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# Discrete convexity and unimodularity—I

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## Abstract

In this paper we develop a theory of convexity for the lattice of integer points  $\mathbb{Z}^n$ , which we call theory of discrete convexity. Namely, we characterize classes of subsets of  $\mathbb{Z}^n$ , which possess the separation property, or, equivalently, classes of integer polyhedra such that intersection of any two polyhedra of a class is an integer polyhedron (need not be in the class). Specifically, we show that these (maximal) classes are in one-to-one correspondence with pure systems. Unimodular systems constitute an important instance of pure systems. Given a unimodular system, we construct a pair of (dual) discretely convex classes, one of which is stable under summation and the other is stable under intersection.

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## 1. Introduction

In this paper we develop a theory of convexity for the lattice of integer points  $\mathbb{Z}^n$ , which we call theory of *discrete convexity*.

What subsets  $X \subset \mathbb{Z}^n$  could be called “convex”? One property seems indisputable:  $X$  should coincide with the set of all integer points of its convex hull  $\text{co}(X)$ . We call such sets *pseudo-convex*. The resulting class  $\mathcal{PC}$  of all pseudo-convex sets is stable under intersection but not under summation. In other words, the sum  $X + Y$  of two

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pseudo-convex sets  $X$  and  $Y$  needs not be pseudo-convex. We should consider subclasses of  $\mathcal{PC}$  in order to obtain stability under summation.

As we show stability under summation is closely related to another question: when the intersection of two integer polytopes is an integer polytope? Beginning from the paper [7], it is known that the class of generalized polymatroids has this property. Let us define a  $PM$ -set in  $\mathbb{Z}^n$  as the set of integer points of some (integer)  $g$ -polymatroid. Then the class of all  $PM$ -sets is a class of discrete convexity (DC-class). Specifically, the sum of  $PM$ -sets is a  $PM$ -set, and non-intersecting  $PM$ -sets can be separated by some linear functional. This class of sets was studied by Murota [14]. In series of papers [10,15,16] was developed discrete convex analysis based on this DC-class and its dual DC-class.

On a way of developing theory of discrete convexity at least two questions arise:

(1) Can we extend the class of  $PM$ -sets without losing in the process the nice properties which precisely made us consider it at the very beginning?

(2) Do other classes exist which exhibit similar properties? If so, how are they to be constructed or described?

Answers on these questions ('No' on the first one and 'Yes' on the second) rest on a relation of the discrete convexity with unimodular systems. The latter are nothing but invariant versions of totally unimodular matrices (we discuss their properties in Section 5). Every unimodular system  $\mathcal{R}$  defines a class  $\mathcal{Pt}(\mathcal{R}, \mathbb{Z})$  of integer polytopes which possesses two properties:

- (a) it is stable under summation;
- (b) the intersection of any two polytopes from  $\mathcal{Pt}(\mathcal{R}, \mathbb{Z})$  is an integer polytope.

The class  $\mathcal{Pt}(\mathcal{R}, \mathbb{Z})$  consists of those integer polytopes all edges of which are parallel to some elements of  $\mathcal{R}$ . Moreover, any ample class of integer polytopes with the properties (a) and (b) has such a form.

For example, the class of  $g$ -polymatroids corresponds to the unimodular system  $\mathbb{A}_n$  in  $\mathbb{Z}^n$  which consists of vectors  $\pm e_i$  and  $e_i - e_j$ ,  $i, j = 1, \dots, n$ . Since this system is maximal as a unimodular system, we obtain the negative answer on the question 1). However, there are many other (maximal) unimodular systems (see [2]) what gives many other classes of discrete convexity.

The classes  $\mathcal{Pt}(\mathcal{R}, \mathbb{Z})$  (as well as the class of integer  $g$ -polymatroids) are stable under summation but not under intersection. Given an unimodular system  $\mathcal{R}$  one can construct another (dual) class of discrete convexity which is stable under intersection (but not under summation). We show in Theorem 3 that the theory becomes enough poor if to require stability DC-classes under summation as well as under intersection.

It is worth to note that in sequel we develop the theory of discrete convexity not only for polytopes but for polyhedra as well. Because of this we find more convenient to work with pure systems instead of unimodular systems. Though, most interesting examples are related to the latter ones.

The paper is organized as follows. In Section 2 we consider several properties which one could want to require from a "good" theory of discrete convexity. We find

that all of them are in essence equivalent. In Section 3 we introduce so called pure systems and discuss their properties. In Section 4 we construct classes of discrete convexity via the pure systems. Sections 5 and 6 are devoted to an important instance of pure systems, namely to unimodular systems. Each such a system enables us to construct a pair of (dual) DC-classes, one of which is stable under summation and the other is stable under intersection, and these classes contain “many” finite sets. In Section 7 we discuss an issue on defining of polytopes from  $\mathcal{P}t(\mathcal{R})$  by means of linear inequalities.

In a separate paper [5] we plan to develop corresponding theory of discretely convex functions based on our theory of discrete convexity.

Finally, we want to point out that recently theory of discrete convexity unexpectedly shown its importance in areas far from discrete mathematics, such as in mathematical economics [6], for a solution of the Horn problem [4], for modules over discrete valuations rings [3], in theory of representation of groups.

**Notations.** In the sequel  $M$  denotes a free Abelian group of finite type.<sup>1</sup>  $V = M \otimes \mathbb{R} \cong \mathbb{R}^n$  denotes the ambient vector space. Elements of  $M$  are called *integer points* of  $V$ . Given a subset  $P \subset V$ , we denote by  $P(\mathbb{Z}) = P \cap M$  the set of integer points of  $P$ .

$M^* = \text{Hom}(M, \mathbb{Z})$  denotes the dual group, that is the group of homomorphisms of Abelian groups  $M \rightarrow \mathbb{Z}$ .  $V^* = M^* \otimes \mathbb{R}$  is the dual vector space to  $V$ . For  $Q \subset V^*$ , we put  $Q(\mathbb{Z}) = Q \cap M^*$ .

Let  $X, Y$  be subsets of  $V$ . Then  $X + Y = \{x + y, x \in X, y \in Y\}$  denotes the (Minkowski) sum of  $X$  and  $Y$ ;  $X - Y$  is understood in a similar fashion.  $\text{co}(X)$  denotes the convex hull of  $X$  in  $V$ .  $\mathbb{Z}(X)$  is the Abelian subgroup in  $V$  generated by  $X$ , that is the set of linear combinations of the form  $\sum_x m_x x$ , where  $x \in X$  and  $m_x \in \mathbb{Z}$ .  $\mathbb{R}X$  denotes the vector subspace generated by  $X$ .

## 2. Discrete convexity: the basics

The issue here is to characterize those subsets  $X$  of a group  $M (\cong \mathbb{Z}^n)$  that we would be willing to call “convex”.

**Definition.** A subset  $X \subset M$  is said to be *pseudo-convex* if  $X = \text{co}(X)(\mathbb{Z})$  and  $\text{co}(X)$  is a polyhedron.

Recall that a polyhedron is the intersection of some finite collection of closed half-spaces of  $V$ . For example, a linear sub-variety of  $V$ , or a polytope (the convex hull of some finite subset in  $V$ ) is a polyhedron. For more details about polyhedra, see [11] or [17].

We denote by  $\mathcal{PC}$  the set of pseudo-convex sets.

<sup>1</sup>Of course,  $M$  is isomorphic to  $\mathbb{Z}^n$  for an appropriate  $n$ , but a general theory does not need to distinguish a basis of the group.

**Definition.** A polyhedron  $P \subset V$  is *rational* if it is given by a finite system of linear inequalities of the form  $p(v) \leq a$  where  $p \in M^*$  and  $a \in \mathbb{Z}$ . A polyhedron  $P$  is called *integer* if it is rational and if every (non-empty) face of  $P$  contains an integer point.

For example, a polytope is integer if and only if all its vertices are integer points.

**Proposition 1.** *Suppose  $X \subset M$ . The following assertions are equivalent:*

- (a)  $X$  is pseudo-convex;
- (b)  $X = P(\mathbb{Z})$  for some integer polyhedron  $P \subset V$ ;
- (c)  $X$  is the set of integer solutions of a finite system of linear inequalities with integer coefficients.

**Proof.** Implication (a)  $\Rightarrow$  (b) is almost obvious; it suffices to take  $P$  to be  $\text{co}(X)$ . Implication (b)  $\Rightarrow$  (c) is obvious. Finally, implication (c)  $\Rightarrow$  (a) is precisely Meyer's theorem (see, for example, [18, Theorem 16.1]).  $\square$

Denote by  $\mathcal{I}Ph$  the class of all integer polyhedra in  $V$ . By Proposition 1, we have the natural bijection between the classes  $\mathcal{I}Ph$  and  $\mathcal{P}\mathcal{C}$ , which is given by the mappings  $P \mapsto P(\mathbb{Z})$  and  $X \mapsto \text{co}(X)$ . Both these classes are stable under integer translations ( $X \mapsto X + m$ ,  $m \in \mathbb{Z}^n$ ), under the reflection ( $X \mapsto -X$ ), and under taking faces ( $X \mapsto X \cap F$ , where  $F$  is a face of the polyhedron  $\text{co}(X)$ ). Furthermore, the class  $\mathcal{P}\mathcal{C}$  is stable under intersection and is not stable under summation, whereas the class  $\mathcal{I}Ph$  is stable under summation and is not stable under intersection (the sum of two pseudo-convex sets needs not be pseudo-convex, while the intersection of integer polyhedra need not be integer).

Indeed, let us consider the following simple example in  $\mathbb{Z}^2$ . Suppose  $X = \{(0, 0), (1, 1)\}$  and  $Y = \{(0, 1), (1, 0)\}$ . Both  $X$  and  $Y$  are pseudo-convex. Despite that  $X$  and  $Y$  do not intersect, they can not be separated by a linear functional (or a hyperplane).

This example suggests that in order to have the separation property in theory of discrete convexity, we need to consider narrower classes of subsets of  $M$  than the class  $\mathcal{P}\mathcal{C}$ .

We say that a class  $\mathcal{K} \subset \mathcal{P}\mathcal{C}$  is *ample* if  $\mathcal{K}$  is stable under (a) integer translations, (b) reflection, and (c) faces. In the same way we understand ampleness of a polyhedral class  $\mathcal{P} \subset \mathcal{I}Ph$ .

