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Tropical Plücker functions and their bases

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Abstract. In this paper we study functions on a subset $B \subset \mathbb{Z}^n$ that obey tropical analogs of classical Plücker relations on minors of a matrix. The most general set B that we deal with is of the form $\{x \in \mathbb{Z}^n : 0 \le x \le a,\ m \le x_1 + \ldots + x_n \le m'\}$ (a rectangular integer box 'truncated from below and above'). We construct a basis for the set $\mathcal{TP}$ of tropical Plücker functions on B, a subset $\mathcal{B} \subseteq B$ such that the restriction map $\mathcal{TP} \to \mathbb{R}^{\mathcal{B}}$ is bijective. Also we characterize, in terms of the restriction to the basis, the classes of submodular, so-called skew-submodular, and discrete concave functions in $\mathcal{TP}$, discuss a tropical analogue of the Laurentness property, and present other results.

Keywords: Plücker relations, tropicalization, octahedron recurrence, submodular function, rhombic tiling, Laurent phenomenon

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1. Introduction

There are well-known algebraic relations on minors of a matrix. For a positive integer n, let $[n]$ denote the ordered set $\{1, 2, \ldots, n\}$. For an $n \times n$ matrix M and a set $J \subseteq [n]$, let Δ_J denote the determinant of the submatrix of M formed by the column set J and the row set $\{1, \ldots, |J|\}$. Then: (i) for any triple $i < j < k$ of elements of $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$,

$$\Delta_{Xik}\Delta_{Xj} = \Delta_{Xij}\Delta_{Xk} + \Delta_{Xi}\Delta_{Xjk};$$

and (ii) for any quadruple $i < j < k < \ell$ in $[n]$ and any $X \subseteq [n] - \{i, j, k, \ell\}$,

$$\Delta_{Xik}\Delta_{Xj\ell} = \Delta_{Xij}\Delta_{Xk\ell} + \Delta_{Xi\ell}\Delta_{Xjk},$$

where for brevity we write Xij instead of $X \cup \{i\} \cup \{j\}$ and so on. These equalities represent simplest cases of so-called *Plücker's relations.* (About classical Plücker's relations see, e.g., [**9**]).

Relations as above can be stated in an abstract form; namely, one can consider a function g on the *Boolean cube* $\{0, 1\}^{[n]}$ (or on an appropriate part of it) and impose the conditions

$$g(Xik)g(Xj) = g(Xij)g(Xk) + g(Xi)g(Xjk),$$

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and/or

$$g(Xik)g(Xjl) = g(Xij)g(Xk\ell) + g(Xi\ell)g(Xjk),$$

for X, i, j, k, ℓ as above (identifying a subset of $[n]$ with the corresponding (0,1)-vector). Such a function is said to be an *algebraic Plücker function*, or an *AP-function.*

Tropical analogs of these relations appear when multiplication is replaced by addition and addition is replaced by taking maximum; they are viewed as

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\}, \tag{1}$$

and

$$f(Xik) + f(Xjl) = \max\{f(Xij) + f(Xk\ell),\ f(Xi\ell) + f(Xjk)\}, \tag{2}$$

(see, e.g., [**1**, Sec. 2]), and a function f obeying (1) and (2) is said to be a *tropical Plücker function*, or a TP-function.

In this paper we do not restrict ourselves by merely the Boolean cube case. We will also deal with functions defined on more general sets, namely: truncated Boolean cubes (generalizing both Boolean cubes and hyper-simplexes), integer boxes, and truncated integer boxes, in which cases relations (1) and (2) are generalized in a natural way (the definitions will be given later).

Functions satisfying algebraic or tropical Plücker relations have been studied in literature. Such functions on Boolean cubes are considered by Berenstein, Fomin and Zelevinsky [**1**] in connection with the total positivity and Lusztig's canonical bases; see also [**12**]. Henriques [**10**] considers AP-functions on the set of integer solutions of the system $0 \le x_i \le m-1$, $x_1 + \ldots + x_n = m$, and refers to the work of Fock and Goncharov [**7**] for results on such functions. The tropical analogs of certain AP-functions form a subclass of polymatroidal concave functions, or M-functions, studied by Murota [**16**]; see also [**14**]. Tropical Plücker functions in dimensions 3 and 4 are considered in [**3, 13, 20**] in connection with the so-called octahedron recurrence. (In fact, a general TP-function is related to a multi-dimensional analog of the octahedron recurrence.) An instance of Plücker relations is a relation on six lengths between four horocycles in the hyperbolic plane with distinct centers at infinity [**17**]. TP-functions on a hyper-simplex form a special case of so-called *valuated matroids* introduced by Dress and Wenzel [**4**].

Main results in this paper concern so-called *bases* of TP-functions. To explain this notion, consider two special cases: the Boolean cube $C_n := 2^{[n]}$ and a hyper-simplex $\Delta_n^m := \{S \subset [n] \colon |S| = m\}$, where $m \in \{1, \ldots, n-1\}$ (in a general case, a basis is defined in a similar way; a precise definition will be given later). Let $\mathcal{TP}(C_n)$ and $\mathcal{TP}(\Delta_n^m)$ denote the sets of TP-functions on C_n and Δ_n^m, respectively.

Definition. For $B = C_n$ or $B = \Delta_n^m$, a subset $\mathcal{B} \subseteq B$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{TP}(B) \to \mathbb{R}^{\mathcal{B}}$ is a bijection. In other words, each TP-function on B is determined by its values on $\mathcal{B}$, and moreover, values on $\mathcal{B}$ can be chosen arbitrarily.

Note that if $\mathbb{B}$ is a basis, then the polyhedral conic complex $\mathcal{TP}(B)$ is PL-isomorphic to the vector space of dimension $|\mathbb{B}|$. In particular, all bases have the same cardinality.

In both cases TP-bases do exist. For the Boolean cube C_n, there is a basis of a quite simple form, namely, the set Int of intervals in $[n]$ (so the dimension of $\mathcal{TP}(C_n)$ is $|Int| = \frac{n(n+1)}{2} + 1$). For a hyper-simplex Δ_m^n, instances of TP-bases are indicated in [**18**] (see also [**19, 21**]); one of them is the collection of all sets $S \in \Delta_n^m$ that are representable as the union of two disjoint intervals I, I' such that I either is empty or contains the element 1.

(Note that an algebraic analog of the notion of bases for AP-functions has encountered in literature as well. A construction of such a basis was announced in [**10**] for the case of a 'simplicial slice' $\{x \in \mathbb{Z}_+^n : \sum x_i = m\}$, with a claim that it could be derived from results on cluster algebras in [**7**].)

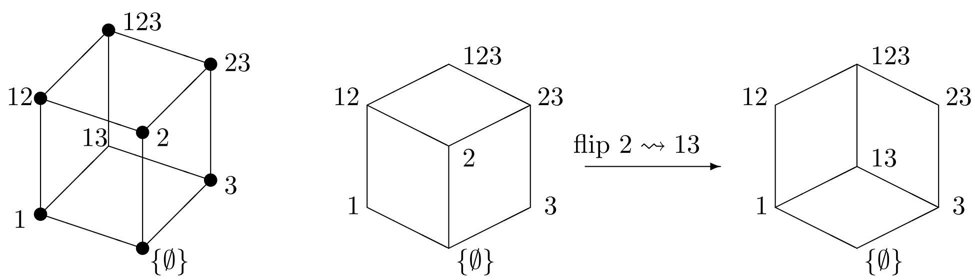
Our main theorem in this paper (Theorem 1) exhibits a TP-basis for a truncated integer box, the most general case of our study. This basis is obtained as a natural generalization of the above-mentioned bases for C_n and Δ_n^m; we call it the *standard* basis.

The proof of Theorem 1 uses only combinatorial tools, and a central role in it is played by a certain *flow model*, which goes back to a method of constructing TP-functions on the Boolean cube in [**1**]. This model generates *any* TP-function f by use of maximum weight flows on a certain weighted digraph. As a by-product, for each set $S \subseteq [n]$, the flow model enables us to represent the value $f(S)$ as a piece-wise linear convex function

$$f(S) = \max_F \left(\sum\nolimits_{I \in Int} \alpha_{F,I} f(I) \right),$$

where F runs over the flows concerning S. (Here for simplicity we consider the Boolean cube case.) Moreover, the coefficients $\alpha_{F,I}$ belong to $\{-1, 0, 1, 2\}$. This can be regarded as a tropical analogue of the so-called *Laurent phenomenon* (for the algebraic or tropical Laurent phenomenon under the octahedron or cube recurrences, see [**8, 11, 20**]).

In the integer box case, the standard basis, as well as many other (but not all) ones, can be associated with rhombus tilings of the regular $2n$-gone, giving a nice visualization of the basis. (For various aspects of rhombus tilings, see, e.g., [**5, 6, 11**].) For an illustration, let us consider the cube $\{0,1\}^3$. The standard basis $\mathcal{B}$ consists of seven intervals, which can be denoted as $\emptyset, 1, 2, 3, 12, 23, 123$. There is only one basis $\mathcal{B}'$ different from $\mathcal{B}$; it is obtained from $\mathcal{B}$ by the replacement (*mutation*) $2 \rightsquigarrow 13$. The cube and the rhombus tilings for $\mathcal{B}$ and $\mathcal{B}'$ are drawn in the picture:



The last group of our results concerns characterizations of special classes of TP-functions. Using the correspondence between certain bases and rhombus tilings, we study the classes of *submodular* and *skew-submodular* TP-functions f on a box $\{x \in \mathbb{Z}^n : 0 \le x \le a\}$, which means that f satisfies the inequalities of the form

$$f(x+1_i)+f(x+1_j) \ge f(x)+f(x+1_i+1_j)$$

in the former case, and of the form

$$f(x+1_i+1_j)+f(x+1_j) \ge f(x+1_i)+f(x+2\cdot 1_j)$$

in the latter case, where 1_q denotes q-th unit base vector in $\mathbb{Z}^n$. It turns out that each class admits a characterization in terms of the restriction of f to the standard basis. More precisely, we show that, for a TP-function f, the above submodular (skew-submodular) inequalities are propagated by the TP3-recurrence, starting from such inequalities within the standard basis.

The paper is organized as follows. In Section 2 we give necessary definitions and preliminary facts about TP-functions on boxes and truncated boxes. The main Theorem 1 is formulated in Section 3. Its proof is long enough and is lasted throughout this section, Section 4 and the Appendix. The proof of injectivity in the theorem and a reduction of truncated boxes to boxes are given in Section 3, while Section 4 introduces the flow model and proves the surjectivity (with one assertion postponed to the Appendix). The Laurent phenomenon for TP-functions is discussed in Section 5. Relations between bases and rhombus tilings are explained in Section 6. Sections 7, 8 and 9 are devoted, respectively, to submodular, skew-submodular and discrete concave TP-functions.

It should be noted that some steps in the proof of Theorem 1 in this paper are alternative to those contained in the preliminary version [**2**], in which also additional results on rhombus tilings are presented.

2. Definition and properties of TP-functions

We start with extending the notion of a tropical Plücker function to sets of integer vectors.

Definition. A function $f : D \to \mathbb{R}$ defined on a subset $D \subset \mathbb{Z}^n$ is said to be a *TP-function* if it satisfies the following TP3- and TP4-relations.

The *TP3-relation* has the form

$$f(x+1_i+1_k)+f(x+1_j) = \max\{f(x+1_i+1_j)+f(x+1_k),\ f(x+1_j+1_k)+f(x+1_i)\} \tag{3}$$

for any x and $1 \le i < j < k \le n$. As before, 1_q denotes q-th unit base vector in $\mathbb{Z}^n$. The *TP4-relations* have the form

$$f(x+1_i+1_k)+f(x+1_j+1_\ell) = \max\{f(x+1_i+1_j)+f(x+1_k+1_\ell),\ f(x+1_i+1_\ell)+f(x+1_j+1_k)\} \tag{4}$$

for any x and $1 \le i < j < k < \ell \le n$. Everywhere in the above we assume that all six vectors occurring in these relations belong to D.

Remark 1. Instead of $\mathbb{R}$ in the definition, one can consider an arbitrary lattice-ordered Abelian group $\mathcal{R}$, e.g., the group $\mathbb{R}^S$ for a set S. All subsequent results remain true for this more general setting. However, for simplicity, we will work with real-valued functions only.

Example. A function f on $\mathbb{Z}^n$ is said to be *quasi-separable* if it is representable as $\varphi_1(x_1)+\ldots+\varphi_n(x_n)+\varphi_0(x_1+\ldots+x_n)$, where φ_0, φ_1, …, φ_n are arbitrary functions in one variable. Clearly any quasi-separable function is a TP-function. Moreover, addition of any quasi-separable function to a TP-function maintains the TP-relations.

