $\mathcal{T}_2$. If X_1 and X_2 do not intersect (i.e., $\mathcal{T}$ is the direct sum of $\mathcal{T}_1$ and $\mathcal{T}_2$), then obviously $\mathcal{T}$ is an antimatroid. Falmagne (2008) asserts this in the particular case when $\mathcal{T}_2$ is isomorphic to $\mathcal{T}_1$, but this restriction is redundant.

In the general case, when X_1 and X_2 do intersect, $\mathcal{T}$ need not be an antimatroid.

Example 4. Let $X_1 = \{a, b, c, c'\}$, and let the pre-topology $\mathcal{T}_1$ on X_1 be given by the following five minimal open sets: a, ab, abc, c', $c'c$ (other open sets are obtained from these by taking unions). It is easy to see that $\mathcal{T}_1$ is an antimatroid.

Let $X_2 = \{b, b', c, d\}$, and let the pre-topology $\mathcal{T}_2$ on X_2 be given by the following five open sets: b', $b'b$, d, dc, dcb. Then $\mathcal{T}_2$ is an antimatroid as well.

On the intersection $\{b, c\}$ both the pre-topologies induce the same discrete topology. However, the pre-topology $\mathcal{T} = \mathcal{T}_1 \star \mathcal{T}_2$ is not an antimatroid. Indeed, consider the set $U = \{a, b, c, d\}$ in X. This set is open. The set $U - a$ is not open; its intersection with X_1 is equal to X_{12} and is not open in X_1. Analogously, $U - d$ is not open. The set $U - b$ also is not open; its intersection with X_1 is equal to $\{a, c\}$ and is not open in X_1. Analogously, $U - c$ is not open.

This example shows that additional requirements have to be imposed in order to guarantee that the amalgam of two antimatroids is an antimatroid again. One such a requirement, related to the notion of a separator, is introduced below.

Separators

Suppose that X_{12} is a separator for the pre-topology $\mathcal{T}_2$. As can be easily shown (see the proof of Theorem 1), X_1 is a separator for the gluing $\mathcal{T}$.

Theorem 2. *Let $\mathcal{T}_i$ $(i = 1, 2)$ be antimatroids on X_i compatible on $X_{12} = X_1 \cap X_2$. Let X_{12} be a separator for $\mathcal{T}_2$. Then the gluing $\mathcal{T} = \mathcal{T}_1 \star \mathcal{T}_2$ is an antimatroid.*

Proof. Suppose U is an open set with respect to $\mathcal{T}$, so $U \cap X_i$ are open with respect to $\mathcal{T}_i$. We have to show that $U - x$ is open for some point $x \in U$.

Consider two cases.

First case: U lies in X_1. Since $\mathcal{T}_1$ is an antimatroid, there exists a point $x \in U$ such that $U - x$ is open with respect to $\mathcal{T}_1$. We assert that $U - x$ is open with respect to $\mathcal{T}$. To prove this we have to show that $(U - x) \cap X_2$ is open with respect to $\mathcal{T}_2$.

If $x \notin X_2$, the latter follows from $(U - x) \cap X_2 = U \cap X_2$ being open with respect to $\mathcal{T}_2$. Therefore we can assume that $x \in X_2$. Then $(U - x) \cap X_2 = (U - x) \cap X_{12}$ is open with respect to $\mathcal{T}_1|X_{12}$, hence with respect to $\mathcal{T}_2|X_{12}$. Since X_{12} is a separator for $\mathcal{T}_2$, this set is open with respect to $\mathcal{T}_2$.

Second case: U does not lie in X_1. So, $U \cap X_2$ is strictly larger than $U \cap X_{12}$. The first set $U \cap X_2$ is open with respect to $\mathcal{T}_2$. Since X_{12} is a separator for $\mathcal{T}_2$, the second set $U \cap X_{12} = (U \cap X_2) \cap X_{12}$ is also open with respect to $\mathcal{T}_2$. Due to Lemma 6, a point $x \in U \cap (X_2 - X_{12})$ exists such that $(U \cap X_2) - x = (U - x) \cap X_2$ is open with respect to $\mathcal{T}_2$. Since $(U - x) \cap X_1 = U \cap X_1$ is open in X_1, $U - x$ is open with respect to $\mathcal{T}$. $\square$

Corollary. *Let $X = X_1 \cup ... \cup X_n$ be a covering of X, and $\mathcal{T}_i$ be pre-topologies on X_i, $i = 1, ..., n$. Suppose that*

a) the pre-topologies $\mathcal{T}_i$ are strictly compatible;

b) every $\mathcal{T}_i$ is an antimatroid on X_i.