**Proposition 2.** *Let  $\mathcal{K} \subset \mathcal{P}\mathcal{C}$  be an ample class. The following four properties of  $\mathcal{K}$  are equivalent:*

- (Add) for every  $X, Y \in \mathcal{K}$  the sets  $X \pm Y$  are pseudo-convex;
- (Sep) if sets  $X$  and  $Y$  of  $\mathcal{K}$  do not intersect, then there exists (integer) linear functional  $p : V \rightarrow \mathbb{R}$  such that  $p(x) > p(y)$  for any  $x \in X$ ,  $y \in Y$ ;
- (Int) if sets  $X$  and  $Y$  of  $\mathcal{K}$  do not intersect, then the polyhedra  $\text{co}(X)$  and  $\text{co}(Y)$  do not intersect as well;
- (Edm) for every  $X, Y \in \mathcal{K}$  the polyhedron  $\text{co}(X) \cap \text{co}(Y)$  is integer.

**Proof.** (Add)  $\Rightarrow$  (Sep): If  $X$  and  $Y$  have an empty intersection, then  $0 \notin X - Y$ . Since the set  $X - Y$  is pseudo-convex,  $0$  does not belong to the polyhedron  $\text{co}(X - Y) = \text{co}(X) - \text{co}(Y)$ . Hence there exists a linear (integer) functional  $p : V \rightarrow \mathbb{R}$  which is strictly positive on  $\text{co}(X - Y)$ . Therefore  $p(x) > p(y)$  for  $x \in X$  and  $y \in Y$ .

(Sep)  $\Rightarrow$  (Int): This one is obvious.

(Int)  $\Rightarrow$  (Add): Let us show that  $X - Y$  is pseudo-convex. Since  $\text{co}(X - Y) = \text{co}(X) - \text{co}(Y)$  is a polyhedron, we need to prove that  $X - Y = \text{co}(X - Y) \cap M$ . Suppose the integer point  $m$  lies in  $\text{co}(X - Y) = \text{co}(X) - \text{co}(Y)$ . Then the polyhedra  $\text{co}(X)$  and  $m + \text{co}(Y) = \text{co}(m + Y)$  intersect. Applying (Int) to the sets  $X$  and  $m + Y$ , we see that these sets also intersect, that is  $m \in X - Y$ .

(Edm)  $\Rightarrow$  (Int): This implication is obvious.

(Int)  $\Rightarrow$  (Edm): Suppose  $X, Y \in \mathcal{K}$ ,  $P = \text{co}(X)$ ,  $Q = \text{co}(Y)$ . We need to show that  $P \cap Q$  is an integer polyhedron. Obviously  $P \cap Q$  is rational. Therefore, we need to establish that every (non-empty) face of  $P \cap Q$  contains an integer point. We assume here, without loss of generality, that the face is minimal.

Suppose  $F$  is a minimal (non-empty) face of the polyhedron  $P \cap Q$ . Let  $P'$  (resp.  $Q'$ ) be a minimal face of  $P$  (resp.  $Q$ ) which contains  $F$ . We claim that  $F = P' \cap Q'$ .

Projecting  $V$  along  $F$ , we may suppose additionally that  $F$  is of dimension 0. That is  $F$  consists of a single point, which is a vertex of  $P \cap Q$ . Suppose, on the contrary, that  $P' \cap Q'$  contains some other point  $a$ . Since the point  $F$  is relatively interior both in  $P'$  and in  $Q'$ , then  $F$  is an interior point of some segment  $[a, b]$ , lying in both  $P'$  and  $Q'$ . But in such a case the segment  $[a, b] \subset P' \cap Q' \subset P \cap Q$ , and  $F$  can not be a vertex of  $P \cap Q$ . Contradiction.

Thus,  $F = P' \cap Q'$ . Since our class  $\mathcal{K}$  is stable under faces, the sets  $P'(\mathbb{Z})$  and  $Q'(\mathbb{Z})$  belong to  $\mathcal{K}$ . The property (Int) implies that the sets  $P'(\mathbb{Z})$  and  $Q'(\mathbb{Z})$  intersect. Because of this,  $F$  is an integer singleton.  $\square$

**Definition.** An ample class  $\mathcal{K} \subset \mathcal{P}\mathcal{C}$  is a *class of discrete convexity* (or a *DC-class*) if it possesses anyone of the properties from Proposition 2.

On the language of integer polyhedra, the definition of discrete convexity is formulated as follows. A class  $\mathcal{P}$  of integer polyhedra is a *polyhedral class of discrete convexity* if it is ample and the following variant of the Edmonds' condition holds:

(Edm') The intersection of any two polyhedra from  $\mathcal{P}$  is an integer polyhedron (not necessarily in  $\mathcal{P}$ ).

According to Proposition 2, the equivalent requirement is:

(Add')  $(P + Q)(\mathbb{Z}) = P(\mathbb{Z}) + Q(\mathbb{Z})$  for every  $P, Q \in \mathcal{P}$ .

Let us give a few examples of DC-classes.

**Example 1.** (*One-dimensional case*). Let  $M \cong \mathbb{Z}$ . Then the class  $\mathcal{P}\mathcal{C}$  of all pseudo-convex sets is a DC-class. This is not the case in higher dimensions of course.

The class of integer rectangles in the plane  $\mathbb{R}^2$  is a DC-class. More generally, if  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are DC-classes in the groups  $M_1$  and  $M_2$ , respectively, then the class of sets of the form  $X_1 \times X_2$  with  $X_i \in \mathcal{K}_i$ ,  $i = 1, 2$ , is a DC-class in  $M_1 \times M_2$  as well.

**Example 2. (Hexagons).** Let us consider a more interesting class  $\mathcal{H}$  of polyhedra in  $\mathbb{R}^2$ . It consists of polyhedra defined by the inequalities  $a_1 \leq x_1 \leq b_1$ ,  $a_2 \leq x_2 \leq b_2$ ,  $c \leq x_1 + x_2 \leq d$ , where  $a_1, a_2, b_1, b_2, c$  and  $d$  are integers. It is easy to check that this hexagon (generally speaking, this hexagon can be degenerated to a polyhedron with smaller number of edges) has integer vertices. Obviously,  $\mathcal{H}$  is stable under integer translations, reflection and faces. Since the intersection of hexagons yields a hexagon, we conclude that  $\mathcal{H}$  is a polyhedral DC-class.

**Example 3. (Base polyhedra).** This is one of the possible high-dimensional generalizations of Example 2. Let  $N$  be a finite set, and  $V = (\mathbb{R}^N)^*$ . We interpret elements of  $V$  as measures on the set  $N$ . Recall, that a function  $b: 2^N \rightarrow \mathbb{R} \cup \{+\infty\}$  is called *submodular* if, for any  $S, T \subset N$ , the following inequality holds

$$b(S) + b(T) \geq b(S \cup T) + b(S \cap T).$$

The elements of  $V$  can be viewed as modular functions, i.e., functions which fulfill the above-written definition of submodularity with equality.

A *base polyhedron* is a polyhedron of the following form:

$$B(b) = \{x \in V \mid x(S) \leq b(S), S \subset N, \text{ and } x(N) = b(N)\},$$

where  $b$  is a submodular function. Obviously, the class  $\mathcal{B}$ , which consists of base polyhedra with integer-valued  $b$ , is stable under integer translations and under reflection. One can show that it is stable under faces, and hence, each base polyhedron has integer vertices. The well-known theorem by Edmonds [7] ensures that property (Edm) obtains, and thus  $\mathcal{B}$  is a polyhedral DC-class. The reader will find details of the proofs of these properties of base polyhedra in [9], or see our Example 13.

**Example 4.** Here we give another high-dimensional generalization of Example 2. Let  $N$  be a finite set, and let  $V = \mathbb{R}^N$  be the space of real-valued functions on  $N$ . Consider the class  $\mathcal{L}$  of polyhedra in  $V$ , given by the inequalities of the form  $a_i \leq x(i) \leq b_i$  and  $a_{ij} \leq x(i) - x(j) \leq b_{ij}$ , where  $i, j \in N$ , and all  $a$ 's and  $b$ 's are integers. We claim that these polyhedra are integer. Indeed, their vertices are given by equalities of the form  $x(i) = c_i$  and  $x(i) - x(j) = c_{ij}$  where  $c$ 's are integers. It is clear that  $x$  is an integer point.

Thus, the class  $\mathcal{L}$  consists of integer polytopes. Since it is stable under intersection, the axiom (Edm') is satisfied automatically, and  $\mathcal{L}$  is a polyhedral DC-class.

We give a general construction of DC-classes in Section 4.

In the classical context, convexity is preserved under summation and under intersection. It would be natural therefore to require these properties for the discrete set-up. For example, both the classes of segments and hexagons and their products

possess these properties. Moreover (see Theorem 3), these cases exhaust DC-classes, stable under both summation and intersection. The class  $\mathcal{B}$  described in Example 3 is stable under summation, but not under intersection (if  $|N| > 3$ ). Similarly, the class  $\mathcal{L}$  described in Example 4 is stable under intersection, but not under summation (if  $|N| > 2$ ). Therefore, when we consider classes stable under summation and classes stable under intersection separately, more interesting theory of discrete convexity is obtained.

**Definition.** An ample class  $\mathcal{K}$  of pseudo-convex sets is called an *S-class* if  $X + Y \in \mathcal{K}$  for any  $X, Y \in \mathcal{K}$ .

In particular,  $X - Y \in \mathcal{P}\mathcal{C}$  for any  $X, Y \in \mathcal{K}$ , and, thus, any *S-class* is a DC-class. However in order to characterize polyhedral *S-classes*, we have to require both that the class be stable under summation and axiom (Add') be satisfied. Note that the intersection of two polyhedra of a polyhedral *S-class* is an integer polyhedron, but need not be a polyhedron of this class.

**Definition.** An ample class  $\mathcal{P}$  of integer polyhedra is called a *polyhedral I-class* if  $P \cap Q \in \mathcal{P}$  for any  $P, Q \in \mathcal{P}$ .

Again any *I-class* is a DC-class, since the axiom (Edm') holds. Let  $P$  and  $Q$  be polyhedra in an *I-class*, then  $P(\mathbb{Z}) + Q(\mathbb{Z})$  is a pseudo-convex set, though  $P + Q$  need not be a polyhedron of this class.