In what follows we assume that the domain D in the above definition of a TP-function is a so-called truncated box, defined as follows.

For an n-tuple $a=(a_1,\ldots,a_n)$ of integers, we refer to $|a|:=a_1+\ldots+a_n$ as the *size* of a. Let a' and a'' be two n-tuples a' with $a'\le a''$. The *box* $B(a',a'')$ consists of the integer vectors $x=(x_1,\ldots,x_n)$ satisfying the box constraints ${a'}_i\le x_i\le {a''}_i$ for all $i\in[n]$. Given integers m' and m'' with $m'\le m''$, by the *truncated box* $B_{m'}^{m''}(a',a'')$ we mean the subset of vectors $x\in B(a',a'')$ such that $m'\le|x|\le m''$. The number $m''-m'$ is regarded as the *width* of the truncated box. For $m'\le m\le m''$, the m-th *layer* of $B_{m'}^{m''}(a',a'')$ is formed by the vectors of the size m.

If $s\in\mathbb{Z}^n$ then the shift $B_{m'}^{m''}(a',a'')+s$ is a truncated box as well. By this reason, we usually assume that $a'=0$, denote a'' simply as a, and write $B(a)$ instead of $B(0,a)$. Note that the Boolean cube $2^{[n]}$ is just the box $B(\mathbf{1})$, where $\mathbf{1}=1^n$ is the all-unit vector.

We also usually assume, w.l.o.g., that all a_i are strictly positive. For if $a_i=0$, then the variable i is redundant and can be excluded.

One more observation is useful. For a truncated box $B=B_{m'}^{m''}(a',a'')$, we can form the reflected box $B^*=B_{-m''}^{-m'}(-a'',-a')$. For a TP-function f on B, take the reflected function f^* on B^*, defined by the relations $f^*(x)=f(-x)$. Then f^* is a TP-function as well.

Three special cases will be important to us. When a is all-unit, we obtain the *truncated Boolean cube* $B_m^{m'}(\mathbf{1})$. When $m=m'$, we obtain a truncated box $B_m^m(a)$ with zero width; it is called a *slice*. When, in addition, $a=\mathbf{1}$, the slice turns into the *hyper-simplex* $\{S\subseteq[n]\colon |S|=m\}$.

The set of TP-functions on a truncated box B is denoted by $\mathcal{TP}(B)$. This is a cone in the space $\mathbb{R}^B$ of all functions on B, containing a large lineal formed by the quasi-separable functions.

To illustrate, let us consider the simplest nontrivial hyper-simplex $B_2^2(1,1,1,1)$. It consists of six two-element subsets in $\{1,2,3,4\}$, which may be denoted as 12, 13, 14, 23, 24, 34. By adding an appropriate quasi-separable function, we can assume that a TP-function is equal to 0 at the points 12,13,14 and 24. Then the unique TP4-relation takes the form $\max\{f(23),f(34)\}=0$. That is, modulo the lineal, the cone of TP-functions is represented as the union of two rays in $\mathbb{R}^2$, namely, $\mathbb{R}_-\times\{0\}$ and $\{0\}\times\mathbb{R}_-$. In particular, the cone is piecewise-linear-morphic to $\mathbb{R}^5$. As we shall see, the latter property holds in a general case: the set $\mathcal{TP}(B)$ is a polyhedral cone PL-morphic to a vector space.

Next we discuss an interrelation between TP3- and TP4-relations. Each TP4-relation concerns vectors of the same layer, while each TP3-one ‘connects’ vectors of two neighboring layers. We assert that the TP4-relations are consequences of TP3-relations provided that the width of the truncated box is nonzero.

Proposition 1. *Let f be a function on a truncated box $B = B_m^{m'}(a)$ and $m < m'$. Suppose f satisfies all TP3-conditions on B. Then f satisfies the TP4-conditions as well.*

Proof. First we show validity of (4) for a cortege $(x; i, j, k, \ell)$ with $m < |x| + 2 \le m'$. We are going to deal with only vectors of the form $x + 1_{i'}$ or $x + 1_{i'} + 1_{j'}$, where $i', j' \in \{i, j, k, \ell\}$ $(i' \ne j')$. For this reason and to simplify notation, one may assume, w.l.o.g., that $x = \mathbf{0}$ and $(i, j, k, \ell) = (1, 2, 3, 4)$ (in which case we, in fact, deal with the truncated Boolean cube $\{S \subset [4] \colon 1 \le |S| \le 2\}$). So we have to prove that

$$f(13) + f(24) = \max\{f(12) + f(34),\ f(14) + f(23)\} \tag{5}$$

(where for brevity qr stands for $1_q + 1_r$).

We use the following three TP3-relations for f:

$$f(24) + f(3) = \max\{f(2) + f(34),\ f(4) + f(23)\}; \tag{6}$$

$$f(13) + f(2) = \max\{f(1) + f(23),\ f(3) + f(12)\}; \tag{7}$$

$$f(14) + f(2) = \max\{f(1) + f(24),\ f(4) + f(12)\}. \tag{8}$$

Adding $f(12)$ to both sides of (6) gives

$$f(24) + f(3) + f(12) = \max\{f(2) + f(34) + f(12),\ f(4) + f(23) + f(12)\}.$$

If in each side of this relation we take the maximum of the expression there and $f(1) + f(23) + f(24)$, we obtain

$$\max\{f(24) + f(3) + f(12),\ f(1) + f(23) + f(24)\}$$
$$= \max\{f(2) + f(34) + f(12),\ f(4) + f(23) + f(12),\ f(1) + f(23) + f(24)\}.$$

This can be re-written as

$$\max\{f(3) + f(12),\ f(1) + f(23)\} + f(24)$$
$$= \max\{f(2) + f(34) + f(12),\ \max\{f(4) + f(12),\ f(1) + f(24)\} + f(23)\}.$$

The maximum in the left hand side is equal to $f(13) + f(2)$, by (7), and the interior maximum in the right hand side is equal to $f(14) + f(2)$, by (8). Therefore, we have

$$f(13) + f(2) + f(24) = \max\{f(2) + f(34) + f(12),\ f(14) + f(2) + f(23)\}$$
$$= \max\{f(34) + f(12),\ f(14) + f(23)\} + f(2).$$

Canceling out $f(2)$ in the left and right sides, we obtain the required equality (5).

Next, let $|x| + 2 = m$. This case is reduced to the previous one by considering the function f^* on the reversed box. $\square$

Thus, in the definition of TP-functions, explicitly imposed TP4-relations are important only when we deal with a slice $B_m^m(a)$, in which case TP3-relations vanish. Note, however, that in this case we could eliminate the variable x_n and obtain a new function f' (of the variables $x_1, \dots, x_{n-1}$) on the new truncated box $B_{m-a_n}^m((a_1, \dots, a_{n-1}))$. This new function f' is a TP-function if and only if f is such. Since $a_n > 0$, the new truncated box has a nonzero width, and we again can restrict ourselves by only TP3-relations. Eventually, only TP3-relations are left.

We conclude this section with one more remark. Let B, B' be truncated boxes such that $B \subset B'$, and f' a TP-function on B'. Obviously, the restriction of f' to B is a TP-function. We shall see later (in Corollary 1 and Proposition 10) that the converse property is also true: any TP-function on B can be extended to a TP-function on B'.

3. Main theorem

Let $B = B_m^{m'}(a)$ be a truncated box. We now introduce an important subset $\mathcal{B}$ of B that we call the standard basis. We need some terminology and notations.

For a nonzero vector $x \in B(a)$, let $c(x)$ and $d(x)$ denote, respectively, the first and last elements (w.r.t. the order in $[n]$) in the support $\mathrm{supp}(x) = \{i \in [n] \colon x_i \neq 0\}$ of x. We say that x is a *fuzzy-interval*, or, briefly, a *fint*, if $x_i = a_i$ for all $c(x) < i < d(x)$. We say that x is a *sesquialteral fuzzy-interval*, or a *sint*, if x is not a fint and is representable as the sum of two fints x', x'' such that $d(x') < c(x'')$, and $x'_i = a_i$ for $i = 1, \dots, d(x') - 1$. When $a = \mathbf{1}$, a fint turns into an *interval* $\{c, c+1, \dots, d\}$ in $[n]$, denoted as $[c..d]$, and a sint turns into a *sesquialteral interval*, a set of the form $[1..d_1] \cup [c_2..d_2]$ with $c_2 > d_1 + 1$.

Let $Int(a;p)$ and $Sint(a;p)$ denote the sets of fints and sints of size p in $B(a)$, respectively. Also we assume by definition that $Int(a;0) = \{0\}$ and $SInt(a;0) = \emptyset$.

Definition. The *standard basis* for a truncated box $B = B_m^{m'}(a)$ is the set $\mathcal{B} = Sint(a;m) \cup Int(a;m) \cup Int(a;m+1) \cup \ldots \cup I(a;m')$.

Observe that the standard basis involves sints only from the lowest layer. In particular, the set $Int(a) := Int(a;0) \cup \ldots \cup Int(a;|a|)$ is the standard basis for the box $B(a)$, and $Sint(a;m) \cup Int(a;m)$ is the standard basis for the slice $B_m^m(a)$.

Theorem 1. *The standard basis $\mathbb{B}$ for a truncated box $B = B_m^{m'}(a)$ is indeed a basis of the set $\mathcal{TP}(B)$ of TP-functions on B, i.e., the restriction map res : $\mathcal{TP}(B) \to \mathbb{R}^{\mathcal{B}}$ is a bijection.*

In other words, each TP-function on B is determined by its values on $\mathcal{B}$, and moreover, values on $\mathcal{B}$ can be assigned arbitrarily. (This reminds a classical property of bases of vector spaces with respect to linear mappings. Later, in Section 6, we will also meet another sort of bases.) Thus, Theorem 1 gives a (piece-wise linear) bijection between the fan (polyhedral cone) $\mathcal{TP}(B)$ and the real vector space $\mathbb{R}^{\mathcal{B}}$.

Corollary 1. *Any TP-function f on a truncated box $B_m^{m'}(a)$ can be extended to a TP-function on the entire box $B(a)$.*

Indeed, first we take the restriction of f to the standard basis for $B_m^{m'}(a)$ and extend it to the standard basis for $B_m^{|a|}(a)$ by assigning arbitrary values on $Int(a;m'+1) \cup \ldots \cup Int(a;|a|)$. This determines a TP-function g on $B_m^{|a|}(a)$ coinciding with f on $B_m^{m'}(a)$. Then we consider the reversed function g^* for g; clearly g^* is a TP-function on $B_{-|a|}^{-m}(-a,0)$. As above, we can extend g^* into a TP-function h on the box $B(-a,0)$. Now h^* is the desired extension of f to $B(a)$.

Our proof of Theorem 1 consists of several stages. First we prove injectivity of the restriction map *res* (Subsection 3.1), which is relatively easy. The proof of the other direction in the theorem, that *res* is surjective, is more complicated. First of all, we reduce the task to the case of an entire (non-truncated) box (Subsection 3.2). Next we have to show, in this special case, that an arbitrary function f_0 on the standard basis is extendable to a TP-function f on the box. A naive approach is to try to propagate f_0 to the box, step by step, by using TP3- or TP4-equalities (when, for some x, i, j, k, the values of f are already constructed for the four arguments in the right hand side of (3) and for one argument in the left hand side, one can determine the value for the remaining argument in the left hand side). By such an approach, a difficulty is to show that the arising function does not depend on the way (route) of propagation (and therefore, the process terminates with a well-defined TP-function). The approach we apply in Section 4 is different; it is based on a certain *flow model* for constructing TP-functions, which is the core of the whole proof.

3.1. Injectivity. Here we show the easier direction in Theorem 1.

Proposition 2. *For $B = B_m^{m'}(a)$, the restriction map $res : \mathcal{TP}(B) \to \mathbb{R}^{\mathcal{B}}$ is injective, i.e., any TP-function on B is determined by its values within $\mathcal{B} = Sint(a;m) \cup Int(a;m) \cup Int(a;m+1) \cup \ldots \cup I(a;m')$.*

Proof. Let $f \in \mathcal{TP}(B)$ and $x \in B$. Using relations TP3 and TP4, we show that $f(x)$ can be expressed via the values of f on $\mathcal{B}$. To provide induction, we assign to x the following four numbers:

$$\begin{aligned}
&\alpha(x) \text{ is the maximal } i \in [n] \text{ such that } x_i > 0;\\
&\beta(x) \text{ is the maximal } i \in [n] \text{ such that } i < \alpha(x) \text{ and } x_i < a_i;\\
&\gamma(x) \text{ is the maximal } i \in [n] \text{ such that } i < \beta(x) \text{ and } x_i > 0;\\
&\delta(x) \text{ is the maximal } i \in [n] \text{ such that } i < \gamma(x) \text{ and } x_i < a_i;
\end{aligned} \tag{9}$$

Observe that all these numbers exist if x is neither a fint (fuzzy-interval) nor a sint (sesquialteral fuzzy-interval), and that $\alpha(x)$ and $\beta(x)$ exist if x is not a fint. Assuming that $x \notin \mathbb{B}$, define

$$\eta(x) := |a|(\gamma(x) + \alpha(x)) + x_{\gamma(x)} + x_{\alpha(x)}, \tag{10}$$

Consider two cases.