Then the gluing $\mathcal{T} = \mathcal{T}_1 \star ... \star \mathcal{T}_n$ is an antimatroid.

Proof. Let $X' = X_1 \cup ... \cup X_{n-1}$ and $\mathcal{T}' = \mathcal{T}_1 \star ... \star \mathcal{T}_{n-1}$. By induction, $\mathcal{T}'$ is an antimatroid on X'. Due to Theorem 1, $\mathcal{T}'$ and $\mathcal{T}_n$ are strictly compatible and $\mathcal{T} = \mathcal{T}' \star \mathcal{T}_n$. Theorem 2 now implies that $\mathcal{T}$ is an antimatroid. □

Decomposition and assembly

Let $\mathcal{T}$ be an antimatroid on X, and let Z be a separator for $\mathcal{T}$. Above we considered the pre-topology $\mathcal{T}_0 = \mathcal{T}|Z$ on Z and, for every $V \in \mathcal{T}_0$, the pre-topology $\mathcal{T}_V = \{W \subset Y,\ W \cup V \in \mathcal{T}\}$ on $Y = X - Z$. We know from Proposition 5 that $\mathcal{T}_0$ is an antimatroid. We assert here that *all pre-topologies $\mathcal{T}_V$ are antimatroids* (this assertion also is contained in Falmagne's Projection Theorem).

Indeed, let U be an open set in Y, with respect to $\mathcal{T}_V$. That is, $U \cup V \in \mathcal{T}$. Since $\mathcal{T}$ is an antimatroid, there exists a point x in $U \cup V$ such that $U \cup V - x$ also belongs to $\mathcal{T}$. If $x \in U$, $U - x$ belongs to $\mathcal{T}_V$. If $x \in V$, $U \in \mathcal{T}_{V-x}$. By induction (on the size of V), there exists $y \in U$ such that $U - y$ belongs to $\mathcal{T}_{V-x}$. But $\mathcal{T}_{V-x} \subset \mathcal{T}_V$, due to the isotonicity of $\mathcal{T}_V$ as a function of V. Therefore $U - y \in \mathcal{T}_V$.

Conversely, let $\mathcal{T}_0$ be an antimatroid on Z, and let $(\mathcal{T}_V, V \in \mathcal{T}_0)$ be an isotone family of antimatroids on $Y = X - Z$. Then *the pre-topology $\mathcal{T}$, given by the formula (2), is an antimatroid.* Indeed, suppose $W = U \cup V$ (where $V \in \mathcal{T}_0$ and $U \in \mathcal{T}_V$) is a non-empty open set with respect to $\mathcal{T}$. We have to check that W contains a point x such that $W - x$ is open as well.

If U is not empty, there exists $y \in U$ such that $U - y \in \mathcal{T}_V$ (since $\mathcal{T}_V$ is an antimatroid). Then $W - y = V \cup (U - y)$ belongs to $\mathcal{T}$.

If U is empty, $W = V$. Since $\mathcal{T}_0$ is an antimatroid, there exists $z \in V$ such that $V - z$ belongs to $\mathcal{T}_0$. Then $W - z = V - z$ belongs to $\mathcal{T}$.

To sum up: The previously described construction gives a natural bijection between

a) the set of antimatroids $\mathcal{T}$ on X for which Z is a separator, and

b) the set of data $(\mathcal{T}_0,\ \varphi : \mathcal{T}_0 \to \mathbf{AMat}(X - Z))$, where $\mathcal{T}_0$ is an antimatroid on Z, and the mapping φ is isotone.