### 3. Pure systems

Linear subspaces are the simplest polyhedra. For a (rational) vector subspace  $F \subset V$  the set  $S = F(\mathbb{Z})$  of all integer points of  $F$  is an Abelian subgroup of  $M$ . Such subgroups of  $M$  are called *pure*. Let us collect some properties of pure subgroups (of  $M$ ) in the following simple

**Lemma 1.** *Let  $S$  be a subgroup of a free Abelian group of finite type  $M$ . The following assertions are equivalent:*

- (1)  *$S$  is a pure subgroup;*
- (2)  *$S$  is a pseudo-convex subset of  $M$ ;*
- (3) *the factor group  $M/S$  is torsion-free;*
- (4) *the factor group  $M/S$  is a free Abelian group.*

In general, the sum of pure subgroups of  $M$  need not be a pure subgroup of  $M$ . For example, if  $M = \mathbf{Z}^2$ ,  $S = \mathbf{Z}(1, 1)$ ,  $S' = \mathbf{Z}(1, -1)$  then the group  $S + S'$  has the index 2 in  $M$ .

**Definition.** Pure subgroups  $S$  and  $S'$  of  $M$  are called *mutually pure* if the sum  $S + S'$  is a pure subgroup of  $M$ . Two (rational) linear subspaces  $L$  and  $L'$  of  $V$  are *mutually pure* if the subgroups  $L(\mathbf{Z})$  and  $L'(\mathbf{Z})$  are mutually pure.

There is the following criterion of the mutual purity.

**Lemma 2.** *Let  $S_1$  and  $S_2$  be two pure subgroups of  $M$ . They are mutually pure if and only if the image of natural homomorphism  $S_1 \rightarrow M/S_2$  is pure.*

**Proof.** The factor group  $(M/S_2)/(\text{Im}(S_1))$  is canonically isomorphic to  $M/(S_1 + S_2)$ .  $\square$

Pure subgroups naturally come in play in the study of DC-classes. Suppose we have a pseudo-convex subset  $X$  in  $M$ . Then we can consider the linear subspace  $\text{Tan}(X) := \mathbb{R}(X - X)$  in  $V$  (the “tangent space” of  $X$ ) and the subgroup  $S = \mathbb{Z}(X - X)$  in  $M$ . Of course,  $S \subset \text{Tan}(X)(\mathbb{Z})$ , and in the general case this inclusion is proper. Hence, in the general case,  $S$  needs not be a pure subgroup of  $M$ . Nevertheless, there is an instance when we can guarantee the purity of  $S$ .

For a natural number  $n$  and  $X \subset M$ , we denote by  $[n]X$  the sum of  $n$  copies of  $X$ ; for example,  $[2]X = X + X$ .

**Proposition 3.** *Let  $X \subset M$ . Suppose that  $[n]X$  is a pseudo-convex set for every  $n = 1, \dots$ . Then the subgroup  $\mathbf{Z}(X - X)$  is pure.*

**Proof.** Changing  $X$  to  $[n]X$  for an appropriate  $n$ , one can assume that  $X$  contain a point  $a$  which belongs to the relative interior of  $\text{co}(X)$ . Changing  $X$  by  $X - a$ , one can assume that  $0$  belongs to the relative interior of  $\text{co}(X)$ . In that case  $\mathbf{Z}(X - X) = \bigcup_{n \geq 1} [n]X$ . It remains to note that an increasing union of pseudo-convex sets is a pseudo-convex set.  $\square$

Given an ample class  $\mathcal{K}$  of pseudo-convex sets, we can associate to it the following system  $\mathcal{U}(\mathcal{K})$  of linear subspaces in  $V$  (the *homogenization* of  $\mathcal{K}$ ). Namely,

$$\mathcal{U}(\mathcal{K}) = \{\text{Tan}(X), X \in \mathcal{K}\}.$$

Similarly we define the system of vector subspaces  $\mathcal{U}(\mathcal{P})$  for an ample polyhedral class  $\mathcal{P}$ .

**Definition.** A collection  $\mathcal{U}$  of linear subspaces in  $V$  is called a *pure system* if any  $F, G \in \mathcal{U}$  are mutually pure subspaces. Elements of a pure system are called *flats*.

The homogenization of DC-classes produces pure systems. Say that an ample class  $\mathcal{P}$  of integer polyhedra is *very ample* if it contains the polyhedron  $nP$  with any integer  $n$  and any polyhedron  $P \in \mathcal{P}$ .

**Proposition 4.** *Let  $\mathcal{P}$  be a very ample DC-class  $\mathcal{P}$  of integer polyhedra. Then  $\mathcal{U}(\mathcal{P})$  is a pure system.*

**Proof.** Let  $F = \text{Tan}(P)$  and  $G = \text{Tan}(Q)$ , where  $P, Q \in \mathcal{P}$ . We have to show that the subgroup  $F(\mathbb{Z}) + G(\mathbb{Z})$  is pure. Of course, this subgroup contains the subgroup  $\mathbb{Z}((A + B) - (A + B))$ , where  $A = P(\mathbb{Z})$  and  $B = Q(\mathbb{Z})$ . According to Proposition 3, it suffices to check that the set  $[n](A + B) = [n]A + [n]B$  is pseudo-convex for any  $n = 1, \dots$

Since the class  $\mathcal{P}$  is discretely convex, the set  $A + A$  is pseudo-convex and coincides with  $2P(\mathbb{Z})$ . Similarly, for any  $n$ ,  $[n]A = (nP)(\mathbb{Z})$  as well as  $[n]B = nQ(\mathbb{Z})$ . At last,  $[n]A + [n]B = nP(\mathbb{Z}) + nQ(\mathbb{Z}) = (nP + nQ)(\mathbb{Z})$  is a pseudo-convex set since  $nP$  and  $nQ$  belong to  $\mathcal{P}$ .  $\square$

In the next section we show how to dehomogenize pure systems.

A pure system  $\mathcal{U}$  is said to be a *pure S-system* (correspondingly, a *pure I-system*) if  $F + G$  (correspondingly,  $F \cap G$ ) belongs to  $\mathcal{U}$  for any  $F, G \in \mathcal{U}$ . It is clear that the homogenization of an S-class is a pure S-system, and the homogenization of an I-class is a pure I-system.

Let us illustrate the homogenization procedure on the class  $\mathcal{B}$  of base polyhedra from Example 3.

**Example 5.** (*The homogenization of base polyhedra*). Recall, that here  $V = (\mathbb{R}^N)^*$  is the space of measures on a finite set  $N$ . Let  $B(b)$  be the base polyhedron defined by a submodular function  $b: 2^N \rightarrow \mathbb{R} \cup \{+\infty\}$ . We are going to show how the corresponding tangent space  $\text{Tan}(B(b))$  looks like. Here we can assume that  $B(b)$  is a symmetric (with respect to the origin 0) base polyhedron. This means that  $b(S) = b(N \setminus S)$ ; in particular,  $b(N) = 0$ . It is clear, that  $nB(b) = B(nb)$ . Therefore the tangent space  $\text{Tan}(B(b))$  is the base polyhedron  $B(\infty b)$ , that is given by the following list of equations

$$x(S) = 0, S \in \mathcal{F}(b),$$

where  $\mathcal{F}(b) = \{S \subset N, b(S) = 0\}$ . Obviously,  $\emptyset, N \in \mathcal{F}(b)$ . The symmetry of  $B(b)$  implies that  $N \setminus S \in \mathcal{F}(b)$  with any  $S \in \mathcal{F}(b)$ . Submodularity of  $b$  implies that  $S \cup T$  and  $S \cap T$  belong to  $\mathcal{F}(b)$  with any  $S, T \in \mathcal{F}(b)$ . Thus,  $\mathcal{F}(b)$  is a Boolean subalgebra of  $2^N$ .

We see that to give a flat of  $\mathcal{U}(\mathcal{B})$  is the same as to give a Boolean subalgebra of  $2^n$ , or is the same as to give an equivalence relation  $\approx$  on  $N$ . The corresponding flat  $F(\approx)$  consists of measures  $x \in V$  such that  $x(S) = 0$  for each equivalence class  $S$  of the relation  $\approx$ . The codimension of this flat  $F(\approx)$  is equal to the number of equivalence classes of  $\approx$ .

Let us consider, for instance, one-dimensional flats. These flats correspond to those equivalence relations which possess a single class of equivalence of cardinality 2, whereas all others classes are of cardinality 1. For example the one-

dimensional flat  $\mathbb{R}(e_i - e_j)$  corresponds to the equivalence relation whose 2-element class of equivalence is  $\{i, j\}$ . Here  $(e_i)$ ,  $i \in N$ , denote the Dirac measure at the point  $i \in N$ .

Similarly, flats of codimension 1 correspond to dichotomous equivalence relations (i.e., relations with only two equivalence classes, say  $T$  and  $N \setminus T$ ).

We denote  $\mathcal{U}(\mathbb{A}(N))$  this pure system.

Let us return to general pure systems. There holds the following finiteness property.

**Proposition 5.** *Any pure system is a finite set.*

**Proof.** Let  $\mathcal{U}$  be a pure system of pure subgroups in  $M$ . Let  $\mathbf{F}_2$  be the 2-elements field. For any pure subgroup  $S$  in  $M$  we can consider the corresponding  $\mathbf{F}_2$ -vector subspace  $S \otimes \mathbf{F}_2$  in the vector space  $M \otimes \mathbf{F}_2$ . It is clear that the dimension of  $S \otimes \mathbf{F}_2$  is equal to the rank of  $S$  (that is the dimension of  $S \otimes \mathbb{R}$ ).