Case 1: $|x| = m$. We show that $f(x)$ is determined, via TP4-relations, by the values of f within $Sint(a;m) \cup Int(a;m)$.

Put $i := \delta(x)$, $j := \gamma(x)$, $k := \beta(x)$, and $\ell := \alpha(x)$. Then $i < j < k < \ell$. Put $x' := x - 1_j - 1_\ell$ and form five vectors $B := x' + 1_i + 1_k$, $C := x' + 1_i + 1_j$, $D := x' + 1_k + 1_\ell$, $E := x' + 1_i + 1_\ell$, and $F := x' + 1_j + 1_k$. From the definitions in (9) it follows that these vectors belong to B (and are of size m). By (4) (with x' instead of x), $f(x)$ is computed from the values of f on B, C, D, E, F. Also one can check that each of the latter vectors either is a fint or is a sint or the value of η on it is strictly less than $\eta(x)$.

So we can apply induction on η (the inductive process of computing f on the lowest layer $B_m(a)$ has as a base the family $Sint(a;m) \cup Int(a;m)$).

Case 2: $|x| > m$. We show that $f(x)$ is determined, via TP3-relations, by the values of f within $Sint(a;m) \cup Int(a;m) \cup \ldots \cup Int(a;|x|)$. We may assume that x is not a fint. Put $i := \gamma(x)$, $j := \beta(x)$, and $k := \alpha(x)$; then $i < j < k$. Put $x' := x - 1_i - 1_k$. By (3) (with x' instead of x), $f(x)$ is computed via the values of f on the vectors

$$B := x' + 1_j, \quad C := x' + 1_i + 1_j, \quad D := x' + 1_k, \quad E := x' + 1_j + 1_k, \quad F := x' + 1_i$$

(each of which belongs to B, in view of (9) and $|x| > m$). One can check that, for each of B, C, D, E, F, at least one of the following is true: it is a fint; it belongs to the preceding layer; the value of η on it is less than $\eta(x)$. So we can apply induction on the layer number and on η. □

3.2. Surjectivity: a reduction to entire box. We start proving the other direction in Theorem 1, i.e., that the restriction map

$$res : \mathcal{TP}(B_m^{m'}(a)) \to \mathbb{R}^{\mathcal{B}}$$

is surjective, where $\mathcal{B} = Sint(a;m) \cup Int(a;m) \cup Int(a;m+1) \cup \ldots \cup Int(a;m')$. Denote this statement as $\mathbf{Surj}(m, m')$.

Observe that $\mathbf{Surj}(m, |a|)$ implies $\mathbf{Surj}(m, m')$. Indeed, let f_0 be a function on $Sint(a;m) \cup Int(a;m) \cup \ldots \cup I(a;m')$. Extend it arbitrarily to the larger set $Sint(a;m) \cup Int(a;m) \cup \ldots \cup I(a;|a|)$. Assuming that $\mathbf{Surj}(m, |a|)$ is valid, this extension can be further extended into a TP-function f on $B_m^{|a|}(a)$. Then the restriction of f to $B_m^{m'}(a)$ is a TP-extension of f_0, yielding $\mathbf{Surj}(m, m')$.

Thus, it suffices to prove validity of $\mathbf{Surj}(m, |a|)$, which we now denote simply as $\mathbf{Surj}(m)$. We prove it by induction on m. The base $\mathbf{Surj}(0)$ of the induction will be proved in the next section, and now we perform an induction step.

Lemma 1. $\mathbf{Surj}(m-1)$ *implies* $\mathbf{Surj}(m)$.

PROOF. Let f_0 be a function on $Sint(a;m) \cup Int(a;m) \cup Int(a;m+1) \cup \ldots$. Our aim is to construct a function g_0 on $Sint(a;m-1) \cup Int(a;m-1) \cup Int(a;m) \cup \ldots$ satisfying the following conditions:

(a) g_0 and f_0 are equal on the set $Int(a;m) \cup \ldots \cup Int(a;|a|)$; and

(b) the TP-function g on $B_{m-1}(a)$ with $res(g) = g_0$ (which exists due to validity of $\mathbf{Surj}(m-1)$ and is unique due to Proposition 2) satisfies

$$g(x) = f_0(x) \qquad \text{for each } x \in Sint(a;m). \tag{11}$$

Then (a),(b) imply that the restriction f of g to $B_m(a)$ is a TP-function possessing the desired property $res(f) = f_0$.

We define the function g_0 as follows.

For $y \in Sint(a;m-1) \cup Int(a;m-1)$, let $p = p(y)$ denote the minimal number such that $y_p < a_p$. We refer to $p(y)$ as the *insertion point* for y and denote the vector $y + 1_p$ by $y^\uparrow$. The vector $y^\uparrow$ has the size m and lies in $B_m(a)$. Moreover, $y^\uparrow$ is either a fint or a sint. Define

$$g_0(y) := f_0(y^\uparrow) + Mt(y),$$

where M is a large positive number (w.r.t. f_0) and $t(y) := y_{p+1} + \ldots + y_n$.

We assert that g_0 defined this way satisfies (11). To show this, consider $x \in Sint(a;m)$. Let $\alpha(x), \beta(x), \gamma(x)$ be defined as in (9) (they exist, since x is not a fint), and assign the parameter $\eta(x)$ as in (10). Put $i := \gamma(x)$, $j := \beta(x)$ and $k = \alpha(x)$. By the TP3-relation for the function g and the cortege $(x-1_i-1_k; i, j, k)$, we have

$$g(x) = \max\{g(C) + g(D), g(E) + g(F)\} - g(B), \tag{12}$$

where $B := x - 1_i + 1_j - 1_k$, $C := x + 1_j - 1_k$, $D := x - 1_i$, $E := x - 1_i + 1_j$, $F := x - 1_k$. We observe the following, letting $\Sigma := x_{i+1} + \ldots + x_n$.

(i) The vectors C and E have the size m, C is either a fint or a sint with $\eta(C) < \eta(x)$, and similarly for E. So, applying induction on η, we have $g(C) = f_0(C)$ and $g(E) = f_0(E)$.

(ii) The vector B has the size $m-1$ and its insertion point is i. Then $B^\uparrow = B + 1_i = C$. Also $t(B) = \Sigma + 1 - 1 = \Sigma$, whence $g(B) = f_0(C) + M\Sigma$.

(iii) The vector D has the size $m-1$ and its insertion point is i. Then $D^\uparrow = D + 1_i = x$. Also $t(D) = \Sigma$, whence $g(D) = f_0(x) + M\Sigma$.

(iv) The vector F has the size $m-1$ and its insertion point is at least i. This and $F_k = x_k - 1$ imply $t(F) \le \Sigma - 1$, whence $g(F) \le f_0(F^\uparrow) + M\Sigma - M$.

Since M is large and $t(D) \ge t(F) + M$ (by (iii),(iv)), the maximum in (12) is attained by the first sum occurring there. Therefore, in view of (i)–(iii),

$$g(x) = g(C) + g(D) - g(B) = f_0(C) + (f_0(x) + M\Sigma) - (f_0(C) + M\Sigma) = f_0(x),$$

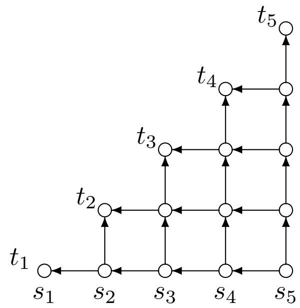
as required, yielding the lemma. $\square$

4. Flow model

In this section we prove surjectivity of the restriction map *res* in Theorem 1 for the case of an entire box (then surjectivity in a general case follows by explanations in Subsection 3.2). The goal is to show that any function on the standard basis can be extended to a TP-function on the box. To construct the required TP-functions we develop a certain *flow model*.

We first describe the model for the case when a box is the Boolean cube $C := 2^{[n]} = B(1, \ldots, 1)$. Our flow method in this case has as a source a construction of instances of tropical Plücker functions in [**1**].

4.1. The case of Boolean cube. We form the following directed graph (digraph) $\Gamma = \Gamma_n = (V, E)$. The vertex set V consists of elements v_{pq} for $p, q \in [n]$ such that $q \le p$. The edge set E consists of the pairs $(v_{pq}, v_{p'q'})$ such that either $p' = p - 1$ and $q' = q$, or $p' = p$ and $q' = q + 1$. We visualize this digraph by identifying each vertex v_{pq} with the point (p, q) in the plane. The vertices $v_{11}, \ldots, v_{n,1}$, located in the bottommost horizontal line of Γ, are referred to as the *sources* and denoted by $s_1, \ldots, s_n$, respectively. The vertices $v_{11}, \ldots, v_{n,n}$, located in the diagonal of Γ, are referred to as the *sinks* and denoted by $t_1, \ldots, t_n$, respectively. Note that Γ is acyclic and any maximal path in it goes from a source to a sink. The digraph Γ_5 is illustrated in the picture:



Definition. By an *(admissible) flow* we mean a collection $\mathcal{F} = (P_1, \ldots, P_k)$ of pairwise disjoint paths $P_1, \ldots, P_k$ in Γ, each path beginning at a source and ending at a sink among the first $|\mathcal{F}|$ sinks $t_1, \ldots, t_k$.

Consider a *weighting* $w : V \to \mathbb{R}$ on the vertices; the weight $w(v_{pq})$ of a vertex v_{pq} is also denoted as w_{pq}. The weight $w(P)$ of a path P is defined to be the sum of weights $w(v)$ of the vertices v of P, and the weight $w(\mathcal{F})$ of a flow $\mathcal{F} = (P_1, \ldots, P_k)$ is $w(P_1) + \ldots + w(P_k)$. For a subset $S \subseteq [n]$, define

$$f_w(S) := \max\{w(\mathcal{F})\}, \tag{13}$$

where the maximum is taken over all admissible flows $\mathcal{F}$ in Γ beginning at the set $\{s_p \colon p \in S\}$.

The following assertion (generalized by Theorem 2′ in Section 4) plays the key role.

Theorem 2. *Let w be a weighting on the vertex set V of Γ as above. Then f_w defined by* (13) *is a TP-function on the Boolean cube $C = 2^{[n]}$.*

This is a special case of Theorem 2.4.6 in [**1**] where, instead of Γ as above, one considers an arbitrary planar digraph (embedded in the plane) in which the sources $s_1, \ldots, s_k$ and the sinks $t_1, \ldots, t_n$ are disjoint (which is not important) and follow in succession clockwise and anti-clockwise, respectively, in the boundary of the digraph. However, the question of possibility of generating *all* TP-functions on C by this method is beyond that paper.

We show that the flow model as above constructs almost all TP-functions. Here 'almost' is because f_w obeys the evident relation $f_w(\emptyset) = 0$ (and this is a unique restriction, in essense). We say that a function f on C is *normalized* if $f(\emptyset) = 0$. (In fact, we can deal with only normalized functions since no TP-relation involves the empty set. Note also that any TP-function can be considered up to adding a constant, but now this is not important to us.) Let $\mathcal{TP}^0(C)$ denote the set of normalized TP-functions on C. Accordingly, we exclude $\{\emptyset\}$ from the standard basis and denote the set of non-empty intervals in $[n]$ by $\mathcal{B}^0$.

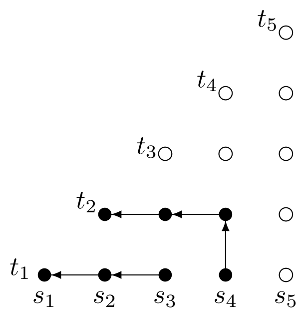
Proposition 3. *For every function $g : \mathcal{B}^0 \to \mathbb{R}$, there exists a weighting w such that $g(I) = f_w(I)$ for all intervals $I \in \mathcal{B}^0$. Moreover, w is unique and the correspondence of g and w gives an isomorphism between the vector spaces $\mathbb{R}^{\mathcal{B}^0}$ and $\mathbb{R}^V$.*

Taken together, Theorem 2 and Propositions 2 and 3 imply that the mapping $\mathbb{R}^{\mathcal{B}^0} \to \mathcal{TP}^0(C)$ is bijective. This gives Theorem 1 in the case of Boolean cube.