5 Neighborhood systems

The concepts and results that we have discussed before hold for objects more primitive and fundamental than pre-topologies. The previous discussion can be generalized in two seemingly different but essentially equivalent languages (see Danilov & Koshevoy, 2009): in the language of neighborhood systems and in that of heritage choice functions. We begin with neighborhood systems as they are more topological in nature.

Filters

Definition. An (order) *filter* on a set X is a subset $\mathcal{F} \subset 2^X$ possessing the following property: if $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

For example, there are three filters on the set $X = \{a\}$:

$$\emptyset,\ \{a\},\ \{\emptyset, a\} = 2^X.$$

There are six filters on $X = \{a, b\}$:

$$\emptyset, \{ab\}, \{a, ab\}, \{b, ab\}, \{a, b, ab\} \text{ and } \{\emptyset, a, b, ab\} = 2^X.$$

Let $\Phi(X)$ denote the set of filters on X. Endowed with the natural order, the poset $\Phi(X)$ is a distributive lattice (with operations $\cup$ and $\cap$). The minimal filter is empty, the maximal is equal to 2^X.

Neighborhood systems

Given a point $x \in X$, we say that a filter $\mathcal{F}$ is a *filter of neighborhoods* of x if $x \in F$ for every $F \in \mathcal{F}$. The set (poset) of such filters is denoted by $\Phi_x(X)$; it is naturally isomorphic to the poset $\Phi(X - x)$.

Definition. A *neighborhood system* (NS) on a set X is a family $\mathcal{N} = (\mathcal{N}_x,\ x \in X)$, where $\mathcal{N}_x \in \Phi_x(X)$. The elements of $\mathcal{N}_x$ are called *neighborhoods* of the point $x \in X$.

The set (poset) of NSs on X is denoted by $\mathbf{NS}(X)$; it is isomorphic to the direct product $\prod_x \Phi_x(X) \simeq \prod_x \Phi(X - x)$. As the direct product of distributive lattices it is a distributive lattice as well. Figure 2 shows the lattice $\mathbf{NS}(X)$ for a two-element set $X = \{a, b\}$.

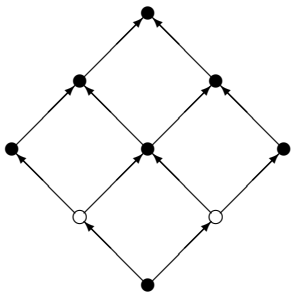

Figure 2. The lattice $\mathbf{NS}(X)$ for a two-element set $X = \{a, b\}$.
Black nodes represent pre-topologies (cf. Figure 1).

Pre-topologies give the most important examples of NSs. Let $\mathcal{T}$ be a pre-topology on X. An *open neighborhood* of a point $x \in X$ is an open set $U \in \mathcal{T}$ containing x. A *neighborhood* of a point x is an arbitrary set $N \subset X$ which contains an open neighborhood of x. Thus, every pre-topology $\mathcal{T}$ generates some NS $\mathcal{N} = \mathcal{N}(\mathcal{T})$; this construction gives a natural (and isotone) mapping

$$\mathbf{PTop}(X) \to \mathbf{NS}(X).$$

It is easy to see that this mapping commutes with $\vee$. In Figure 2, black nodes represent those NSs (on the two-element set X) that are generated by pre-topologies.

One can also consider the reverse situation. Let $\mathcal{N}$ be an NS on X. We say that a subset $U \subset X$ is *open* (with respect to $\mathcal{N}$) if $U \in \mathcal{N}_x$ for every point $x \in U$. It is easy to

see that arbitrary unions of open sets are open; so we obtain a pre-topology $\mathcal{T} = \mathcal{T}(\mathcal{N})$ on X. This gives a mapping (which commutes with $\wedge$)

$$\mathbf{NS}(X) \to \mathbf{PTop}(X).$$

It can be shown directly that the second mapping is a retraction of the first one; so $\mathcal{T} = \mathcal{T}(\mathcal{N}(\mathcal{T}))$ for every pre-topology $\mathcal{T}$. In particular, the mapping $\mathbf{PTop}(X) \to \mathbf{NS}(X)$ is an embedding; this allows one to consider NSs a natural generalization of pre-topologies. Of course, there are many more NSs than pre-topologies. 'Pre-topological' NSs can be characterized by the following property (see Danilov & Koshevoy, 2009): for every neighborhood N of a point x, the *interiority* $\mathrm{int}(N) = \{y \in N,\ N \in \mathcal{N}_y\}$ of N is a neighborhood of x as well.