We assert that for different  $S, S' \in \mathcal{U}$  their images  $S \otimes \mathbf{F}_2$  and  $S' \otimes \mathbf{F}_2$  are also different. Suppose that  $S \otimes \mathbf{F}_2 = S' \otimes \mathbf{F}_2$ . Then  $(S + S') \otimes \mathbf{F}_2 = (S \otimes \mathbf{F}_2) + (S' \otimes \mathbf{F}_2) = S \otimes \mathbf{F}_2$ . Since  $S + S'$  is pure then the rank of  $S + S'$  is equal to the rank of  $S$  (and is equal to the rank of  $S'$ ). Therefore  $S = S + S' = S'$ .  $\square$

**Dualization.** Now we discuss a construction of a dual (or orthogonal) pure system. For a vector subspace  $L$  of  $V$ , let  $L^\perp$  denote the orthogonal vector subspace of the dual vector space  $V^*$ , that is

$$L^\perp = \{p \in V^*, p(v) = 0 \text{ for any } v \in L\}.$$

**Theorem 1.** *Let  $L$  and  $L'$  be mutually pure subspaces of  $V$ . Then  $L^\perp$  and  $L'^\perp$  are mutually pure subspaces of  $V^*$ .*

For proving this theorem, it is convenient to use the notion of a pure homomorphism. Let  $M$  and  $N$  be free Abelian groups of finite type. Let us say that a homomorphism  $f : M \rightarrow N$  is *pure* if the factor group  $N/f(M)$  is a free (or torsion-free) Abelian group. This means, of course, that  $f(M)$  is a pure subgroup in  $N$ .

**Lemma 3.** *A homomorphism  $f : M \rightarrow N$  is pure if and only if the dual homomorphism  $f^* : N^* \rightarrow M^*$  is pure.*

**Proof.** Let us consider the canonical decomposition of the homomorphism  $f : M \rightarrow N$  in two exact sequences

$$0 \rightarrow K \rightarrow M \rightarrow H \rightarrow 0, \quad \text{and} \quad 0 \rightarrow H \rightarrow N \rightarrow C \rightarrow 0.$$

Since  $f$  is pure,  $C$  is a free Abelian group. The group  $H$  is free as a subgroup of the free group  $N$ . Therefore, both sequences are split. Hence the dual sequences

$$0 \rightarrow C^* \rightarrow N^* \rightarrow H^* \rightarrow 0, \quad 0 \rightarrow H^* \rightarrow M^* \rightarrow K^* \rightarrow 0$$

are exact (where  $X^* = \text{Hom}(X, \mathbb{Z})$ ). Since  $K^*$  is free, we obtain that  $f^* : N^* \rightarrow M^*$  is pure.  $\square$

**Proof of Theorem 1.** Set  $S = L(\mathbb{Z})$ , and similarly  $S' = L'(\mathbb{Z})$ . It is obvious that  $L^\perp(\mathbb{Z})$  is equal to  $S^\perp = \{p \in M^*, p(s) = 0 \ \forall s \in S\}$ . That is that  $S^\perp$  is the kernel of the canonical projection  $M^* \rightarrow S^*$  being dual to the inclusion  $S \rightarrow M$ . It is clear from this, that  $(S^\perp)^*$  can be identified to  $M/S$ .

We have to show that the subgroup  $S^\perp + S'^\perp$  is pure in  $M^*$ . That is, by Lemma 2, that the canonical homomorphism  $S^\perp \rightarrow M^*/S'^\perp$  is pure. By Lemma 3, it suffices to check that the dual homomorphism  $(M^*/S'^\perp)^* \rightarrow (S^\perp)^*$  is pure. The latter homomorphism can be identified to the canonical homomorphism  $S'^\perp \rightarrow M/S$ . But this homomorphism is pure because  $S$  and  $S'$  are mutually pure subgroups.  $\square$

**Corollary.** *Let  $\mathcal{U}$  be a pure system in  $V$ . Then the collection  $\mathcal{U}^\perp := \{L^\perp, L \in \mathcal{U}\}$  is a pure system in  $V^*$ .*

#### 4. Construction of DC-classes

In the previous section, we constructed pure systems via the homogenization of (very ample) DC-classes. Here we shall go in the opposite direction.

Let  $\mathcal{U}$  be a pure system in  $V$ . If we collect all integer translations of flats of  $\mathcal{U}$ , we get a polyhedral DC-class. However, this class is of a little interest. For instance, it contains no polytopes (except, may be, zero-dimensional ones). Below we define a more interesting (maximal) DC-class  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  of integer polyhedra associated to a given pure system  $\mathcal{U}$ .

**Definition.** Let  $\mathcal{U}$  be a collection of (rational) vector subspace in  $V$ . A polyhedron  $P$  is said to be  $\mathcal{U}$ -convex (or  $\mathcal{U}$ -polyhedron) if, for any face  $F$  of  $P$ , the tangent space  $\text{Tan}(F) = \mathbb{R}(F - F)$  belongs to  $\mathcal{U}$ .

Let  $\mathcal{Ph}(\mathcal{U})$  be the set of  $\mathcal{U}$ -polyhedra, and let  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  be the set of integer  $\mathcal{U}$ -polyhedra. Note that the class  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is stable under integer translations, reflection and faces. In other words, it is an ample (and even very ample) class of integer polyhedra. The homogenization of  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  returns us back to  $\mathcal{U}$ .

The following result will be used in the sequel.

**Proposition 6.** *Let  $P \in \mathcal{Ph}(\mathcal{U}, \mathbb{Z})$ , and  $L$  be an integer vector subspace in  $V$ . Suppose that  $L$  is mutually pure with any subspace of  $\mathcal{U}$ . Then the intersection  $P \cap L$  is an integer polyhedron.*

**Proof.** Let  $\gamma$  be a minimal face of  $P \cap L$ ; we have to show that  $\gamma$  is an integer polyhedron. In fact,  $\gamma$  is an affine subspaces in  $V$  because it has no faces. Changing  $P$  to its minimal face containing  $\gamma$ , we may assume that  $\gamma = P \cap L$ . Let  $\text{aff}(P)$  denote the affine span of  $P$ . Then we have  $\text{aff}(P) \cap L = \gamma$ . But  $\text{aff}(P)$  is an integer translation of the linear subspace  $\mathbb{R}(P - P)$ . Therefore we can assume that  $P$  is an integer translation of a linear subspace  $L'$  in  $V$ , say  $P = L' + m$  for some  $m \in M$ .

Now we can repeat the reasoning from Proposition 2. If  $L$  and  $L' + m$  do not intersect, then the assertion is obviously true. Let  $x \in L \cap (L' + m)$ , that is  $x \in L$  and  $x = x' + m$ ,  $x' \in L'$ . Then  $m = x - x'$  is an integer point of  $L - L'$ . Since  $L$  and  $L'$  are mutually pure,  $m \in L(\mathbb{Z}) - L'(\mathbb{Z})$ . That is, there exists an integer point  $l \in L$  such that  $l + m \in L'$ .  $\square$

Now we show that if  $\mathcal{U}$  is a pure system, then  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is a DC-class.

**Theorem 2.** *A class  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is a DC-class of integer polyhedra if and only if  $\mathcal{U}$  is a pure system.*

**Proof.** Since  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is a very ample, the “only if” part of this theorem was proven in Proposition 4. Let us prove the “if” part.

More precisely, we shall show that the intersection of two polyhedra from the class  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is an integer polyhedron. For this, we use a trick known in Algebraic Geometry as “reduction to the diagonal”. Namely, we replace the intersection of two polyhedra  $P$  and  $Q$  by the intersection of their direct product  $P \times Q$  with the linear subspace  $\Delta$  being the diagonal in  $V \times V$ .

Let us consider in  $V \times V$  the following system  $\mathcal{U} \times \mathcal{U}$  of subspaces of the form  $L \times L'$  where  $L, L' \in \mathcal{U}$ . Obviously,  $P \times Q$  is  $\mathcal{U} \times \mathcal{U}$ -polyhedron. The intersection  $P \times Q$  with the diagonal  $\Delta$  consists of points of the form  $(v, v)$  such that  $v$  belongs to  $P \cap Q$ . Therefore to prove that  $P \cap Q$  is an integer polyhedron in  $V$  is the same as to prove that  $(P \times Q) \cap \Delta$  is an integer polyhedron in  $V \times V$ .

By virtue of Proposition 6, it suffices to check that the diagonal  $\Delta$  is mutually pure with any subspace  $L \times L'$  where  $L, L' \in \mathcal{U}$ . But this is equivalent to the mutual purity of the subspaces  $L$  and  $L'$ . The latter property holds by the definition of pure systems.  $\square$

**Remark.** Indeed the above arguments allow us to obtain the following more general result. Let  $\mathcal{U}$  and  $\mathcal{U}'$  be two systems of subspaces in  $V$ . Suppose that for every  $L \in \mathcal{U}$  and  $L' \in \mathcal{U}'$  the subspaces  $L$  and  $L'$  are mutually pure. Then the intersection of any integer  $\mathcal{U}$ -polyhedron with any integer  $\mathcal{U}'$ -polyhedron is an integer polyhedron.

Of course, if a pure system  $\mathcal{U}$  is stable under summation (intersection) then the corresponding class  $\mathcal{Ph}(\mathcal{U}, \mathbb{Z})$  is an  $S$ -class ( $I$ -class).

## 5. Unimodular systems

We have shown above, that pure systems play a crucial role in the description and construction of DC-classes (of integer polyhedra in  $V$  or of pseudo-convex subsets in  $M$ ). The corresponding DC-classes contain, for example, all integer translations of flats. However, if we want that a DC-class contains *polytopes*, we have to provide that the corresponding pure system has “sufficiently many” one-dimensional flats. This means that every flat of our system is generated (as a vector subspace) by one-dimensional flats. Here we explain how to construct pure  $S$ -systems (and the dual pure  $I$ -systems, see the next section) by means of unimodular systems.

**Definition.** A subset  $\mathcal{R} \subset M$  is called *unimodular* if, for any subset  $B \subset \mathcal{R}$  the subgroup  $\mathbb{Z}B \subset M$  is pure. A *unimodular system* is a pair  $(M, \mathcal{R})$  where  $\mathcal{R}$  is a unimodular set in  $M$ . Non-zero elements of  $\mathcal{R}$  are called *roots*.

We call *flats* (or  $\mathcal{R}$ -*flats*) subspaces  $\mathbb{R}B$ , where  $B \subset \mathcal{R}$ . It is obvious that the set  $\mathcal{U}(\mathcal{R})$  of all  $\mathcal{R}$ -flats is a pure  $S$ -system.

Unimodular systems are closely related to totally unimodular matrices, that is matrices whose minors are equal to 0 or  $\pm 1$ . Suppose that a unimodular set  $\mathcal{R}$  is of full dimension, or, equivalently, spans  $V$ . If we pick a basis  $B \subset \mathcal{R}$  and represent vectors of  $\mathcal{R}$  as linear combinations of the basis vectors, then the matrix of coefficients is totally unimodular. In particular, the coefficients of this matrix are either 0 or  $\pm 1$ , which proves finiteness of any unimodular set. Conversely, columns of a totally unimodular  $n \times m$  matrix yield a unimodular set in  $\mathbb{Z}^n$ . Thus unimodular systems are nothing but coordinate-free representations of totally unimodular matrices. The reader might find many other characterizations of totally unimodular matrices in [18].