Proposition 3 is easy. Indeed, for each interval $I = [c..d]$, there exists only one admissible flow $\mathcal{F}$ having the source set $\{s_p, p \in I\}$. This flow consists of the paths $P_1, \dots, P_{d-c+1}$, where each P_i begins at the source $s_{\bar{i}}$ for $\bar{i} := c + i - 1$, ends at the sink t_i, and is of the form

$$P_i = (s_{\bar{i}} = v_{\bar{i},1}, v_{\bar{i},2}, \dots, v_{\bar{i},i}, v_{\bar{i}-1,i}, \dots, v_{i,i} = t_i).$$

(Hereinafter we use notation for a path without indicating its edges.) The picture below illustrates the flow $\mathcal{F}$ for the interval $I = \{3, 4\}$ in the case $n = 5$.



Thus, for an interval $I = [c..d]$, the value $f_w(I)$ is equal to sum of weights of those vertices of Γ that lie in the trapezoid within $[d] \times [d - c + 1]$:

$$f_w(I) = \sum (w_{pq} \colon q \le p \le d,\ q \le d - c + 1). \tag{14}$$

This defines a linear mapping from the space $\mathbb{R}^V$ of weights to the space $\mathbb{R}^{\mathbb{B}^0}$ of functions on the set of non-empty intervals. The fact that this mapping is an isomorphism follows from two observations.

I. *These vector spaces have the same dimension.* Indeed, there exists a natural bijection between the sets V and $\mathbb{B}^0$, namely, $(p, q) \mapsto I_{p,q} := [p - q + 1..p]$.

II. *The mapping is injective.* Indeed, let $w \ne 0$ and let (p, q) be a minimal pair in V (w.r.t. the natural partial order on $\mathbb{Z}^2$) such that $w_{pq} \ne 0$. Then (14) implies

$$f_w(I_{pq}) = \sum_{i \le p,\ j \le q} w_{ij} = w_{pq} \ne 0,$$

and therefore, $f_w \ne 0$.

This gives Proposition 3.

Remark 2. One can explicitly define the weighting w by the following formula:

$$w_{pq} = f(I_{pq}) - f(I_{p-1,q}) - f(I_{p,q-1}) + f(I_{p-1,q-1}), \tag{15}$$

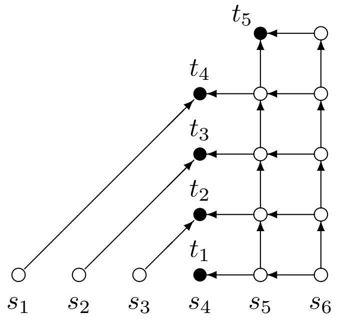
letting $f(I_{pq}) := 0$ if $q = 0$ or $p < q$.

Remark 3. One can propose a direct flow model for the case of truncated Boolean cube $B_m^{m'}(1^n)$. The vertices are the integer points (p, q) of the plane such that:

(a) either $q = 1$ and $p = 1, \dots, m - 1$ (the initial part);

(b) or $m \le p \le n$, $1 \le q \le m'$ and $q \le p$ (the main body).

The edges in the main body are assigned as before. Besides, there is an edge from each vertex $(p,1)$ of the initial part to the vertex (m,p). The digraph for $n = 6, m = 4$ and $m' = 5$ is drawn in the picture:



The set of sources is $\{(p,1), 1 \le p \le n\}$. The set of sinks consists of the points (m,q) for $1 \le q \le m$ and the points (p,p) for $m < p \le m'$ (in the above picture they are indicated by black circles). Weights w_{pq} are arbitrary for (p,q) with $p > m$ and zero for $p \le m$. Admissible flows and the function f_w are defined similarly to the above case. Note that if $m \le |S| \le m'$ then an admissible flow from $\{s_p \colon p \in S\}$ does exist; so the value $f_w(S)$ is well-defined.

One can check that if S is an interval of size between m and m' or a sesquialteral interval of size m, then there exists a unique flow for it. This implies that the weighting w is determined by the values $f_w(S)$ for $S \in Sint_m \cup Int_m \cup \ldots \cup Int_{m'}$, and vice versa. The arguments below are applicable to this model as well, yielding surjectivity for truncated cubes.

4.2. The case of an entire box. Let $B = B(a)$ be an arbitrary box. We associate to it an auxiliary digraph $\Gamma_a = (V,E)$. To define the latter, we need some notation and terminology.

For $i = 0,1,\ldots,n$, denote $a_1 + \ldots + a_i$ by $\bar{a}_i$ (in particular, $\bar{a}_0 = 0$), and let $N := \bar{a}_n = |a|$. The ordered set $[N]$ is naturally partitioned into intervals (*blocks*) $L_1,\ldots,L_n$, where L_i is the interval from $\bar{a}_{i-1}+1$ to $\bar{a}_i$.

The vertex set V consists of the pairs $(p,q) \in [N] \times [N]$ such that:

(a) $q \le p$, and

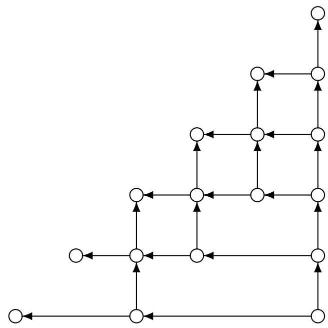
(b) if p lies in a block L_i then $q \ge p - \bar{a}_{i-1}$.

We assign an edge from a vertex (p,q) to a vertex (p',q') in the following two cases:

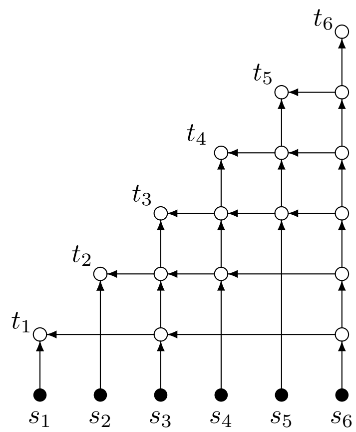
(c) (p',q') is either $(p+1,q)$ or $= (p,q+1)$;

(d) $p = \bar{a}_i + 1$, $q < a_i$, $p' = \bar{a}_{i-1} + q$, and $q' = q$.

(If $a = (1,\ldots,1)$, we just obtain the previous digraph Γ_n.) The picture illustrates Γ_a for $a = (2,3,1)$.



As before, the vertices on the diagonal are assigned to be the *sinks*; these are $t_1 = (1,1), t_2 = (2,2), \ldots, t_N = (N,N)$. As to the *sources*, one can assign them to be the vertices $(\bar a_{i-1}+q, q)$ for $1 \le q \le a_i$, $i = 1, \ldots, n$ (lying on the diagonals of squares whose lower sides correspond to blocks in the bottommost horizontal line). We prefer, however, to consider the sources $s_1, \ldots, s_N$ as extra vertices, place them at the points $(1,0), \ldots, (N,0)$, respectively, and connect each source s_p by outgoing edge to the vertex $(p, p-\bar a_{i-1})$ if p lies in i-th block L_i. The extended digraph is denoted by $\tilde\Gamma_a$. Note that this graph remains planar as before though in the visualization (see the picture) some edges are crossing.



As before, an (admissible) flow consists of pairwise disjoint paths in $\tilde\Gamma_a$ going from a set $S \subseteq [N]$ of sources to the sinks $t_1, \ldots, t_{|S|}$. (Speaking of a flow from S, we mean a flow from $\{s_p \colon p \in S\}$.) Given a weighting $w : V \to \mathbb{R}$, the function f_w on the set $2^{[N]}$ is defined as in (13).

At this point, there is one important difference from the Boolean case: for some subsets $S \subset [N]$, no flow from S exists; in this case we formally define $f_w(S) := -\infty$. Nevertheless, our embedding of the box $B(a)$ to the cube $2^{[N]}$ is arranged so that a flow exists for the image of any element of the box.

More precisely, we associate to a vector $x \in B(a)$ the subset $[x] = (\bar a_0 + [x_1]) \cup \ldots \cup (\bar a_{n-1} + [x_n])$ of $[N]$ (letting $[0] := \emptyset$). In other words, $[x]$ consists of x_i *beginning* elements of each block L_i. We call such a set *left-squeezed*. One can check that a flow does exist for any left-squeezed subset $S \subseteq [N]$, whence $f_w(S)$ is finite. The desired function on $B(a)$, for which we use the same notation f_w, is defined in a natural way: $f_w(x) := f_w([x])$ for each $x \in B(a)$.

Theorem 2′ (a generalization of Theorem 2). *Let w be a weighting on Γ_a. Then the function f_w on $B(a)$ is a TP-function.*

(Note that this does not follow from Theorem 2.4.6 in [**1**] since in general we cannot embed $\tilde\Gamma_a$ in the plane so that all sources and sinks occur in the *boundary* of the graph.) Theorem 2′ together with the next proposition implies the surjectivity of *res* in the box case.

Proposition 3′ (a generalization of Proposition 3). *For any function g on the set $Int^0(a)$ of non-zero fuzzy-intervals, there exists a weighting w on $\Gamma_a = (V, E)$ such that $g(x) = f_w(x)$ holds for all $x \in Int^0(a)$. Moreover, w is unique and the correspondence of g and w gives an isomorphism between the vector spaces $\mathbb{R}^{Int^0(a)}$ and $\mathbb{R}^V$.*

This is proved similarly to the proof of Proposition 3. Let x be a fint and let $[c..d]$ be its support (in $[n]$). If $c = d$ of if $x_c = a_c$, then the set $[x]$ is an interval in $[N]$. In a general case, $[x]$ consists of two intervals: $[x] \cap L_c$ and $[x] \cap [\bar a_c + 1..N] = [\bar a_c + 1..\bar a_{d-1} + x_d]$. Analysing the construction of Γ_a, one can realize that in all cases there exists a unique admissible flow from the set of sources corresponding to $[x]$. This gives a linear mapping from the space $\mathbb{R}^V$ of weightings to the space of functions on $Int^0(a)$. These spaces have the same dimension and the mapping is injective. Hence, the mapping is an isomorphism. □

Proof of Theorem 2′ It is based on a technique of flow rearrangements.

We have to show validity of TP3-relation (3) for the function f_w on $B(a)$ and a cortege $(x; \tilde i, \tilde j, \tilde k)$ in $B(a)$ (with $\tilde i < \tilde j < \tilde k$). The operator [] transfers the six vectors occurring as arguments in this relation into six left-squeezed sets in $[N]$. Moreover, one can see that the latter sets are of the form $Xik, Xj, Xij, Xk, Xi, Xjk$ with $i < j < k$ and $X \subseteq [N] - \{i, j, k\}$, and that the above relation turns into the Boolean TP3-relation (1) involving these sets and the function $f = f_w$ on the cube $2^{[N]}$ determined by the weighting w on Γ_a. So our aim is to show validity of the latter TP3-relation. We will use the following result, which will be proved in the Appendix.

Proposition 4. Let X, i, j, k be as above.

(a) *Let $\mathcal{F}$ be a flow from Xij, and $\mathcal{F}'$ a flow from Xk in $\tilde\Gamma_a$. Then the union of these flows can be rearranged as the union of a flow $\mathcal{F}_1$ from Xik and a flow $\mathcal{F}_2$ from Xj. A similar property is true for flows from the sets Xjk and Xi.*

(b) *Let $\mathcal{F}$ be a flow from Xik, and $\mathcal{F}'$ a flow from Xj in $\tilde\Gamma_a$. Then the union of these flows can be rearranged as the union of flows $\mathcal{F}_1$ and $\mathcal{F}_2$ such that: either $\mathcal{F}_1$ goes from Xij and $\mathcal{F}_2$ goes from Xk, or $\mathcal{F}_1$ goes from Xjk and $\mathcal{F}_2$ goes from Xi.*

(Here the rearrangement means that each vertex (or edge) of Γ_a is covered by the flows in the first union as many times as it is covered by the flows in the second union.)