In the following, we aim at transferring to NSs the notions of restriction, extension, gluing and separator developed before for pre-topologies.

Restriction, extension

Let Z be a subset of X. Since an NS is a family of filters, the restriction and extension constructions are reduced to corresponding operations with filters. How does one go from filters on X to filters on Z, and vice versa?

Let $\mathcal{F}$ be a filter on X. Denote by $\mathcal{F}|Z$ the filter on Z consisting of subsets of the form $F \cap Z$, where $F \in \mathcal{F}$. If $\mathcal{F}|Z \subset \mathcal{F}$ (i.e., $F \cap Z \in \mathcal{F}$ for every $F \in \mathcal{F}$), we say that the filter $\mathcal{F}$ *is concentrated* on Z.

Let $\mathcal{G}$ be a filter on Z. Denote by $\mathcal{F} = j_{\#}(\mathcal{G})$ the filter on X consisting of sets $F \subset X$ such that $F \cap Z \in \mathcal{G}$. Clearly, $\mathcal{G} \subset \mathcal{F}$; in fact, $\mathcal{F}$ is the minimal filter on X which contains $\mathcal{G}$. It is clear that $\mathcal{F}|Z = \mathcal{G}$.

Lemma 7. *The filter $j_{\#}(\mathcal{G})$ is concentrated on Z. Conversely, if a filter $\mathcal{F}$ is concentrated on Z, then $\mathcal{F} = j^{\#}(\mathcal{F}|Z)$.* $\square$

Let us go from filters to NSs.

Restriction. Suppose that $\mathcal{N} = (\mathcal{N}_x,\ x \in X)$ is an NS on X, and let $Z \subset X$. The *restriction* of $\mathcal{N}$ on Z is the family $\mathcal{N}|Z = (\mathcal{N}_z|Z,\ z \in Z)$. In other words, we retain points only from Z and change every neighborhood filter $\mathcal{N}_z$ to its restriction $\mathcal{N}_z|Z$. It is easy to see that this operation is compatible with the corresponding operation for pre-topologies.

Extension. First, we introduce the maximal extension $j_{\#}$. Let $\mathcal{M} = (\mathcal{M}_z,\ z \in Z)$ be an NS on Z. Its *extension* to X is the following NS on X: $\mathcal{N} = j_{\#}(\mathcal{M}) = (\mathcal{N}_x,\ x \in X)$, where

$$\mathcal{N}_x = j_{\#}(\mathcal{M}_x) \text{ if } x \in Z, \text{ and } \mathcal{N}_x = \{N \subset X,\ x \in N\} \text{ if } x \notin Z.$$

It is clear that $\mathcal{N}$ is the maximal NS on X satisfying the property $\mathcal{N}|Z = \mathcal{M}$. This operation is compatible with the corresponding operation for pre-topologies.

Similarly, one can define the minimal extension j_*. For an NS $\mathcal{M} = (\mathcal{M}_z,\ z \in Z)$ on Z and a point $x \in X$,

$$j_*(\mathcal{M})_x = \begin{cases} j_{\#}(\mathcal{M}_x), & \text{if } x \in Z, \\ \emptyset, & \text{if } x \notin Z. \end{cases}$$

Gluing

Let $(X_i, i \in I)$ be a covering of X, and let $\mathcal{N}_i$ be NSs on X_i compatible on the intersections $X_{ij} = X_i \cap X_j$ (in the sense that $\mathcal{N}_i|X_{ij} = \mathcal{N}_j|X_{ij}$). Define the following NS $\mathcal{N} = (\mathcal{N}_x, x \in X)$ on X:

$$\mathcal{N}_x = \{N \subset X, \text{ such that } N \cap X_i \in \mathcal{N}_{ix} \text{ for every } i \in I\}.$$

Proposition 7. *It holds that $\mathcal{N}|X_i = \mathcal{N}_i$, and $\mathcal{N}$ is the largest NS on X with this property.*

Proof. The inclusion $\mathcal{N}|X_i \subset \mathcal{N}_i$ is obvious, so we focus on the inverse inclusion. Let $x \in X_i$. We have to prove that $\mathcal{N}_{ix} \subset \mathcal{N}_x|X_i$. Let F be a neighborhood of x in X_i; we have to show that there exists a neighborhood G of the point x in X such that $G \cap X_i = F$.