Consider some important examples of unimodular systems.

**Example 6.** In Example 5, we introduced the pure system  $\mathbb{A}(N)$ , which is spanned by one-dimensional flats  $\mathbb{Z}(e_i - e_j)$ ,  $i, j \in N$ . Therefore, the set of vectors  $e_i - e_j$ ,  $i, j \in N$ , is a unimodular set in  $(\mathbb{Z}^N)^*$ . Let us denote this system as well by  $\mathbb{A}(N)$ . Note that it is not of full dimension, since it spans the subspace  $\{x, x(N) = 0\}$ , which is orthogonal to the vector  $\mathbf{1}_N \in \mathbb{R}^N$ . We shall show in Section 7 that the class  $\mathcal{P}h(\mathbb{A}(N))$  coincides with the class of base polyhedra  $\mathcal{B}$  from Example 3.

If we project the set  $\mathbb{A}(N \cup \{0\})$  along the axis  $\mathbb{R}e_0$  onto the space  $(\mathbb{R}^N)^*$ , we obtain the full-dimensional unimodular system consisting of the vectors  $\pm e_i$  and  $e_i - e_j$ ,  $i, j \in N$ , in  $(\mathbb{Z}^N)^*$ . Of course, we could construct this system simply by adding the basic system  $(\pm e_i, i \in N)$  to the system  $\mathbb{A}(N)$ . We denote this system by  $\mathbb{A}_N$ . We shall show that  $\mathbb{A}_N$ -polyhedra are nothing but generalized polymatroids.

Sub-systems  $\mathcal{R} \subset \mathbb{A}_N$  (more precisely, symmetrical sub-systems, which contain 0 and  $-r$  for any  $r \in \mathcal{R}$ ) are called *graphic* unimodular systems.

**Example 7.** To any graph  $G$  one can associate another unimodular system, the so called *cographic* unimodular system  $\mathbb{D}(G)$ . It is located in the cohomology group  $H^1(G, \mathbb{Z})$  of the graph  $G$  and consists of the cohomology classes  $\pm[e]$ , corresponding to oriented edges of the graph  $G$ . The proof of the unimodularity of the system  $\mathbb{D}(G)$  is based on the fact that this system is, in some (matroidal) sense, dual to the graphic system associated with  $G$ .

Cubic (or 3-valent) graphs give the most interesting examples of cographic systems. The simplest example of such a graph is the complete graph  $K_4$  with 4 vertices. The corresponding system  $\mathbb{D}(K_4)$  is isomorphic to  $\mathbb{A}_3$ . The bipartite graph  $K_{3,3}$  yields a more interesting example. The system  $\mathbb{D}(K_{3,3})$  consists of the following 19 vectors in  $\mathbb{R}^4: \{0, \pm e_i, i = 1, \dots, 4, \pm(e_1 + e_2), \pm(e_2 + e_3), \pm(e_3 + e_4), \pm(e_4 + e_1), \pm(e_1 + e_2 + e_3 + e_4)\}$ .

One can check that  $\mathbb{D}(K_{3,3})$  is not a graphic system.

**Example 8.** There is an exceptional unimodular system  $\mathbb{E}_5$  in dimension 5 which is neither graphic no cographic. It consists of the following 21 vectors:  $0, \pm e_i, i = 1, \dots, 5, \pm(e_1 - e_2 + e_3), \pm(e_2 - e_3 + e_4), \pm(e_3 - e_4 + e_5), \pm(e_4 - e_5 + e_1), \pm(e_5 - e_1 + e_2)\}$ .

According to the Seymour theorem [19], every unimodular system can be constructed via graphic systems, cographic systems, and the system  $\mathbb{E}_5$ .

Let  $(M, \mathcal{R})$  and  $(M', \mathcal{R}')$  be unimodular systems. A homomorphism of Abelian groups  $\varphi: M \rightarrow M'$  is called a *morphism of unimodular systems* if  $\varphi(\mathcal{R}) \subset \mathcal{R}'$ . For example, if  $\varphi$  is the projection of  $M$  onto  $M' = M/\mathbb{Z}r$ , where  $r \in \mathcal{R}$ , then  $\varphi(\mathcal{R})$  is a unimodular set in  $M'$ . The *direct sum* of unimodular systems  $(M, \mathcal{R})$  and  $(M', \mathcal{R}')$  is a unimodular system  $(M \oplus M', \mathcal{R} \oplus \mathcal{R}')$ , where  $\mathcal{R} \oplus \mathcal{R}' = \mathcal{R} \cup \mathcal{R}'$ .

The following theorem characterizes unimodular systems  $\mathcal{R}$  whose pure systems  $\mathcal{U}(\mathcal{R})$  are stable under intersection. For such a system the corresponding DC-class  $\mathcal{P}(\mathcal{U})$  is simultaneously  $S$ -class and  $I$ -class.

**Theorem 3.** *Let  $\mathcal{R}$  be a unimodular set such that the pure system  $\mathcal{U}(\mathcal{R})$  is stable under intersection. Then  $\mathcal{R}$  is the direct sum of copies of  $\mathbb{A}_1$  and  $\mathbb{A}_2$ .*

**Proof.** The proof is by induction on the dimension of unimodular systems. The assertion is obvious in dimensions 1 and 2.

Consider first of all the case of dimension 3. Assume  $\mathcal{R}$  contains a flat  $S$  isomorphic to  $\mathbb{A}_2$ . Denote by  $e_1, e_2$  and  $e_1 + e_2$  the vectors of  $\mathcal{R} \cap S$ . We claim that there is at most one more vector of  $\mathcal{R}$  (up to collinearity). Suppose there are two non-collinear vectors. Clearly we may denote them by  $e_3$  and  $e_1 + e_3$ . Then, since  $\mathcal{U}(\mathcal{R})$  is stable under intersection,  $e_2 - e_3$  and  $e_1 + e_2 + e_3$  belong to  $\mathcal{R}$ . But this contradicts unimodularity of  $\mathcal{R}$ , and the claim is proven. Therefore,  $\mathcal{R}$  is isomorphic to  $\mathbb{A}_1 \oplus \mathbb{A}_2$ .

One can similarly check that if  $\mathcal{R}$  does not contain flats isomorphic to  $\mathbb{A}_2$ , then  $\mathcal{R}$  is isomorphic to  $\mathbb{A}_1 \oplus \mathbb{A}_1 \oplus \mathbb{A}_1$ . Thus, in the three-dimensional case, the proposition is verified.

General case. Let  $\mathcal{U}(\mathcal{R})$  contain a flat  $S$  isomorphic  $\mathbb{A}_2$ . This means that  $S$  is a plane of  $V$  such that  $\mathcal{R} \cap S \cong \mathbb{A}_2$ . We will show that there exists a flat  $T$  of codimension 2 in  $V$  such that

$$\mathcal{R} = (\mathcal{R} \cap S) \cup (\mathcal{R} \cap T). \tag{1}$$

By induction  $\mathcal{R} \cap T$  is equal to the sum of copies  $\mathbb{A}_1$  and  $\mathbb{A}_2$ , and we have  $\mathcal{R} \cap S \cong \mathbb{A}_2$ , so if (1) is true, the proposition is also true.

Pick a flat  $T$  of  $\mathcal{U}(\mathcal{R})$  of codimension 2 (in  $V$ ) such that  $T \cap S = 0$ . Obviously such a flat exists.

**Claim.**  $\mathcal{R} \subset S \cup T$ .

Let us consider the projection  $\pi : V \rightarrow S$  which has  $T$  as the kernel (the projection along  $T$ ). Then  $\pi(\mathcal{R})$  is a unimodular system of  $S$  which contains  $\mathcal{R} \cap S$ . Because  $\mathcal{R} \cap S \cong \mathbb{A}_2$  and the  $\mathbb{A}_2$  is a maximal unimodular system, any vector  $r \in \mathcal{R}$ , which does not belong to  $S \cup T$ , is projected into some vector  $r_1 \in \mathcal{R} \cap S$ . Therefore, we have  $r - r_1 \in T$ . On the other hand,  $r - r_1$  belongs to the flat  $\mathbb{R}r + \mathbb{R}r_1$ . Since  $\mathcal{U}(\mathcal{R})$  is closed under intersection, the line  $(\mathbb{R}r + \mathbb{R}r_1) \cap T$  is an one dimensional flat of  $\mathcal{U}(\mathcal{R})$ , and, hence, there exists a vector  $r_2 \in \mathcal{R}$  which spans this flat.

Now we consider the three-dimensional subspace  $S + \mathbb{R}r_2$  of  $V$  and the unimodular system  $\mathcal{R} \cap (S + \mathbb{R}r_2)$ . Obviously, the pure system of this unimodular system is closed under intersection. Therefore,  $\mathcal{R} \cap (S + \mathbb{R}r_2)$  is isomorphic to  $\mathbb{A}_2 \oplus \mathbb{A}_1$ . Thus, there can be at most one generator outside of  $\mathcal{R} \cap S$ : the vector  $r_2$ . However, we have another one: the vector  $r \neq \pm r_2$ . A contradiction. Therefore  $\mathcal{R} \subset S \cup T$  and the claim is proven.

Finally, suppose that  $\mathcal{U}(\mathcal{R})$  contains no flats isomorphic to  $\mathbb{A}_2$ . In such a case, we assert that  $\mathcal{R}$  equals the sum of  $n$  ( $= \dim V$ ) exemplars  $\mathbb{A}_1$ . Let  $r_1, \dots, r_n$  be linear independent elements of  $\mathcal{R}$ . We show that there holds  $\mathcal{R} = \{\pm r_1, \dots, \pm r_n\}$ . Assume some  $r \in \mathcal{R} \setminus \{\pm r_1, \dots, \pm r_n\}$ . Clearly we may assume that there holds  $r = r_1 + \dots + r_n$  (i.e.  $r$  does not belong to the coordinate hyperplanes). Let us consider the intersection of flats  $\mathbb{R}r_1 + \mathbb{R}r_2$  and  $\mathbb{R}r + \mathbb{R}r_3 + \dots + \mathbb{R}r_n$ . This intersection is a line  $\mathbb{R}(r_1 + r_2)$  and it is a flat of  $\mathcal{U}(\mathcal{R})$ . Therefore, we have  $r_1 + r_2 \in \mathcal{R}$  and, hence,  $\{\pm r_1, \pm r_2, \pm(r_1 + r_2)\} \subset \mathcal{R}$ , but  $\{\pm r_1, \pm r_2, \pm(r_1 + r_2)\}$  is isomorphic to  $\mathbb{A}_2$ . A contradiction.  $\square$

Of course, the largest possible DC-classes are of the most interest. Such DC-classes correspond to maximal pure systems and maximal unimodular systems.