Using this proposition, we finishes the proof of Theorem 2′ as follows. Let $\mathcal{F}$ be a flow from Xij such that $f(Xij) = w(\mathcal{F})$, and $\mathcal{F}'$ a flow from Xk such that $f(Xk) = w(\mathcal{F}')$, where $f := f_w$. By (a) in Proposition 4, there exist a flow $\mathcal{F}_1$ from Xik and a flow $\mathcal{F}_2$ from Xj such that $w(\mathcal{F}_1)+w(\mathcal{F}_2) = w(\mathcal{F})+w(\mathcal{F}')$. Since $f(Xik) \ge w(\mathcal{F}_1)$ and $f(Xj) \ge w(\mathcal{F}_2)$, we have

$$f(Xik) + f(Xj) \ge f(Xij) + f(Xk).$$

Similarly we have the inequality

$$f(Xik) + f(Xj) \ge f(Xjk) + f(Xi).$$

Arguing in a similar way and using (b) in Proposition 4, we obtain

$$f(Xik) + f(Xj) \le \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\}.$$

These three inequalities give the desired TP3-relation.

This completes the proof of Theorem 2′ and the proof of Theorem 1. □ □

5. The tropical Laurent phenomenon

Consider a TP-function f on the Boolean cube $2^{[n]}$, and a subset $S \subseteq [n]$. We have seen that the value $f(S)$ can be computed (by use of the operations of addition, subtraction and taking the maximum) via the values $f(I)$, where I runs over the set Int of intervals in $[n]$. Thus, $f(S)$ can be regarded as a function of variables $f(I)$, $I \in Int$. Moreover, it is a piecewise linear function. A remarkable feature is that the function is convex! Equivalently, this function is a tropical Laurent polynomial. Such a behavior of TP-functions w.r.t. the standard basis is a sample of the so-called *tropical Laurent phenomenon* (cf. [**8**]).

More precisely, a *tropical Laurent polynomial* (of variables ξ_e) is the maximum of a finite collection of tropical Laurent monomials. A tropical Laurent monomial is an integer linear form of ξ_e. So a tropical Laurent polynomial is expressed as

$$P(\xi) = \max_j \left(\sum\nolimits_e a_{j,e}\xi_e\right),$$

where the coefficients $a_{j,e}$ are integer.

Proposition 5. *Let f be a TP-function f on the Boolean cube $2^{[n]}$, and $S \subseteq [n]$. There exists a tropical Laurent polynomial P_S of variables associated with intervals $I \in Int$ such that*

$$f(S) = P_S(f(I),\, I \in Int).$$

Moreover, all coefficients of linear forms involved in P_S belong to $\{-1, 0, 1, 2\}$.

(Note that the lower and upper bounds –1 and 2 on the 'tropical monomial' coefficients in this expression are similar to those on the exponents of face variables established by Speyer and stated in the Main Theorem of [**20**], where algebraic Laurent polynomials are considered.)

Proof. We know (from Theorem 2 and Proposition 3) that f is determined by a weighting w, a function on the set V of vertices of the digraph Γ. More precisely,

$$f(S) = \max\{w(\mathcal{F})\colon \mathcal{F} \in \Phi_S\}, \tag{16}$$

where Φ_S is the set of admissible flows going from S. Thus, we have a representation of $f(S)$ as a tropical polynomial, but of variables w_v for $v \in V$. Here each monomial corresponds to a flow in Φ_S. Recall that each variable w_v is linearly expressed via the values of f on intervals (see (15)):

$$w_v = \sum\nolimits_{I\in Int} h_v(I)f(I), \tag{17}$$

where each coefficient $h_v(I)$ is 0, 1 or –1. Taking the sum of weights w_v over the set of vertices v covered by a flow $\mathcal{F}$ and substituting it into (16), we obtain the desired tropical Laurent polynomial:

$$f(S) = \max_{\mathcal{F}\in\Phi_S} \left(\sum\nolimits_{I\in Int} h_{\mathcal{F}}(I)f(I)\right), \tag{18}$$

where $h_{\mathcal{F}}(I) := \sum_{v\in\mathcal{F}} h_v(I)$.

It remains to show that the coefficients $h_{\mathcal{F}}(I)$ are between –1 and 2.

To show this, consider a path P in a flow $\mathcal{F}$. For an intermediate vertex $v = v_{pq}$ of P, we say that P makes *right turn* at v if the edge e of P entering v is horizontal (i.e., $e = (v_{p+1,q}, v)$) while the edge e' leaving v is vertical (i.e., $e' = (v, v_{q+1})$), and say that P makes *left turn* at v if e is vertical while e' is horizontal. Also if the first edge of P is horizontal, we (conditionally) say that P makes left turn at its beginning vertex as well. Let h_P denote the sum of functions h_{pq} over the vertices v_{pq} contained in P. The values of h_P on the intervals can be calculated by considering relations in (15) and making corresponding cancelations when moving along the path P. More precisely, one can see that

(i) if P makes left turn at v_{pq}, then $h_P([p-q+1..p]) = 1$ and $h_P([p-q+1..p-1]) = -1$ (unless $q = 1$, in which case the interval $[p-q+1..p-1]$ vanishes);

(ii) if P makes right turn at v_{pq}, then $h_P([p-q+1..p]) = -1$ and $h_P([p-q+1..p-1]) = 1$; and

(iii) $h_P(I) = 0$ for the remaining intervals I in $[n]$.

This enables us to estimate the values of $h_{\mathcal{F}}$, i.e., of the sum of the functions h_P over the paths P in $\mathcal{F}$. Consider an interval $I = [c..d]$. Since the paths in $\mathcal{F}$ are disjoint, (i)–(iii) show that there are at most two paths P such that $h_P(I) \neq 0$. Therefore, $|h_{\mathcal{F}}(I)| \le 2$. Suppose $h_{\mathcal{F}}(I) = -2$. Then $h_P(I) = h_{P'}(I) = -1$ for some (neighboring) paths P, P' in $\mathcal{F}$. In view of (i)–(iii), this can happen only if one of these paths makes right turn at the vertex v_{pq} with $p = d$ and $q = d-c+1$, while the other path makes left turn at the vertex $v_{p+1,q+1}$. But then P, P' must intersect (see Fig. 1(a)); a contradiction.

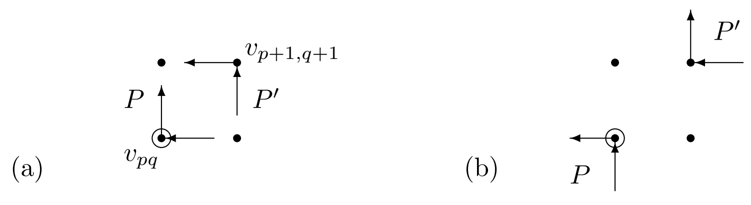


FIGURE 1. (a) $h_{\mathcal{F}}(I = [p-q+1..p]) = -2$; (b) $h_{\mathcal{F}}([p-q+1..p]) = 2$.

Thus, $-1 \le h_{\mathcal{F}}(I) \le 2$, as required. (In fact, $h_{\mathcal{F}}(I) = 2$ is possible; in this case there are two paths in $\mathcal{F}$, one making left turn at $v_{d,d-c+1}$, and the other making right turn at $v_{d+1,d-c+2}$. See Fig. 1(b).) □

Remark 4. Each flow $\mathcal{F}$ in expression (16) (or (18)) is essential. Indeed, if in the digraph Γ we put unit weights w_v for the vertices v covered by $\mathcal{F}$ and zero weights for the other vertices, then $\mathcal{F}$ is the unique maximum-weight flow in Φ_S for this weighting. So the number of linear pieces (slopes) in (18) is just $|\Phi_S|$. Next, adding an appropriate expression to each sum in the maximum, one can re-write (18) in the form

$$f(S) = \max\left\{\sum\nolimits_{I\in Int} h'_{\mathcal{F}}(I)f(I) \colon \mathcal{F} \in \Phi_S\right\} \\ - \sum \left(f(I) \colon I \in Int,\, I \subseteq [\min(S)+1..\max(S)-1]\right),$$

where all coefficients $h'_{\mathcal{F}}(I)$ are nonnegative integers not exceeding 3.

Remark 5. The admissible flows figured in (18) can be replaced by somewhat simpler objects. For $k \in \mathbb{Z}_+$, let us say that a triangular array $A = (a_{ij})$, $1 \le j \le i \le k$, is a *semi-strict Gelfand-Tsetlin pattern* of size k if $a_{i,j-1} < a_{ij} \le a_{i+1,j-1}$ holds for all i, j. (Classical Gelfand-Tsetlin patterns are defined by the non-strict inequalities in both sides.) The tuple $a_{11} < a_{21} < \ldots < a_{k1}$ is called the *shape* of A. For each $S \subseteq [n]$, there is a bijection between the set Φ_S of admissible flows going from S and the set of semi-strict GT-patterns of size $|S|$ with the shape $p_1 < \ldots < p_{|S|}$, where $S = \{p_1, \ldots, p_{|S|}\}$.

Indeed, given $\mathcal{F} \in \Phi_S$, let P_i be the path in $\mathcal{F}$ beginning at s_{p_i}. Let V_i be the set of vertices entered by vertical edges of P_i plus the source s_{p_i}. The second coordinate of the vertices in V_i runs from 1 through i (along P_i) and we denote these vertices as $v_{a_{ij},j}$, $j = 1, \ldots, i$. Then the admissibility of $\mathcal{F}$ implies that the arising triangular array (a_{ij}) is a semi-strict GT-pattern of size $|S|$. Conversely, given a semi-strict pattern A of size k with $a_{k1} \le n$, one can uniquely construct an admissible flow $\mathcal{F}$ in which the vertices entered by vertical edges are just $v_{a_{ij},j}$ for $i = 1, \ldots, k$ and $j = 2, \ldots, i$, and the sources are $v_{a_{i1},1} = s_{a_{i1}}$, $i = 1, \ldots, k$.

For a semi-strict GT-pattern A of shape $p_1 < \ldots < p_k \le n$ and a TP-function f on $2^{[n]}$, define

$$\hat{f}(A) := \sum\nolimits_{i,j} \Delta f([a_{ij} - j + 1..a_{ij}]),$$

where for an interval $I = [c..d]$,

$$\Delta f(I) := f(I) + f(I - \{c, d\}) - f(I - \{c\}) - f(I - \{d\})$$

if $c < d$, and $\Delta f(I) := f(I)$ if $c = d$ (assuming $f(\emptyset) = 0$). One can check that for an admissible flow $\mathcal{F}$ and its corresponding semi-strict GT-pattern A, $\hat{f}(A)$ is equivalent to $\sum_{I\in Int} h_{\mathcal{F}}(I)$. This and Proposition 5 give the following

Corollary 2. *For a TP-function f on $2^{[n]}$ and a subset $S = \{p_1, \ldots, p_{|S|}\} \subseteq [n]$ with $p_1 < \ldots < p_{|S|}$, one holds*

$$f(S) = \max\{\hat{f}(A)\},$$

where the maximum is taken over all semi-strict GT-patterns A with the shape $p_1 < \ldots < p_{|S|}$.

Remark 6. In the case of a truncated Boolean cube $B_m^{m'}(1^n)$, the tropical Laurentness property for the TP-functions w.r.t. the standard basis $\mathcal{B}$ is shown in a similar way as for the entire cube $2^{[n]}$. Then Proposition 5 is generalized as follows (see [**2**]).

Proposition 6. *Let f be a TP-function on a truncated Boolean cube $B = B_m^{m'}(1^n)$, and let $S \in B$. Then*

$$f(S) = \max_{\mathcal{F}} \left(\sum\nolimits_{X \in \mathcal{B}} h_{\mathcal{F}}(X) f(X) \right),$$

where $\mathcal{B}$ is the standard basis for B. Also each coefficient $h_{\mathcal{F}}(X)$ in this expression is in $\{-1, 0, 1, 2\}$.

Finally, using a similar approach and considering flows in the digraph Γ_a defined in Subsection 4.2, one can show the tropical Laurentness property for the TP-functions on a box (see [**2**] for details).

Proposition 7. *Let f be a TP-function on a box $B(a)$, and let $x \in B(a)$. Then the value $f(x)$ is expressed as*

$$f(x) = \max_{\mathcal{F}} \left(\sum\nolimits_{I \in Int(a)} h_{\mathcal{F}}(I) f(I) \right)$$

(where $\mathcal{F}$ concerns flows in Γ_a), and all coefficients $h_{\mathcal{F}}(I)$ are in $\{-1, 0, 1, 2\}$.

6. Bases and rhombus tilings

So far we have been concerned only with the standard basis. In this section we confine ourselves by considering an entire n-dimensional box $B(a)$ and deal with a class of bases that can be produced from the standard basis by a series of elementary transformations and have a nice graphical representation. (Recall that a (TP-)basis for $B(a)$ is a subset $\mathcal{B} \subseteq B(a)$ such that the restriction map $res : \mathcal{TP}(B(a)) \to \mathbb{R}^{\mathcal{B}}$ is a bijection.)