To construct G, consider the intersection $F \cap X_{ij}$ for an index $j \in I$. Since the filters $(\mathcal{N}_i)_x$ and $(\mathcal{N}_j)_x$ are compatible on X_{ij}, $F \cap X_{ij} = F \cap X_j$ belongs to the restriction of $(\mathcal{N}_j)_x$ on X_{ij}. This means that there exists $G_j \in (\mathcal{N}_j)_x$ such that $G_j \cap X_{ij} = G_j \cap X_i$ is equal to $F \cap X_j$. Therefore $(F \cup G_j) \cap X_i = F$ and $(F \cup G_j) \cap X_j = G_j$.

Define $G = F \cup (\cup_{j \neq i} G_j)$. Then, $G \cap X_j \supset G_j$ and $G \cap X_j$ belongs to $(\mathcal{N}_j)_x$ for every $j \neq i$. Also, $G \cap X_i \supset F$ and $G \cap X_i$ belongs to $(\mathcal{N}_i)_x$. Hence $G \in \mathcal{N}_x$. Moreover, $G \cap X_i = \cup_{j \neq i}(F \cup G_j) \cap X_i = F$. This proves that $\mathcal{N}|X_i = \mathcal{N}_i$.

The second assertion of the proposition is obvious. □

Thus, in contrast to pre-topologies, the gluing of NSs always exists.

Separators

Let $\mathcal{N}$ be an NS on X. A subset $Z \subset X$ is called a *separator* for $\mathcal{N}$ if, for every point $z \in Z$, the neighborhood filter $\mathcal{N}_z$ is concentrated on Z. The intersection (of a finite number) and the union (of an arbitrary number) of separators are separators as well. This notion is compatible with the corresponding notion for pre-topologies.

Proposition 8. *Let $\mathcal{N}$ be a neighborhood system generated by a pre-topology $\mathcal{T}$. Then, a subset $Z \subset X$ is a separator for $\mathcal{T}$ if and only if Z is a separator for $\mathcal{N}$.*

Proof. Let Z be a separator for $\mathcal{T}$, and $x \in Z$. We have to check that, for every neighborhood $N \in \mathcal{N}_x$, the set $N \cap Z$ is a neighborhood of the point x. The neighborhood N contains an open neighborhood U, $x \in U \subset N$. Since Z is a separator for $\mathcal{T}$, $U \cap Z$ is open. Since $x \in U \cap Z \subset N \cap Z$, $N \cap Z$ is a neighborhood of the point x.

Conversely, suppose that all filters $\mathcal{N}_x$ (at $x \in Z$) are concentrated on Z. Let U be an open set, $U \in \mathcal{T}$. We have to check that $U \cap Z \in \mathcal{T}$, that is, $U \cap Z \in \mathcal{N}_x$ for every point $x \in U \cap Z$. Since U is open, $U \in \mathcal{N}_x$ for every point $x \in U \cap Z$. Since the filter $\mathcal{N}_x$ is concentrated on Z, $U \cap Z$ is a neighborhood of x. □

Decomposition

As in the case of pre-topologies, separators allow decomposing NSs into 'blocks.' Previously, we have confined ourselves to the simplest situation with one separator. Now we elaborate the general case.

Let $\mathbf{S}$ be a finite set of separators for an NS $\mathcal{N}$ on X, which contains the whole set X and is stable with respect to intersection (cf. Proposition 3). For any $S \in \mathbf{S}$, denote by S° the set S minus all strictly smaller separators in $\mathbf{S}$. The family $(S^\circ, S \in \mathbf{S})$ forms a partition of X, $X = \coprod_{S \in \mathbf{S}} S^\circ$, where $\coprod$ denotes disjoint union. Assign to every $S \in \mathbf{S}$ the following NS $\mathcal{N}^S$ on S: $\mathcal{N}^S_x = \mathcal{N}_x|S$ if $x \in S^\circ$, and $\mathcal{N}^S_x = \emptyset$ otherwise.