**Definition.** A pure system  $\mathcal{U}$  in  $M$  is said to be *maximal* if for any subspace  $F$ , not of  $\mathcal{U}$ , the system  $\mathcal{U} \cup \{F\}$  is not a pure system. A unimodular system  $\mathcal{R}$  is *maximal* if for any  $r \notin \mathcal{R}$  the system of vectors  $\mathcal{R} \cup r$  is not a unimodular.

**Example 9.** (*The unimodular system  $\mathbb{A}_n$  is maximal*). Suppose that  $r = (r_1, \dots, r_n)$  is an integer vector such that  $\mathbb{A}_n \cup r$  is a unimodular system. Since  $\mathbb{A}_n$  contains the

basic system  $\{\pm e_i, i = 1, \dots, n\}$ , all  $r_i$  are equal to 0 or  $\pm 1$ . We assert that for any  $i \neq j$  there holds either  $r_i r_j = 0$  or  $r_i r_j = -1$ . In fact, assume that  $r_i r_j = 1$  holds for some  $i \neq j$ . Then consider the Abelian subgroup  $S$  generated by  $e_k, k \neq i, j, e_i - e_j$ , and  $r$ . The index of the subgroup  $S$  in  $M := \mathbb{Z}^n$  is equal to the determinant of the matrix

$$\begin{pmatrix} r_i & 1 \\ r_j & -1 \end{pmatrix},$$

that is  $\pm 2$ . That contradicts to the purity of  $S$ . Therefore,  $r$  has at most two non-zero coordinates, and in such a case these coordinates are of opposite signs. That is  $r \in \mathbb{A}_n$ .

Another useful way to say this is as follows. Let  $L$  be a (rational) line in  $\mathbb{R}^n$  and let  $\rho$  be the canonical projection of  $\mathbb{R}^n$  onto  $V' = \mathbb{R}^n/L$ . Suppose  $L$  is such that the image  $\rho(\mathbb{A}_n)$  is a unimodular system in  $V'$  (with respect to the integer structure  $\rho(M)$ ). Then  $L$  is generated by some  $r \in \mathbb{A}_n$ , and the unimodular system  $\rho(\mathbb{A}_n)$  is isomorphic to  $\mathbb{A}_{n-1}$ .

We assert that  $\mathbb{A}_n$  is not only maximal as a unimodular system but also *the corresponding pure system*  $\mathbb{U} = \mathcal{U}(\mathbb{A}_n)$  is maximal. For this we consider a vector subspace  $F$  and suppose that  $F$  is mutually pure with any flat of  $\mathbb{U}$ . We have to show that  $F$  also is a flat of  $\mathbb{U}$ .

Let us consider the canonical projection  $\phi$  of  $\mathbb{R}^n$  onto the vector space  $V' = \mathbb{R}^n/F$ . As above, the image  $\phi(\mathbb{A}_n)$  is a unimodular system in  $V'$ . Let now  $k, 1 \leq k \leq n$ , be such that  $R = \mathbb{R}^k \cap F$  is an one-dimensional subspace. Let us consider the restriction of  $\phi$  to  $\mathbb{R}^k$ . Since the image of  $\mathbb{A}_k$  is a unimodular set (as a subset of the unimodular set  $\phi(\mathbb{A}_n)$ ), we conclude that  $R$  is generated by some non-zero vector  $r \in \mathbb{A}_k \subset \mathbb{A}_n$ . Thus,  $F$  contains some root  $r$  of  $\mathbb{A}_n$ .

Now we consider the projection  $\rho$  of  $\mathbb{R}^n$  along  $\mathbb{R}r$ . For the space  $\mathbb{R}^n/\mathbb{R}r$  we have a similar case: a unimodular system  $\mathcal{R} = \rho(\mathbb{A}_n)$  isomorphic to  $\mathbb{A}_{n-1}$ , and a vector subspace  $F' = \rho(F)$  which is mutually pure with flats of  $\mathcal{R}$ . By induction,  $F'$  is a flat of  $\mathcal{R}$ . Therefore its pre-image  $F$  is a flat of  $\mathbb{A}_n$ .

As a consequence, we obtain that the DC-class  $\mathcal{P}h(\mathbb{A}_n, \mathbb{Z})$  of integer g-poly-matroids is maximal.

**Example 10.** The unimodular system  $\mathbb{E}_5$  is maximal too. However, the corresponding pure system is not maximal. In order to see this, consider the following homomorphism  $\phi: \mathbb{Z}^5 \rightarrow \mathbb{Z}$ ,  $\phi(x_1, \dots, x_5) = x_1 + \dots + x_5$ . It is clear that  $\phi(r) = \pm 1$  for any root  $r \in \mathbb{E}_5$ . Therefore the kernel of  $\phi$ , that is the hyperplane  $H = [x_1 + \dots + x_5 = 0]$ , is mutually pure with any flat of  $\mathbb{R}_5$ .

One can show that the  $S$ -class  $\mathcal{P}h(\mathbb{E}_5)$  consists of *zonohedra*, that is the sum of segments (bounded or not) every of which is parallel to some root  $r \in \mathbb{E}_5$ . We obtain

that the intersection of two such integer zonohedra, or the intersection of a zonohedron and the hyperplane  $H$ , is an integer polyhedron.

The pure system corresponding the maximal unimodular system  $\mathbb{D}(K_{3,3})$  also is not maximal. It can be expanded by adding some two-dimensional subspace.

$\mathbb{A}_n$  is a unique maximal unimodular system of dimension  $\leq 3$ . In dimension 4, besides  $\mathbb{A}_4$ , there is another maximal unimodular system  $\mathbb{D}(K_{3,3})$ . In dimension 5, there are 4 non-isomorphic maximal unimodular systems; there are 11 in dimension 6. For more details, we refer to the article [2], which contains a complete description of maximal unimodular systems.

Let  $\mathcal{R}$  be a unimodular system. Element  $r$  of  $\mathcal{R}$  can be identified with morphisms of  $\mathbb{A}_1$  to  $\mathcal{R}$ . Conversely, morphisms of  $\mathcal{R}$  to  $\mathbb{A}_1$  are called *co-roots*. In other words, a co-root is a homomorphism of groups  $\phi: M \rightarrow \mathbb{Z}$  such that  $|\phi(r)| \leq 1$  for any root  $r \in \mathcal{R}$ . The set of co-roots is denoted by  $\mathcal{R}^*$ .

A polyhedron is an  $\mathcal{R}$ -polyhedron if every of its face is parallel to some  $\mathcal{R}$ -flat. Denote by  $\mathcal{Ph}(\mathcal{R}, \mathbb{Z})$  the  $S$ -class of integer  $\mathcal{R}$ -polyhedra. A pseudo-convex set  $X$  in  $M$  is said to be  $\mathcal{R}$ -convex set if  $\text{co}(X)$  is a  $\mathcal{R}$ -polyhedron.

## 6. Dual DC-classes associated to unimodular systems

Besides the  $S$ -class of  $\mathcal{R}$ -polyhedra, we can associate to a unimodular system  $\mathcal{R}$  a dual  $I$ -class of integer  $*\mathcal{R}$ -polyhedra (in the dual vector space  $V^*$ ).

Let  $\mathcal{R}$  be a unimodular set in  $M$ , and let  $\mathcal{U} = \mathcal{U}(\mathcal{R})$  be the corresponding pure  $S$ -system in  $V$ . A polyhedron  $P$  in  $V^*$  is called  $*\mathcal{R}$ -convex (or  $*\mathcal{R}$ -polyhedron) if it belongs to  $\mathcal{Ph}(\mathcal{U}^\perp)$ , that is any face of  $P$  is orthogonal to some  $\mathcal{R}$ -flat. In other words, a  $*\mathcal{R}$ -polyhedron is given by a system of linear inequalities (where  $p$  is a linear functional on  $V$ )

$$p(r) \leq a(r), \text{ where } r \in \mathcal{R} \text{ and } a(r) \in \mathbb{R} \cup \{+\infty\}.$$

The inverse is also true. If all entries  $a(r)$  are integer, the corresponding polyhedron is integer. In fact, since the class  $\mathcal{U}^\perp$ -polyhedra is  $I$ -class, we have to prove that every hyperplane  $H_r(a) = \{p \in V^*, p(r) = a\}$ , where  $r \in \mathcal{R}$  and  $a \in \mathbb{Z}$ , contains an integer point. But this is a consequence of primitiveness of  $r$  in  $M$ . (This is a kind of the Hoffman–Kruskal theorem [12].)

Thus, the set of all integer  $*\mathcal{R}$ -polyhedra is an  $I$ -class of discrete convexity. For example, the class from Example 4 is the dual  $I$ -class corresponding to the unimodular system  $\mathbb{A}_n$ .

In order to “visualize” integer  $*\mathcal{R}$ -polyhedra, it is convenient to use the notion of a dicing [8]. A dicing is the following regular polyhedral decomposition of  $V^*$ . Let us consider the following countable (but locally finite) collection of hyperplanes  $H_r(a) = \{p \in V^*, p(r) = a\}$ , where  $r \in \mathcal{R}$  and  $a \in \mathbb{Z}$ . These hyperplanes split the space  $V^*$  into connected parts, the *regions* of the dicing. Regions are bounded sets if  $\mathcal{R}$  is of full dimension. The closure of any region, as well as any its face, is called a *chamber* of the dicing. The set  $\mathcal{D}(\mathcal{R})$  of the chambers form a polyhedral decomposition of  $V^*$ ,

that is the chambers intersect by faces and cover the whole space  $V^*$ . If  $\mathcal{R}$  is of full dimension, then the nodes of the dicing (that is zero-dimensional chambers) are integer points of  $V^*$ , i.e., are elements of  $M^*$ .