Suppose that a basis $\mathcal{B}$ for $B(a)$ contains the four vectors occurring in the right hand side of an instance of TP3-relation (3) and one vector in the left hand side, i.e., vectors $x + 1_i + 1_j$, $x + 1_k$, $x + 1_j + 1_k$, $x + 1_i$ and $x' \in \{x + 1_i + 1_k, x + 1_j\}$ for some $x \in B(a)$ and $i < j < k$. It is easy to see that replacing in $\mathbb{B}$ the element x' by the other element x'' of $\{x + 1_i + 1_k, x + 1_j\}$ (which is, obviously, not in $\mathbb{B}$) makes a basis as well. Such a transformation is said to be a *(TP3-)mutation*, or a *flip*, of $\mathbb{B}$ and we use notation $x' \rightsquigarrow x''$ for it.

Thus, starting from the standard basis $Int(a)$ (consisting of the fuzzy-intervals), one can produce other bases by making arbitrary sequences of flips. Let $\mathcal{M} = \mathcal{M}(a)$ be the set of all bases obtained in this way. (It is open for us whether $\mathcal{M}$ contains *all* bases for $B(a)$.)

We are interested in a certain subclass of $\mathcal{M}$. It concerns a special sort of mutations. More precisely, for x, i, j, k as above, we allow to apply the mutation $x' \rightsquigarrow x''$ only if the vectors x and $x+1_i+1_j+1_k$ belong to the basis $\mathbb{B}$ as well. In this case we (conditionally) call the mutation (flip) *normal* and use the same adjective for a basis obtained by a series of such mutations from $Int(a)$. It turns out that the normal bases one-to-one correspond to the rhombus tilings of the zonogon related to $B(a)$. A rhombus tiling is constructed as follows.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take n-vectors $\xi_1, \dots, \xi_n$ so that: (i) these vectors have Euclidean norm 1 and are ordered clockwise around $(0, 0)$, and (ii) all integer combinations of these vectors are different. Then the set

$$Z(a) := \{\lambda_1\xi_1 + \dots + \lambda_n\xi_n : 0 \leq \lambda_i \leq a_i,\ i = 1, \dots, n\}$$

is a $2n$-gone (when $a_i > 0$ for all i). Moreover, it is a zonogon, that is, the sum of n segments $[0, a_i\xi_i]$, $i = 1, \dots, n$. Also it is the image of a linear projection π of the convex box $\prod_{i\in[n]}[0, a_i] = \mathrm{conv}B(a)$ into the plane, namely, $\pi(x) = x_1\xi_1 + \dots + x_n\xi_n$.

A *rhombus tiling* D is a subdivision of the zonogon $Z(a)$ into rhombi with side length 1. It is easy to see that these rhombi have the form $q + \{\lambda_i\xi_i + \lambda_j\xi_j : 0 \leq \lambda_i, \lambda_j \leq 1\}$ for some $i < j$ and a point $q \in \pi(B(a))$. The tiling D can also be regarded as a directed planar graph whose vertices and edges are the vertices and side segments of the rhombi, respectively. Each edge e corresponds to a parallel translation of some vector ξ_i and is directed accordingly. Two instances are illustrated in Fig. 2.

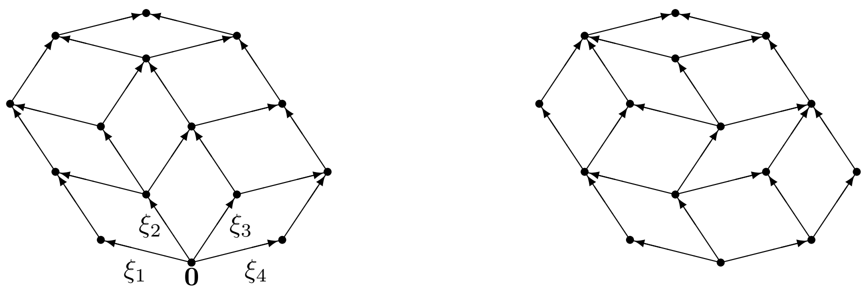
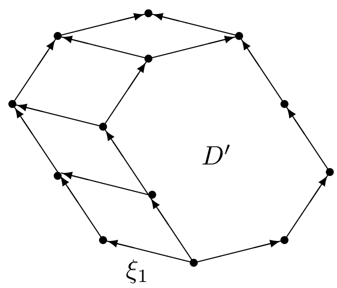


FIGURE 2. Two instances of tilings for $n = 4$ and $a = (1, 1, 2, 1)$.

Especially we are interested in the vertex set $V(D)$ of D because, as we shall see later, its pre-image $\pi^{-1}(V(D))$ gives a basis for $B(a)$. For example, the standard basis for $B(a)$ corresponds to the 'standard rhombus tiling' drawn in the left side of Fig. 2.

The *standard rhombus tiling* can be constructed by induction. Assume $a_1 > 0$ and let D' be the standard rhombus tiling of the zonogon $Z(a - 1_1)$. The zonogon $Z(a)$ is the sum of the zonogon $Z(a - 1_1)$ and the segment $[0, \xi_1]$. In other words, $Z(a)$ is the union of $Z(a - 1_1)$ and the strip $L + [0, \xi_1]$, where L is the left half of the boundary of $Z(a - 1_1)$. Now the standard tiling D of $Z(a)$ is obtained by adding to D' all rhombi of the form $E + [0, \xi_1]$, where E runs over all unit segments of L. See the picture below. If $a_1 = 0$, we work with a_2, and so on.

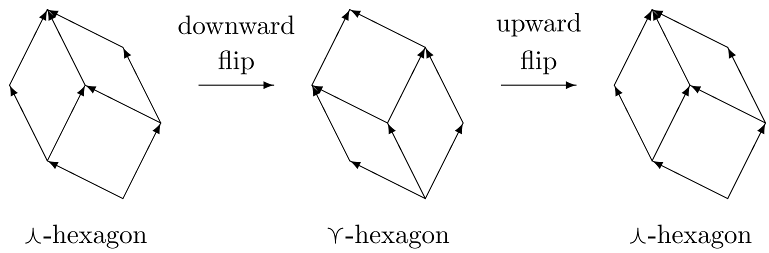


A similar induction is used to show that the vertices of the standard tiling correspond to the fuzzy-intervals in $B(a)$. Such a correspondence takes place in a general case.

Theorem 3. *For any rhombus tiling D, the set $V(D)$ is a basis for the box $B(a)$.*

To prove this theorem, we define the operation of *flip* for rhombus tilings. Then we show that for every rhombus tiling D, there is a series of downward flips which transform D into the standard tiling.

Let us first consider the case of 3-dimensional Boolean cube $C_3 = B(1,1,1)$. The corresponding zonogon $Z(1,1,1)$ is a hexagon. There are two rhombus tilings for it, and a flip is just the transformation of one to the other. See the picture:



The *upward flip* changes the point ξ_2 by the point $\xi_1+\xi_3$ whereas the *downward flip* makes the inverse change (these terms are borrowed from [**11**]). The Υ-tiling of the hexagon corresponds to the standard basis $\{\varnothing, 1, 2, 3, 12, 23, 123\}$ for C_3, whereas the ⅄-tiling corresponds to the basis $\{\varnothing, 1, 3, 12, 13, 23, 123\}$. Thus, flips on the tilings match flips on the bases for C_2.

Return to a general box $B(a)$. Suppose that a rhombus tiling D of $Z(a)$ contains a (little) hexagon H; then H is the zonogon of a 3-dimensional cube $x+B(1_i, 1_j, 1_k)$ (where $i < j < k$) lying in $B(a)$. This H is subdivided into three rhombi in D, and the flip in H results into another tiling D' of $Z(a)$. If the subdivision of H has the Υ-configuration and $V(D)$ is a basis for $B(a)$, then $x+1_j \rightsquigarrow x+1_i+1_k$ is a normal mutation (in view of $x, x+1_i+1_j+1_k \in V(D)$), and therefore, $V(D')$ is again a basis for $B(a)$ (we identify a subset of $B(a)$ and its image by π in $Z(a)$). Similarly for the ⅄-subdivision.

It remains to make the last step and to prove the following

Proposition 8. *Starting from any tiling D, one can reach the standard tiling by a series of downward flips.*

This assertion is proved in [**11**]; for completeness of our description we give a proof following a method in that paper. (An alternative proof of Proposition 8 using wirings and additional results on rhombus tilings are given in [**2**].)

If a tiling D contains a hexagon H of $\curlywedge$-form, then the downward flip in H results in a tiling whose total height of vertices is smaller, where the *height* $h(v)$ of a vertex $v = a'_1\xi_1 + \ldots + a'_n\xi_n$ is $a'_1 + \ldots + a'_n$ (or the length of a path from the minimal vertex 0 to v). So we can consider a tiling without $\curlywedge$-hexagons.

Claim. *Let a tiling D of $Z(a)$ have no $\curlywedge$-hexagon. Then D is the standard rhombus tiling of $Z(a)$.*

Proof of the claim Consider a vertex v of D, and let $(u_1, v), \ldots, (u_k, v)$ be its entering edges, in this order around v. Suppose that $k \geq 3$ and consider the second edge (u_2, v). One can see that the vertex u_2 has only one leaving edge, namely, (u_2, v), and that u_2 cannot belong to the boundary of $Z(a)$. These facts imply that u_2 has at least two entering edges. Moreover, the number of these edges is more than two; for otherwise u_2 belongs to exactly three rhombi, and these rhombi form a $\curlywedge$-hexagon. So we can take u_2 instead of v, and so on (every time decreasing the height of a vertex). This implies that D has no vertices with three or more entering edges at all.

Now let $\mathcal{R}$ be the set of rhombi of D that have a common edge with the left half L of the boundary of $Z(a)$. Going along L from the maximal vertex $a_1\xi_1+\ldots+a_n\xi_n$ to the minimal vertex 0 and using the above property, it is not difficult to conclude that the union of rhombi in $\mathcal{R}$ forms the strip $L + [0, \xi_1]$ (assuming $a_1 > 0$). Then the rest of $Z(a)$ is the zonogon $Z(a - 1_1)$, and now the claim follows by induction on $|a|$. □

This completes the proofs of Proposition 8 and Theorem 3. □□

Next we show one more fact about rhombus tilings (it will be used in next sections). Imagine that the box $B(a)$ contains a sub-box $B(p, p') := p+B(a')$, where $a' = p' - p$. Projecting it to the plane, we obtain the sub-zonogon $Z' = \pi(p)+Z(a')$ in $Z(a)$.

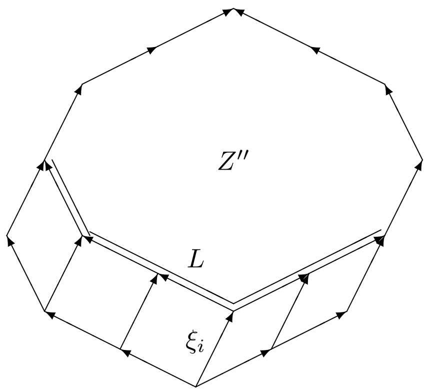
Proposition 9. *Any tiling D' of the sub-zonogon Z' corresponding to $B(p, p')$ (with $0 \leq p < p' \leq a$) can be extended to a tiling of $Z(a)$.*

(In [**11**], this is proved for the case when Z' is a hexagon.)

Proof. One may assume that the sub-box $B(p, p')$ is smaller than $B(a)$. Then $B(p, p')$ is contained in one of the following sub-boxes:

$$B'_i := B(1_i, a) \quad \text{or} \quad B''_i := B(0, a - 1_i), \quad i = 1, \ldots, n.$$

Let for definiteness $B(p, p')$ be contained in some B'_i. By induction on the size of a box, the tiling D' of Z' can be extended to a tiling D'' of the zonogon $Z'' = Z(1_i, a)$. So it suffices to extend D'' to a tiling of $Z(a)$. Note that $Z(a)$ is $Z'' + [-\xi_i, 0]$. Then the desired tiling for $Z(a)$ is obtained by adding to D'' the (unique) tiling of the strip $L + [-\xi_i, 0]$, where L is the part of boundary of Z'' 'visible in the direction ξ_i' (see the picture).



When $B(p, p')$ is contained in a box B''_i, we argue in a similar way. □

Proposition 10. *Let $B \subset B'$ be two truncated boxes. Then any TP-function f on B can be extended to a TP-function f' on B'.*

Proof. We may assume that B' is an entire box. Due to Corollary 1, one may assume that B is an entire box as well. These B' and B correspond to a zonogon Z' and its sub-zonogon Z, respectively.