Proposition 9. *It holds that* $\mathcal{N} = \bigcup_S j_{S*}(\mathcal{N}^S)$*, where* S *ranges over* $\mathbf{S}$ *and* j_S *is the canonical embedding of* S *in* X*.*

Proof. Let x be an arbitrary point of X. We have to show that $\mathcal{N}_x = \bigcup_S j_{S*}(\mathcal{N}^S)_x$. If $x \notin S^\circ$, the corresponding summand $j_{S*}(\mathcal{N}^S)_x$ is empty. Therefore only one summand $j_{S*}(\mathcal{N}^S)_x$, the one for S with $x \in S^\circ$, remains. This summand is equal to $\mathcal{N}_x$, as we show next.

If $N \in \mathcal{N}_x$, $N \cap S \in \mathcal{N}_x^S$; hence $N \in j_{S*}(\mathcal{N}^S)_x$. Conversely, let $N \in j_{S*}(\mathcal{N}^S)_x$, that is, $N \cap S \in \mathcal{N}_x^S = \mathcal{N}_x|S$. The latter means that $N \cap S$ is equal to $N' \cap S$ for some neighborhood N' of x (i.e., $N' \in \mathcal{N}_x$). Since S is a separator for the filter $\mathcal{N}_x$, $N' \cap S$ belongs to $\mathcal{N}_x$, and $N \cap S \in \mathcal{N}_x$. Since N contains $N \cap S$, $N \in \mathcal{N}_x$. $\square$

One can consider the equality $\mathcal{N} = \bigcup_S j_{S*}(\mathcal{N}^S)$ as a decomposition of the NS $\mathcal{N}$ into simpler blocks $j_{S*}(\mathcal{N}^S)$, which correspond to the separators S in $\mathbf{S}$. Each block $j_{S*}(\mathcal{N}^S)$ is situated on S° (in the sense that the filters $j_{S*}(\mathcal{N}^S)_x$ are non-empty only if $x \in S^\circ$). Moreover, each filter $j_{S*}(\mathcal{N}^S)_x$ is concentrated on S.

Synthesis

One can construct (assemble) a large NS from finer NSs. Suppose we have a finite topology $\mathbf{S}$ on X. For any $S \in \mathbf{S}$, let S° denote the set S minus all strictly smaller sets in $\mathbf{S}$. Let $\mathcal{N}^S$ be an NS on S for any $S \in \mathbf{S}$. Let $\mathcal{N}_x^S = \emptyset$ for $x \notin S^\circ$. Define the following NS on X:

$$\mathcal{N} = \bigcup_{S \in \mathbf{S}} j_{S*}(\mathcal{N}^S).$$

Proposition 10. a) *Every* $S \in \mathbf{S}$ *is a separator for* $\mathcal{N}$*.*

b) *For* $x \in S^\circ$*, it holds that* $\mathcal{N}_x = j_{S*}(\mathcal{N}_x^S)$*.*

Proof. Part b) is proved as before. To prove a), let $S \in \mathbf{S}$ and $x \in S$. We have to show that the filter $\mathcal{N}_x$ (which is equal to $j_{S*}(\mathcal{N}_x^S)$ due to b)) is concentrated on S. By definition of $(j_S)_*$, we have $(j_S)_*(\mathcal{N}_x^S) = (j_S)_\#(\mathcal{N}_x^S)$. The statement then follows from Lemma 7. $\square$

A particular case

To better understand Propositions 9 and 10, let us return to the simplest situation when $\mathbf{S}$ consists of three elements: the empty set $\emptyset$, the whole set X, and a subset Z. Suppose that $\mathcal{N}$ is an NS on X and Z is a separator for $\mathcal{N}$. Then $\mathcal{N}^Z = \mathcal{N}|Z$, and $\mathcal{N}^X$ is the following NS on X: $\mathcal{N}_x^X = \emptyset$ if $x \in Z$, and $\mathcal{N}_x^X = \mathcal{N}_x$ if $x \in X - Z$. We obtain the decomposition

$$\mathcal{N} = j_*(\mathcal{N}^Z) \cup \mathcal{N}^X.$$

The initial NS $\mathcal{N}$ 'is composed' of the NS $\mathcal{N}^Z$ on Z and the NS $\mathcal{N}^X$ on X which is trivial on Z (i.e., $\mathcal{N}_z^X = \emptyset$ for $z \in Z$).