Each chamber of the dicing  $\mathcal{D}(\mathcal{R})$  is an integer  $\ast\mathcal{R}$ -polyhedron. Conversely, any integer  $\ast\mathcal{R}$ -polyhedron is a union of chambers of  $\mathcal{D}(\mathcal{R})$ . Thus, an integer  $\ast\mathcal{R}$ -polyhedron is nothing but a convex set composed of chambers.

**Example 11.** Let us consider the *dicing star*  $\mathbf{St}(\mathcal{R})$ . It is composed of the chambers of the dicing  $\mathcal{D}(\mathcal{R})$ , which contain the origin 0. In order to establish the convexity of  $\mathbf{St}(\mathcal{R})$ , we show that

$$\mathbf{St}(\mathcal{R}) = \{p \in V^*, r(p) \leq 1, \text{ where } r \in \mathcal{R}\}.$$

For the time being, we call  $\mathbf{St}'$  the polyhedron appearing on the right hand of the formula. Obviously any chamber which contains 0 belongs to  $\mathbf{St}'$ . Hence  $\mathbf{St}(\mathcal{R}) \subset \mathbf{St}'$ .

Conversely, let  $p \in \mathbf{St}' \setminus \mathbf{St}(\mathcal{R})$ . Assume we move from  $p$  to 0 along the segment  $[0, p]$ . At some time  $t, 0 < t < 1$ , the point  $tp$  will be on the boundary of  $\mathbf{St}(\mathcal{R})$ . Hence, there exists  $r \in \mathcal{R}$  with  $r(tp) = 1$ . This implies that  $r(p) = 1/t > 1$ , a contradiction.

From this description of  $\mathbf{St}(\mathcal{R})$  we see that integer points of  $\mathbf{St}(\mathcal{R})$  are the co-roots of  $\mathcal{R}$ ,

$$\mathbf{St}(\mathcal{R})(\mathbb{Z}) = \mathcal{R}^*.$$

Reversely,  $\mathbf{St}(\mathcal{R}) = \text{co}(\mathcal{R}^*)$ .

The dual pure system  $\mathcal{U}^\perp = \mathcal{U}(\mathcal{R})^\perp$  has the following structure. It consists of the hyperplanes-mirrors  $H_r(0) = (\mathbb{R}r)^\perp$  and all possible intersections of the mirrors. As well as a dicing, the mirrors cut the space  $V^*$  onto a finite number of cones (the *cameras*) which constitute the fan  $\Sigma(\mathcal{R})$  or  $\mathcal{R}^\perp$ . One-dimensional flats of  $\mathcal{U}^\perp$  are called *crossings* as well as their primitive generators (of  $M^*$ ). (Of course, the crossings exist only if the unimodular system  $\mathcal{R}$  is of full dimension.) As an element of  $M^*$ , a crossing is a surjective homomorphism of Abelian groups  $\xi: M \rightarrow \mathbb{Z}$  such that the kernel of  $\xi$  is a flat of  $\mathcal{R}$ . Let us denote  $\mathcal{R}^\vee$  the set of crossings in  $M^*$ .

**Lemma 4.**  $\mathcal{R}^\vee \subset \mathcal{R}^*$ .

**Proof.** If  $\mathcal{R}$  is not of full dimension, the set  $\mathcal{R}^\vee$  is empty. Therefore we can assume that  $\mathcal{R}$  is of full dimension. Let  $\xi$  be a crossing, that is a surjective homomorphism  $M \rightarrow \mathbb{Z}$ . Since the kernel  $\xi^{-1}(0)$  of  $\xi$  is a flat, the image  $\xi(\mathcal{R})$  is a unimodular system in  $\mathbb{Z}$ , that is  $\xi$  is a co-root.  $\square$

**Remark.** As Example 9 shows, for  $\mathcal{R} = \mathbb{A}_n$  we have the equality  $\mathcal{R}^\vee = \mathcal{R}^*$ . For other unimodular systems (such as  $\mathbb{E}_5$ ) the crossings constitute a proper subset of  $\mathcal{R}^*$ .

In general case, the set  $\mathcal{R}^\vee$  is not a unimodular system in  $M^*$  (see Theorem 3). However, if we can find a unimodular system  $\mathcal{Q}$  in  $\mathcal{R}^\vee$  (we call such  $\mathcal{Q}$  a *laminarization* of  $\mathcal{R}$ ), this brings us an advantage. Namely, in such a case we can construct  $\mathcal{R}$ -polyhedra simply as  $\ast\mathcal{Q}$ -polyhedra. That is to define them by systems of linear inequalities

$$\{v \in V, \zeta(v) \leq a(\zeta), \zeta \in \mathcal{Q}\}$$

with arbitrary “right parts”  $a(\zeta)$ . Of course, when  $a(\zeta)$  are integer, the corresponding polyhedron is integer too. Let us give a more precise realization of this idea.

**Example 12.** (See also [9]). A family  $\mathcal{T}$  of subsets of a finite set  $N$  is called *laminar* if for any  $A, B \in \mathcal{T}$ , either  $A \subset B$ , or  $B \subset A$ , or  $A \cap B = \emptyset$ . Without loss of generality, we can assume that any singleton belongs to  $\mathcal{T}$ .

Let  $\mathcal{T}$  be a laminar family. We assert that the set  $\mathcal{Q} = \{\pm \mathbf{1}_T, T \in \mathcal{T}\}$  is a unimodular set in the space  $\mathbb{R}^N$ . That is  $\mathcal{Q}$  is indeed a laminarization of the system  $\mathbb{A}_N$ . Since the orthogonal hyperplanes  $(\mathbf{1}_T)^\perp$  are  $\mathbb{A}_N$ -flats, we have to check that intersections of such hyperplanes are also  $\mathbb{A}_N$ -flats.

Let us recall (see Example 5) that an  $\mathbb{A}_N$ -flat takes the form

$$F(A_1, \dots, A_k) := \{x \in (\mathbb{R}^N)^*, x(A_j) = 0 \text{ for } j = 1, \dots, k\},$$

where  $A_1, \dots, A_k$  are disjoint subsets of  $N$ . (The codimension of  $F(A_1, \dots, A_k)$  is equal to the number of non-empty  $A_j$ 's.) In particular, the hyperplane  $(\mathbf{1}_T)^\perp$  is equal to  $F(T)$ . Let us show that the intersection of hyperplanes  $F(T_1), \dots, F(T_k)$ , where  $T_j \in \mathcal{T}$ , takes the form  $F(A_1, \dots, A_k)$ . For this we write  $A_j$ 's explicitly. Namely,  $A_j$  is equal to  $T_j$  minus the union of those of  $T_i$  which are contained in  $T_j$ . Indeed, using the laminarity of  $\mathcal{T}$ , we can assume that the  $T_i$ 's do not intersect. Therefore,  $x(T_j) = 0$  is equivalent to  $x(A_j) = 0$ .

In particular, for a laminar family  $\mathcal{T}$  in  $N$ , the polyhedron given by the inequalities

$$a(S) \leq x(S) \leq b(S), \quad S \in \mathcal{T},$$

is an  $\mathbb{A}_N$ -polyhedron for any functions  $a, b: \mathcal{T} \rightarrow \mathbb{R} \cup \{\infty\}$ , and is an integer  $\mathbb{A}_N$ -polyhedron for integer-valued  $a$  and  $b$ .

### 7. Exterior description of $\mathcal{U}$ -polytopes

In this section we characterize support functions of  $\mathcal{U}$ -polyhedra, where  $\mathcal{U}$  is a pure system. As we know support functions of base polyhedra are closely related to submodularity. Because of this, support functions of  $\mathcal{R}$ -polyhedra give rise to a generalization of submodularity.

Recall that the support function of a (non-empty) closed convex set  $A \subset V$  is the function  $\phi(A; \cdot) : V^* \rightarrow \mathbb{R} \cup \{+\infty\}$  on the dual space  $V^*$  given by the following formula

$$\phi(A; p) = \sup_{x \in A} p(x), \quad p \in V^*. \tag{2}$$

Let us work in a setting with compact sets in order to avoid messing up with infinite values. In this setting the support function is defined on whole the space  $V^*$  and is homogeneous and convex. Conversely, every homogeneous convex function  $f$  on  $V^*$  is the support function of the *subdifferential* of  $f$ ,

$$\partial(f) := \{x \in V \mid x(p) \leq f(p) \quad \forall p \in V^*\}. \tag{3}$$

The set  $\partial(f)$  is non-empty, convex, and compact; and the operations  $\phi$  and  $\partial$  are dual:  $\partial(\phi(A)) = A$  and  $\phi(\partial(f)) = f$  (see, for example, [17]).

Support functions of polytopes are characterized by a “piece-wise linearity” property. It is convenient to use the notion of fan here.

A *fan* (in  $V^*$ ) is a finite collection  $\Sigma$  of polyhedral cones possessing the following three properties: (a) the cones  $\sigma \in \Sigma$  cover  $V^*$ ; (b) every face of any  $\sigma \in \Sigma$  is also in  $\Sigma$ ; (c) the intersection of two cones of  $\Sigma$  is a face of each of them. For example, in the previous section we have defined the fan  $\Sigma(\mathcal{R})$ .

A convex function  $f$  on  $V^*$  is *compatible* with a fan  $\Sigma$  if  $f$  is linear on every cone  $\sigma$  from  $\Sigma$ . In this case, it is easy to show that  $\partial(f)$  is a polytope. More precisely, let  $\sigma$  be a full-dimensional cone of the fan  $\Sigma$ ; denote by  $v_\sigma$  a (unique) linear function on the space  $V^*$ , which coincides with  $f$  on the cone  $\sigma$ . Then  $v_\sigma$  (being considered as an element of  $V$ ) is a vertex of the polytope  $\partial(f)$ . And all vertices of the polytope are of that form. In particular, a polytope  $P$  is integer if and only if its support function  $\phi(P, \cdot)$  has integer values in integer points. However, in this section, we shall not deal with the integer-valuedness.