Let D be a tiling of Z, e.g., the standard one. By Theorem 3, $V(D)$ is a basis for B; let f_0 be the restriction of f to this basis. By Proposition 9, there is a tiling D' of Z' extending D; then $V(D) \subset V(D')$. Again by Theorem 3, $V(D')$ is a TP-basis for B'. Extend f_0 to a function g_0 on $V(D')$. Then g_0 determines a TP-function g on B', and this g coincides with f within B. □

Remark 7. An interesting open problem is: given a subset $X \subseteq B(a)$, decide whether or not X can be extended to a TP-basis for $B(a)$. A similar problem concerning *normal* bases has a solution (recall that the normal bases are those corresponding to the rhombus tilings). More precisely, it is shown in [**15**] (see also [**2**]) that X is extendable to a normal basis (equivalently, $\pi(X)$ is extendable to the vertex set of a tiling of $Z(a)$) if and only if X satisfies the following *betweenness condition*:

(Btw) for any two points $x, x' \in X$ and any $i, k \in [n]$, if $x_i < x'_i$ and $x_k < x'_k$, then $x_j \le x'_j$ holds for each j between i and k (i.e., $\min\{i,k\} < j < \max\{i,k\}$).

The simplest example of violation of this condition is the set consisting of the points 2 and 13 in the Boolean cube $2^{[3]}$; we know that they cannot simultaneously occur in the same rhombus tiling.

7. Submodular TP-functions

In this section we consider TP-functions on a box $B(a)$ with the additional property of submodularity. We demonstrate that a TP-function is submodular if and only if its restriction to the standard basis (the set of fuzzy-intervals) $Int(a)$ is such.

Recall that a function f on a lattice $\mathcal{L}$, with meet operation $\wedge$ and join operation $\vee$, is called *submodular* if it satisfies the *submodular inequality*

$$f(\alpha) + f(\beta) \geq f(\alpha \wedge \beta) + f(\alpha \vee \beta)$$

for each pair $\alpha, \beta \in \mathcal{L}$. (When a part $\mathcal{L}'$ of the lattice is considered, the submodular inequality is imposed whenever all $\alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta$ occur is $\mathcal{L}'$.)

The lattice operations on a box $B(a)$ are defined in a natural way (coordinatewise). A simple fact is that a function f on the lattice $B(a)$ is submodular if and only if

$$f(x + 1_i) + f(x + 1_j) \geq f(x) + f(x + 1_i + 1_j) \tag{19}$$

holds for all x, i, j $(i \neq j)$ such that all four vectors involved belong to $B(a)$.

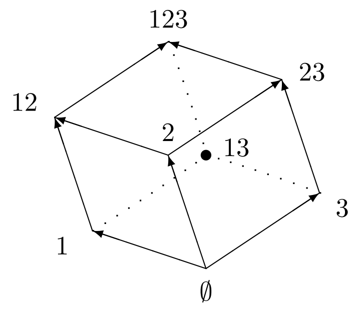
Theorem 4. *A TP-function f on a box $B(a)$ is submodular if and only f it is submodular on the standard basis $Int(a)$. The latter means that* (19) *holds whenever $i \neq j$ and the four vectors occurring in it belong to $Int(a)$.*

Proof. We use results on rhombic tilings from Section 6.

Consider elements $x, x+1_i, x+1_j, x+1_i+1_j$ of $B(a)$ $(i \neq j)$. Their images in the zonogon $Z(a)$ form a (little) rhombus, and by Proposition 10, this rhombus belongs to some tiling of $Z(a)$. In other words, the above four elements are contained in some normal basis for $B(a)$. In light of this, we can reformulate the theorem (and thereby slightly strengthen it) by asserting that if a TP-function f is submodular with respect to some normal basis $\mathcal{B}$ (or its corresponding tiling), then f is submodular w.r.t. any other normal basis. (When saying that f is *submodular w.r.t. $\mathcal{B}$*, we mean that (19) holds whenever the four vectors there belong to $\mathcal{B}$. The theorem considers as $\mathcal{B}$ the standard basis $Int(a)$.)

Next, we know (see Proposition 8) that making flips, one can reach any normal basis from a fixed one. Therefore, it suffices to show that the submodularity is maintained by flips.

In other words, it suffices to prove the theorem for the simplest case when $B(a)$ is the 3-dimensional Boolean cube $C = 2^{[3]}$. In this case, the standard basis Int consists of the sets $\emptyset, 1, 2, 3, 12, 23, 123$, the submodularity on Int involves the three rhombi of the corresponding tiling, and one has to check the submodularity for the three rhombi arising under the mutation $2 \rightsquigarrow 13$; see the picture.



Let f be a TP-function on C, i.e., f satisfies

$$f(2)+f(13)=\max\{f(1)+f(23),f(3)+f(12)\}. \tag{20}$$

The submodularity on Int reads as:

$$f(\emptyset)+f(23) \leq f(2)+f(3); \tag{21}$$
$$f(\emptyset)+f(12) \leq f(1)+f(2); \tag{22}$$
$$f(2)+f(123) \leq f(12)+f(23). \tag{23}$$

We show that (20)–(23) imply the submodular inequalities for the other three rhombi, as follows. Adding $f(1)$ to (both sides of) (21) gives

$$f(1)+f(23)\leq f(2)+f(3)-f(\emptyset)+f(1).$$

Adding $f(3)$ to (22) gives

$$f(3)+f(13)\leq f(1)+f(2)-f(\emptyset)+f(3).$$

Substituting these inequalities into (20), we obtain

$$f(2)+f(13)=\max(f(1)+f(23),f(3)+f(12))\leq f(1)+f(2)+f(3)-f(\emptyset),$$

which implies the submodular inequality for the rhombus on $\emptyset,1,3,13$:

$$f(\emptyset)+f(13)\leq f(1)+f(3).$$

Arguing similarly, one obtains the submodular inequalities for the rhombi on $1,12,13,123$ and on $3,13,23,123$. More precisely:

$$\begin{aligned} f(1)+f(123) &\leq f(1)+f(12)+f(23)-f(2) && \text{(by (23))}\\ &\leq f(2)+f(13)+f(12)-f(2) && \text{(by (20))}\\ &= f(13)+f(12); \end{aligned}$$

and

$$\begin{aligned} f(3)+f(123) &\leq f(3)+f(12)+f(23)-f(2) && \text{(by (23))}\\ &\leq f(2)+f(13)+f(23)-f(2) && \text{(by (20))}\\ &= f(13)+f(23). \end{aligned}$$

(Note that if needed, one can reverse the arguments to obtain (21)–(23) from the other three inequalities.) □

Remark 8. If we replace in Theorem 4 the submodularity condition by the corresponding *supermodularity* condition (i.e., replace $\geq$ by $\leq$), then the TP-function f need not be supermodular globally, even in the Boolean case with $n=3$. A counterexample is the function on $2^{[3]}$ taking value 0 on $\{\emptyset\},1,2,3,12$ and value 1 on $13,23,123$ (the supermodularity is violated for the sets 13 and 23). On the other hand, one can show that a version of the theorem concerning *modular* TP-functions is valid.

8. Skew-submodular TP-functions

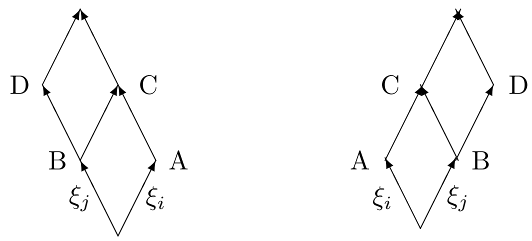
In this section we show that another important property can also be TP-propagated from the standard basis to the entire box.

Definition. We say that a function f on a box $B(a)$ is *skew-submodular* if

$$f(x+1_i+1_j)+f(x+1_j)\ge f(x+1_i)+f(x+2_j) \qquad (24)$$

for all x,i,j $(i\ne j)$ such that all four vectors involved are in $B(a)$.

Here 2_j stands for $2{\cdot}1_j$, and i,j need not be ordered. So the skew-submodularity imposes a restriction on f within each sub-box of the form $B(x,x+1_i+2_j)$ in $B(a)$. The picture below illustrates the corresponding tiling of the zonogon $Z(1_i+2_j)$ when $i<j$ (on the right) and $j<i$ (on the left); here the skew-submodular condition reads as $f(B)+f(C)\ge f(A)+f(D)$.



In fact, one can regard (24) as a degenerate form of the TP3-relation (3). Indeed, putting $j=k$ in (3), we obtain

$$f(x+1_i+1_j)+f(x+1_j)=\max\{f(x+1_i+1_j)+f(x+1_j),f(x+2_j)+f(x+1_i)\},$$

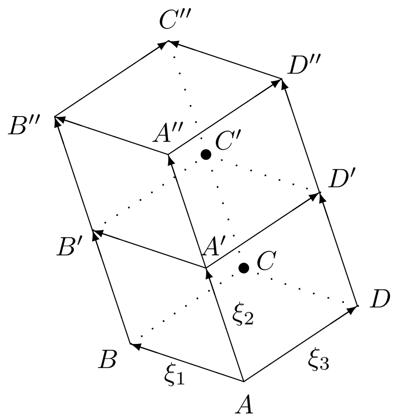
which is just equivalent to (24).

Theorem 5. *A TP-function f on a box $B(a)$ is skew-submodular if and only if its restriction to the standard basis $Int(a)$ is skew-submodular (in the sense that holds whenever $i\ne j$ and the four vectors occurring in it belong to $Int(a)$). Furthermore, a skew-submodular f satisfies the additional relation*

$$f(x+1_i+1_j)+f(x+1_j+1_k)\ge f(x+1_i+1_k)+f(x+2_j), \qquad (25)$$

where i,j,k are different.

Proof. Arguing as in the previous section and using Propositions 8 and 9, we reduce the task to examination of the 3-dimensional boxes $B(1,1,2)$, $B(1,2,1)$ and $B(2,1,1)$. Below we consider the case $B(1,2,1)$ (in the other two cases, the proof is analogous and we leave it to the reader as an exercise). This case is illustrated in the picture:



There are two TP3-relations in the box $B(1,2,1)$, namely:

$$f(A')+f(C)=\max(f(B)+f(D'),f(B')+f(D)) \tag{26}$$

and

$$f(A'')+f(C')=\max(f(B')+f(D''),f(B'')+f(D')). \tag{27}$$

The face (parallelogram) $AA''B''B$ gives the skew-submodular inequality in the standard basis:

$$f(A'')+f(B)\le f(A')+f(B'). \tag{28}$$

The face $AA''D''D$ gives one more skew-submodular inequality

$$f(D)+f(A'')\le f(D')+f(A'). \tag{29}$$

First of all we prove inequality (25) (with $(i,j,k)=(1,2,3)$); it is viewed as

$$f(B')+f(D')\ge f(A'')+f(C). \tag{30}$$

Adding $f(D')$ to (28) gives

$$f(A'')+f(B)+f(D')\le f(A')+f(B')+f(D').$$

Adding $f(B')$ to (29) gives

$$f(D)+f(A'')+f(B')\le f(A')+f(D')+f(B').$$

These inequalities together with (26) result in

$$f(A'')+f(A')+f(C)\le f(A')+f(B')+f(D').$$

Now the desired inequality (30) is obtained by canceling $f(A')$ in both sides.

Next we show validity of the other two skew-submodular inequalities in the box, namely, those concerning the faces $BB''C''C$ and $DD''C''C$.

Adding (30) and the inequality $f(B'')+f(D')\le f(A'')+f(C')$ (which is a consequence of (27)), we obtain

$$f(A'')+f(C)+f(B'')+f(D')\le f(B')+f(D')+f(A'')+f(C').$$

Canceling $f(A'')+f(D')$ in this inequality gives

$$f(C)+f(B'')\le f(B')+f(C'),$$

which is just the skew-submodular inequality for the face $BB''C''C$. The skew-submodular inequality $f(C)+f(D'')\le f(D')+f(C')$ for the face $DD''C''C$ is obtained in a similar way. □

9. Discrete concave TP-functions

In this section we combine the above submodular and skew-submodular conditions on TP-functions.

Let us say that a TP-function f on a box $B(a)$ is a *DCTP-function* if

$$f(x+1_i+1_j)+f(x+1_j+1_k)\ge f(x+2_j)+f(x+1_i+1_k) \tag{31}$$

holds for all $x\in B(a)$ and $i,j,k\in\{0\}\cup[n]$ such that the four vectors in this relation belong to $B(a)$. Here 1_0 means the zero vector. Note that i,j,k need not be ordered and some of them may coincide.

Remark 9. The meaning of the abbreviation 'DC' is that the TP-functions obeying (31) possess the property of *discrete concavity.* More precisely, one can check that such functions satisfy requirements in a discrete concavity theorem from [**16**, Ch. 6], and therefore, they form a subclass of *polymatroidal concave functions*, or $M^{\#}$-concave functions, in terminology of that book.