Conversely, suppose we have an NS $\mathcal{N}^Z$ on Z and an NS $\mathcal{N}^X$ on X which is trivial on Z. Define the NS $\mathcal{N} = j_*(\mathcal{N}^Z) \cup \mathcal{N}^X$ on X. Then,

a) Z is a separator for $\mathcal{N}$,

b) $\mathcal{N}|Z = \mathcal{N}^Z$,

c) $\mathcal{N}_y^X = \mathcal{N}_y$ for every $y \in X - Z$.

6 Choice functions

Next we recapitulate the preceding section in the language of choice functions.

A *choice function* (CF) on a set X is a mapping $f : 2^X \to 2^X$ such that $f(A) \subset A$ for every $A \subset X$. A CF f is called *heritage* if $B \subset A$ implies $f(A) \cap B \subset f(B)$. In other words, if an element b belongs to the smaller set B and is chosen from the larger set A, $b \in f(A)$, then it should be chosen from the smaller set B as well, $b \in f(B)$. In the following we will focus on heritage CFs (HCFs).

Lemma 8. *A choice function f is heritage if and only if $f(A) \cap B \subset f(A \cap B)$ for every A and B.*

Proof. Suppose that F is a heritage CF, and $A, B \subset X$. Since $f(A) \subset A$, we have the equality $f(A) \cap B = f(A) \cap (A \cap B)$. Applying the heritage condition to the sets $A \cap B \subset A$ we obtain the inclusion $f(A) \cap (A \cap B) \subset f(A \cap B)$.

Conversely, let $B \subset A$. Then $f(A) \cap B \subset f(A \cap B) = f(B)$. $\square$

The set (poset) $\mathbf{H}(X)$ of HCFs on X is stable with respect to operations $\cup$ and $\cap$; hence it is a distributive lattice.

Let $\mathcal{N} = (\mathcal{N}_x, x \in X)$ be a neighborhood system on X. Define the following CF $f = f_{\mathcal{N}}$, setting (for $A \subset X$)

$$f(A) = \{a \in A, \text{ there exists } N \in \mathcal{N}_a \text{ such that } N \cap A = \{a\}\}.$$

In other words, we choose 'isolated' points from A. More precisely, a point a is chosen from A if the filter $\mathcal{N}_a|A$ contains $\{a\}$ (or is the maximal filter). A direct inspection shows that f is an HCF. It is more important that every HCF can be obtained in this way. Indeed, let f be an HCF on X, and $x \in X$. An *f-neighborhood* of the point x is a subset $N \subset X$ such that $x \in N$ and $x \in f(x \cup (X - N))$. The heredity of f gives that the set $\mathcal{N}(f)_x = \mathcal{N}_x$ of all f-neighborhoods of x is a filter. It is easy to see that this construction is inverse to the previous, and that it gives a bijection (and isomorphism of posets) between $\mathbf{H}(X)$ and $\mathbf{NS}(X)$. Thus, all above considered notions for NSs can be transferred to HCFs.

Restriction

Let f be an HCF on X and $Z \subset X$. The *restriction* of f to Z, $f|Z$, is given by the formula (where B is a subset of Z):

$$(f|Z)(B) = f(B),$$

and $f|Z$ also is an HCF.

Extension

Let g be an HCF on Z. The extension $j_\#(g)$ is given by

$$j_\#(g)(A) = g(A \cap Z) \cup (A - Z).$$

The CF $j_\#(g)$ is the maximal HCF on X which extends g. The minimal extension $j_*(g)$ is given by $j_*(g)(A) = g(A \cap Z)$.