Observe that if $j=0\ne i,k$ and $i\ne k$, then (31) turns into the submodular condition (cf. (19)). If $k=0\ne i,j$ and $i\ne j$, then (31) turns into the skew-submodular condition (24). And if $i=k=0$, then (31) turns into the concavity inequality

$$2f(x+1_j)\ge f(x)+f(x+2_j).$$

One easily shows that this inequality follows from submodular and skew-submodular relations.

Now assume that none of i,j,k is 0. If all i,j,k are different, then (31) is a consequence of the skew-submodularity, due to Theorem 5. Finally, if $i=k$, then (31) turns into

$$2f(x+1_i+1_j)\ge f(x+2_i)+f(x+2_j),$$

which again is easily shown to follow from skew-submodular relations.

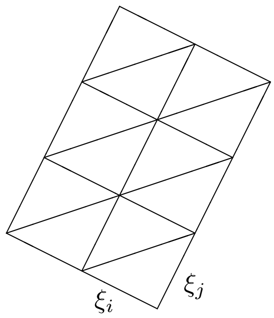
The above observations are summarized as follows.

Proposition 11. *A TP-function on a box is a DCTP-function if and only if it is submodular and skew-submodular.*

This proposition and Theorems 4 and 5 give the following

Corollary 3. *A TP-function f on a box $B(a)$ is a DCTP-function if and only if it is submodular and skew-submodular on the standard basis $Int(a)$.*

One can visualize this corollary by considering the standard tiling of the zonogon $Z(a)$. It contains 'big' parallelograms $P(i,j)$ for $i<j$, where $P(i,j)$ is the sub-zonogon $Z(a_{i+1}\xi_{i+1}+\ldots+a_{j-1}\xi_{j-1}\ ;\ a_i1_i+a_j1_j)$. Subdivide each ij-rhombus $[x,x+\xi_i,x+\xi_j,x+\xi_i+\xi_j]$ in $P(i,j)$ into two triangles by drawing the diagonal $[x+\xi_i,x+\xi_j]$. This gives a triangulation of $P(i,j)$; see the picture where $a_i=2$ and $a_j=3$.



In terms of such triangulations, the submodular and skew-submodular conditions on f say that for any two adjacent triangles ABC and BCD, one holds $f(B)+f(C)\ge f(A)+f(D)$. In other words, the affine interpolation of f within each little triangle produces a globally concave function.

Appendix. Flow rearrangements

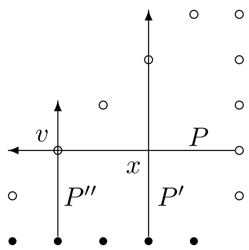
In this section we prove Proposition 4 from Section 4. Recall that we deal with the extended auxiliary digraph $\tilde\Gamma_a$ with the sources $s_1,\dots,s_N$ of sources and the sinks $t_1,\dots,t_N$. We are interested in (admissible) flows from left-squeezed subsets S of $[N]$ (identifying S with the subset $\{s_p\colon p\in S\}$ of sources). Let X,i,j,k be as in the hypotheses of Proposition 4. We start with the first part of this proposition.

Proposition 4a. *Let $\mathcal F$ be a flow from Xij, and $\mathcal F'$ a flow from Xk. Then the union of these flows can be rearranged as the union of a flow $\mathcal F_1$ from Xik and a flow $\mathcal F_2$ from Xj. A similar property is true for flows from the sets Xjk and Xi.*

Proof. It essentially uses the facts that the graph Γ_a is planar and the flows in question are left-squeezed.

Under the visualization of the extended digraph $\tilde\Gamma_a$ as in Section 4, a path P in $\tilde\Gamma_a$ is represented as a (piecewise linear) curve in the plane; denote it by $\zeta(P)$. Also for each block L_i, let T_i denote the triangle in the plane with the vertices $(\bar a_{i-1}+1,1)$, $(\bar a_i,1)$ and $(\bar a_i,a_i-1)$.

We observe that for any flow $\mathcal F''$ from a left-squeezed set S of sources, the curves $\zeta(P)$, $P\in\mathcal F''$, are pairwise non-intersecting. Indeed, suppose there are $P,P'\in\mathcal F''$ such that $\zeta(P),\zeta(P')$ meet at a point x. Then x is the point $(\bar a_i+r,q)$ for some i and $1\le q<r\le a_{i+1}$ (lying in the interior of T_{i+1}). Therefore, one of P,P' contains the vertex $v=(\bar a_i+q,q)$ of Γ_a. Since $\mathcal F''$ is left-squeezed, the source $s_{\bar a_i+q}$ is in S, and therefore, there is a path $P''\in\mathcal F''$ beginning at this source. But then P'' passes the vertex v as well, which is impossible. See the picture.



Next, we associate to a path P in $\tilde{\Gamma}_a$ beginning at a source $s=(p,0)$ and ending at a sink $t=(q,q)$ the closed south-west region in the plane bounded by $\zeta(P)$, the horizontal ray from t to $(-\infty,q)$, and the vertical ray from s to $(p,-\infty)$. We call it the *lower region* of P and denote by $\mathcal{R}(P)$. From the above observation it follows that if a flow from a left-squeezed set of sources consists of paths $P_1,\ldots,P_r$, where P_i ends at t_i, then $\mathcal{R}(P_1)\subset\mathcal{R}(P_2)\subset\ldots\subset\mathcal{R}(P_r)$.

Now consider $X,i,j,k,\mathcal{F},\mathcal{F}'$ as in the hypotheses of the proposition. Let $r:=|X|+2$, and let $\mathcal{F}=\{P_1,\ldots,P_r\}$ and $\mathcal{F}'=\{P'_1,\ldots,P'_{r-1}\}$. We combine the flows $\mathcal{F},\mathcal{F}'$ into one family $\mathcal{P}=(P_1,\ldots,P_r,P'_1,\ldots,P'_{r-1})$ (possibly containing repeated paths). Observe that

(i) each vertex belongs to at most two paths in $\mathcal{P}$;

(ii) for $p\in[N]$, the source s_p is the beginning of exactly one path in $\mathcal{P}$ if $p\in\{i,j,k\}$, and the beginning of exactly two paths if $p\in X$;

(iii) each of the sinks $t_1,\ldots,t_{r-1}$ is the end of exactly two paths in $\mathcal{P}$, and t_t is the end of exactly one path.

Also one can see that

(iv) for any two members P,P' of $\mathcal{Q}$, the intersection of $\zeta(P)$ and $\zeta(P')$ is (the image of) a *subgraph* of $\tilde{\Gamma}_a$ (i.e., these curves cannot cross in the interiors of $T_1,\ldots,T_n$).

Using a standard planar flow decomposition technique and relying on (iv), one can rearrange paths in $\mathcal{P}$ so as to obtain a family $\mathcal{Q}=\{Q_1,\ldots,Q_{2r-1}\}$ of paths from sources to sinks in $\tilde{\Gamma}_a$ having properties (ii), (iii) as above (with $\mathcal{Q}$ in place of $\mathcal{P}$), and in addition:

(v) for each vertex v of $\tilde{\Gamma}_a$, the numbers of occurrences of v in paths of $\mathcal{Q}$ and in paths of $\mathcal{P}$ are equal;

(vi) $\mathcal{R}(Q_1)\subseteq\mathcal{R}(Q_2)\subseteq\ldots\subseteq\mathcal{R}(Q_{2r-1})$.

(Such a $\mathcal{Q}$ is constructed uniquely.) Partition $\mathcal{Q}$ into two subfamilies:

$$\mathcal{F}_1:=\{Q_p\colon p \text{ is odd}\}\quad\text{and}\quad\mathcal{F}_2:=\{Q_p\colon p \text{ is even}\}.$$

We assert that each of these subfamilies consists of pairwise disjoint paths. Indeed, suppose this is not so. Then, in view of (vi), some subfamily contains 'consecutive' paths Q_p,Q_{p+2} that share a common vertex v. But now the inclusions $\mathcal{R}(Q_p)\subseteq\mathcal{R}(Q_{p+1})\subseteq\mathcal{R}(Q_{p+2})$ imply that v must belong to the third path Q_{p+1} as well, which is impossible by (i) and (v).

This assertion together with (ii),(iii),(v) easily implies that both $\mathcal{F}_1,\mathcal{F}_2$ are admissible flows, that the set of the beginning vertices of paths in $\mathcal{F}_1$ consists of the sources s_i,s_k and s_p for all $p \in X$, and that the set of the beginning vertices of paths in $\mathcal{F}_2$ consists of the sources s_j and s_p for all $p \in X$. Here we use the fact that, due to $i < j < k$, the paths in $\mathcal{Q}$ beginning at s_i, s_j, s_k have odd, even and odd indices, respectively.

The proof for flows from Xjk and Xi is similar. □

Proposition 4b. *Let $\mathcal{F}$ be a flow from Xik, and $\mathcal{F}'$ a flow from Xj. Then the union of these flows can be rearranged as the union of flows $\mathcal{F}_1$ and $\mathcal{F}_2$ such that: either*

(1) $\mathcal{F}_1$ goes from Xij and $\mathcal{F}_2$ goes from Xk, or

(2) $\mathcal{F}_1$ goes from Xjk and $\mathcal{F}_2$ goes from Xi.

Proof. In fact, this assertion can be extracted from a result in [**16**, p. 60]. We give a direct proof by arguing in a similar spirit.

Regarding $\mathcal{F}$ as a graph, we modify it as follows. Each vertex v of $\mathcal{F}$ is replaced by edge $e_v = (v', v'')$; each original edge (u, v) of $\mathcal{F}$ is transformed into edge (u'', v'). The resulting graph, consisting of pairwise disjoint paths as before, is denoted by $\gamma(\mathcal{F})$. The graph $\mathcal{F}'$ is modified into $\gamma(\mathcal{F}')$ in a similar way. Corresponding edges of $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$ are identified.

Next we construct an auxiliary graph H by the following rule:

(a) if e is an edge in $\gamma(\mathcal{F})$ but not in $\gamma(\mathcal{F}')$, then e is included in H;

(b) if $e = (u, v)$ is an edge in $\gamma(\mathcal{F}')$ but not in $\gamma(\mathcal{F})$, then the edge (v, u) *reverse* to e is included in H.

(Common edges of $\gamma(\mathcal{F}), \gamma(\mathcal{F}')$ are not included in H.) One can see that H has the following properties: each vertex has at most one incoming edge and at most one outgoing edge; the vertices having one outgoing edge and no incoming edge are exactly s'_i, s'_k; the vertices having one incoming edge and no outgoing edge are exactly s'_j, t''_r, where $r = |X| + 2$. This implies that H is represented as the disjoint union of cycles, isolated vertices and two paths P, Q, where either P is a path from s'_i to s'_j and Q is a path from s'_k to t''_r (*Case 1*), or P is a path from s'_k to s'_j and Q is a path from s'_i to t''_r (*Case 2*).

We use the path P to rearrange the graphs $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$ as follows: for each edge $e = (u, v)$ of P,

(c) if e is in $\gamma(\mathcal{F})$, then we delete e from $\gamma(\mathcal{F})$ and add to $\gamma(\mathcal{F}')$;

(d) if e is not in $\gamma(\mathcal{F})$, and therefore, the edge $\bar{e} = (v, u)$ reverse to e is in $\gamma(\mathcal{F}')$, then we delete $\bar{e}$ from $\gamma(\mathcal{F}')$ and add to $\gamma(\mathcal{F})$.

Let $\mathcal{G}$ and $\mathcal{G}'$ be the graphs obtained in this way from $\gamma(\mathcal{F})$ and $\gamma(\mathcal{F}')$, respectively (if there appear isolated vertices, we ignore them). In these graphs we shrink each edge of the form $e_v = (v', v'')$ into one vertex v. This produces subgraphs $\mathcal{F}_1$ and $\mathcal{F}_2$ of Γ, where the former corresponds to $\mathcal{G}$, and the latter to $\mathcal{G}'$.

It is not difficult to deduce from (a)–(d) that each of $\mathcal{F}_1, \mathcal{F}_2$ consists of pairwise disjoint paths, and moreover: in Case 1, $\mathcal{F}_1$ is a flow from Xjk and $\mathcal{F}_2$ is a flow from Xi, while in Case 2, $\mathcal{F}_1$ is a flow from Xij and $\mathcal{F}_2$ is a flow from Xk. Also one can see that for each vertex v of Γ, the numbers of occurrences of v in paths of $\{\mathcal{F}_1, \mathcal{F}_2\}$ and in paths of $\{\mathcal{F}, \mathcal{F}'\}$ are the same. □

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