Gluing

Suppose $X = \cup_i X_i$, and let f_i be HCFs on X_i compatible on the intersections $X_{ii'} = X_i \cap X_{i'}$ (i.e., $f_i|X_{ii'} = f_{i'}|X_{ii'}$). The formula

$$f = \cap_i j_{i\#}(f_i) \tag{4}$$

gives the *gluing* of the choice functions f_i. The CF f is heritage and in fact is the maximal HCF which extends f_i.

Separators

Let f be an HCF on X. A subset $Z \subset X$ is called a *separator* for f if

$$f(A) \cap Z = f(A \cap Z)$$

for every subset $A \subset X$. Due to Lemma 8, this condition is equivalent to the condition: $f(A \cap Z) \subset f(A)$ for every $A \subset X$.

Examples. 1) The empty subset and the whole set X are trivial separators.

2) Every subset $Z \subset f(X)$ is a separator for f. Indeed, for every A we have $f(A \cap Z) \subset A \cap Z \subset A \cap f(X) \subset f(A)$.

3) The *support* $\mathrm{supp}(f) := \{s \in X,\ s \in f(\{s\})\}$ of an HCF f is not a separator in general. Here is an example: $X = \{x, y\}$, $f(x) = x$, $f(y) = f(x, y) = \emptyset$. The CF f is heritage with the support $S = \{x\}$. If we take $A = X$, then

$$f(X) \cap S = \emptyset,\ f(X \cap S) = f(S) = S.$$

4) The complementation to the support is a separator.

The assertion follows from the following remark. Let a set Z do not intersect the support of an HCF f. Then $f(Z) = \emptyset$. Indeed, for every $z \in Z$, $f(Z) \cap \{z\} \subset f(\{z\}) = \emptyset$.

5) Let g be an HCF on Z and $f = j_*(g)$, where $j : Z \to X$ is the canonical embedding. Then Z is a separator for f. Indeed,

$$f(A) = g(A \cap Z) = f(A \cap Z).$$

Similarly, Z is a separator for the maximal extension $j_\#(g)$.

6) If Z is a separator for HCFs f and g, then Z is a separator for $f \cup g$ and $f \cap g$.

Decomposition and assembly

The set of separators is a topology. This topology reflects a structure of an HCF f and allows to decompose f into finer parts (here we assume that X is a finite set). Assign to every separator S the following CF f^S on S: for $B \subset S$,

$$f^S(B) = f(B) \cap S^\circ.$$

Obviously, every f^S is an HCF, and the following formula of decomposition holds (where A is an arbitrary subset of X):

$$f(A) = \coprod_S f^S(A \cap S). \tag{5}$$

Indeed, $f^S(A \cap S) = f(A \cap S) \cap S^\circ = (f(A) \cap S) \cap S^\circ = f(A) \cap S^\circ$ (in the second equality we use that S is a separator for f). Since $X = \coprod_S S^\circ$, we have $f(A) = f(A) \cap (\coprod_S S^\circ) = \coprod_S (f(A) \cap S^\circ) = \coprod_S f^S(A \cap S)$.

In other words, $f(A)$ is represented as a disjoint 'sum' of smaller pieces $f^S(A \cap S)$, and each piece corresponds to own separator S.

Now about a synthesis. Suppose we have a topology on X with open sets S, and an HCF f^S on every S which takes values in S°. Then the formula (5) gives HCF f on X, and all S are its separators.

Here is the simplest particular case: f is an HCF on X and $Z \subset X$ is a separator for f. Then (for $B \subset Z$) $f^Z(B) = f(B)$, and (for $A \subset X$) $f^X(A) = f(A) - Z$. The decomposition takes the form

$$f(A) = f(A \cap Z) \cup (f(A) - Z).$$

Conversely, suppose we have an HCF f^Z on Z, an HCF f^X on X, and $f^X(A)$ does not intersect Z for every $A \subset X$. Define a choice function f by the formula (where $A \subset X$)

$$f(A) = f^Z(A \cap Z) \cup f^X(A).$$

Then f is an HCF and Z is a separator for f.

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