The support function of any polytope  $P$  is compatible with the following fan  $\mathcal{N}(P)$ . Given a point  $x \in P$ , the following cone in the dual space  $V^*$

$$\text{Con}^*(P, x) = \{p \in V^*, p(x) \geq p(y) \quad \forall y \in P\}$$

is said to be the *cotangent* cone to  $P$  at  $x$ . The collection of all cotangent cones  $\text{Con}^*(P, x)$ ,  $x \in P$ , forms the *cotangent fan* (or the *normal fan*)  $\mathcal{N}(P)$  of the polytope  $P$ . For example, the cotangent fan of the zonotope  $\sum_{r \in \mathcal{R}} \text{co}(\{-r, r\})$  coincides with the arrangement fan  $\Sigma(\mathcal{R})$ . Cones of normal fan  $\mathcal{N}(P)$  one-to-one correspond to faces of  $P$ . Moreover, they are orthogonal one to other.

In particular, this establishes the following

**Proposition 7.** *Let  $\mathcal{U}$  be a pure system in  $V$ , and let  $P \subset V$  be a convex polytope. The following assertion are equivalent:*

- (a)  $P$  is a  $\mathcal{U}$ -convex polytope;
- (b) the normal fan  $\mathcal{N}(P)$  consists of  $\mathcal{U}^\perp$ -cones.

When a pure system  $\mathcal{U}$  is generated by a unimodular system  $\mathcal{R}$ , we can say a bit more. In this case there is the finest  $\ast\mathcal{R}$ -convex fan  $\Sigma(\mathcal{R})$ . And a polytope  $P$  is  $\mathcal{R}$ -convex if and only if its support function is compatible with the fan  $\Sigma(\mathcal{R})$ .

One can get also the following characterization of  $\mathcal{R}$ -polytopes.

**Proposition 9.** *A polytope  $P$  is  $\mathcal{R}$ -convex if and only if there exists a polytope  $P'$  such that  $P + P'$  is an  $\mathcal{R}$ -zonotope.*

**Proof.** It is clear that any edge of  $P$  is parallel to some edge of  $P + P'$ . Therefore  $P$  is an  $\mathcal{R}$ -polytope. This prove the “if” part of the statement.

Conversely, let  $P$  be a  $\mathcal{R}$ -polytope. Then the arrangement fan  $\Sigma(\mathcal{R})$  is a refinement of the normal fan  $\mathcal{N}(P)$ . Since the normal fan of an  $\mathcal{R}$ -zonotope is  $\Sigma(\mathcal{R})$ , the assertion follows from the following

**Lemma 5** (Grünbaum [11]). *For polytopes  $P$  and  $Q$  the following assertions are equivalent:*

- (a)  $\mathcal{N}(Q)$  is a refinement of  $\mathcal{N}(P)$ ,
- (b) there exists a polytope  $P'$  such that  $P + P' = kQ$ , for some  $k \geq 0$ .

Assume now that  $\mathcal{R}$  is a full-dimensional unimodular system, and that  $\mathcal{R}^\vee$  is the set of crossings in  $M^\ast$ . A function  $f$ , compatible with the fan  $\Sigma(\mathcal{R})$  is uniquely determined by its restriction on  $\mathcal{R}^\vee$ , that is by the family of real numbers  $(f(\xi), \xi \in \mathcal{R}^\vee)$ . However, the values  $f(\xi), \xi \in \mathcal{R}^\vee$ , are not arbitrary. Being the restriction of a convex function, they have to satisfy some kind of “sub-modularity” relations. These relations may be divided into two groups. The first group of relations addresses the functions’ linearity on each cone of the fan. The second group of the relations yields convexity. Let us formulate these relations more explicitly:

I. Suppose that crossings  $\xi_1, \dots, \xi_m \in \mathcal{R}^\vee$  belong to a cone  $\sigma \in \Sigma(\mathcal{R})$ . Then any linear relation  $\sum_i \alpha_i \xi_i = 0$  should imply the similar relation  $\sum_i \alpha_i f(\xi_i) = 0$ .

Of course, if the cone  $\sigma$  is simplicial (as in the case of  $\mathbb{A}_n$ ), these relations disappear.

II. Suppose that we have two adjacent (full-dimensional) cones  $\sigma$  and  $\sigma'$  of the fan, separated by a wall  $\tau$ . Let  $\tau$  be spanned by the crossings  $\xi_1, \dots, \xi_m$ , and let  $\xi, \xi'$  be crossings from  $\sigma, \sigma'$  respectively, which do not belong to the wall  $\tau$ . Then any relation  $\alpha\xi + \alpha'\xi' = \sum_i \alpha_i \xi_i$ , where  $\alpha, \alpha' > 0$ , implies the relation  $\alpha f(\xi) + \alpha' f(\xi') \geq \sum_i \alpha_i f(\xi_i)$ . (According to Lemma 4, we can assume that  $\alpha = \alpha' = 1$ .)

However these relations do not look too inspiring. In effect, it is neither easy to provide a collection of numbers  $(f(\xi), \xi \in \mathcal{R}^\vee)$  satisfying the relations I and II, nor easy to check that a given collection of numbers satisfies these relations.

Let us illustrate the above said for the unimodular systems  $\mathbb{A}(N)$  and  $\mathbb{A}_N$ .

**Example 13.** (*Base polytopes*). We show here that the class  $\mathcal{B}$  of base polytopes (see Example 3) coincides with the class of  $\mathbb{A}(N)$ -polytopes (a similar assertion is also true for polyhedra; a proof, however, would involve support functions with infinite values), where  $\mathbb{A}(N)$  is the unimodular system from Example 6.

Recall that the set  $\mathbb{A}(N) \subset (\mathbb{R}^N)^*$  consists of differences  $e_i - e_j$ ,  $i, j \in N$ . Consider now how the arrangement fan  $\Sigma := \Sigma(\mathbb{A}(N))$  in the space  $\mathbb{R}^N$  of functions on  $N$  looks like. Given the root  $r = e_i - e_j$ , the corresponding mirror  $r^\perp$  consists of functions  $p \in \mathbb{R}^N$  satisfying the relation  $p(i) = p(j)$ . This mirror divides the space of functions in two halfspaces  $\{p : p(i) \geq p(j)\}$  and  $\{p : p(i) \leq p(j)\}$ . We see that cones of the fan  $\Sigma$  correspond to (weak) orders on  $N$ . If  $\preceq$  is an order, then the corresponding cone  $\sigma(\preceq)$  consists of monotone functions  $p : (N, \preceq) \rightarrow (\mathbb{R}, \leq)$ . For example, full-dimensional cones of  $\Sigma$  correspond to linear orderings; the line of constant functions  $\mathbb{R}\mathbf{1}_N$  corresponds to the total indifference relation on  $N$ .

The set  $\mathbb{A}(N)$  has full dimension in the hyperplane  $[x(N) = 0]$  orthogonal to the constant function  $\mathbf{1}_N \in \mathbb{R}^N$ . Therefore we should consider the fan  $\Sigma$  in the factor space  $\mathbb{R}^N / \mathbb{R}\mathbf{1}_N$ . The crossings correspond to dichotomous orders on  $N$ , which splits  $N$  into two classes  $S$  and  $N \setminus S$  ( $S$  is different from  $\emptyset$  and  $N$ ). Therefore, crossings have the form  $\mathbf{1}_S$ ,  $S \neq \emptyset, N$ .

Let now  $f$  be a convex function compatible with the fan  $\Sigma$ . Define the set-function  $b : 2^N \rightarrow \mathbb{R}$ ,  $b(S) = f(\mathbf{1}_S)$  for  $S \subset N$ . We assert that  $b$  is submodular. Indeed, let  $S$  and  $T$  be subsets of  $N$ . Then, by convexity of  $f$ ,

$$b(S) + b(T) = f(\mathbf{1}_S) + f(\mathbf{1}_T) \geq 2f((\mathbf{1}_S + \mathbf{1}_T)/2).$$

On the other hand, since  $S \cap T \subset S \cup T$ , the points  $\mathbf{1}_{S \cap T}$  and  $\mathbf{1}_{S \cup T}$  belong to a cone of  $\Sigma$ , and therefore

$$b(S \cap T) + b(S \cup T) = f(\mathbf{1}_{S \cap T}) + f(\mathbf{1}_{S \cup T}) = 2f((\mathbf{1}_{S \cap T} + \mathbf{1}_{S \cup T})/2).$$

Since  $\mathbf{1}_S + \mathbf{1}_T = \mathbf{1}_{S \cap T} + \mathbf{1}_{S \cup T}$ , we have

$$b(S) + b(T) \geq b(S \cap T) + b(S \cup T),$$

that is  $b$  is submodular function.

Conversely, any set-function  $b$ , considered as a function on the set of vectors  $\{\mathbf{1}_S, S \subset N\}$ , has the unique extension  $f = \tilde{b}$  on whole  $\mathbb{R}^N$  compatible with the fan  $\Sigma$ . This extension coincides with the Choquet integral (see [1]) of the non-additive measure  $b$ ,  $\tilde{b}(p) = \int p db$ . If  $b$  is submodular function then  $\tilde{b}$  is convex (see [13]).

The corresponding polytope  $\partial \tilde{b}$  is given by the following system of inequalities

$$\mathbf{1}_S(x) = x(S) \leq b(S), \quad S \subset N, \quad x(N) = b(N),$$

and is a base polytope. Thus, we prove

**Proposition 10.** *The class  $\mathcal{P}t(\mathbb{A}(N))$  of  $\mathbb{A}(N)$ -polytopes coincides with the class of base polytopes.*

Of course, the class of  $\mathbb{A}(N)$ -polyhedra coincides with the class of base polyhedra, and the class of integer  $\mathbb{A}(N)$ -polyhedra coincides with the class of integer base polyhedra.

**Example 14.** (*Generalized polymatroids*). In the same spirit, we can check that the class of generalized polymatroids in  $(\mathbb{R}^N)^*$  coincides with the class of  $\mathbb{A}_N$ -polyhedra. The arrangement  $\mathcal{A}(\mathbb{A}_N)$  consists of hyperplanes  $p(i) = 0$ ,  $i \in N$ , and  $p(i) = p(j)$ ,  $i, j \in N$ . The collection of vectors  $\{\pm \mathbf{1}_S, S \subset N\}$  is the set of crossings. Cones of  $\Sigma(\mathbb{A}_N)$  are in a one-to-one correspondence with pairs of orders  $(\preceq_W, \preceq_{W'})$  on partitions  $(W, W')$  of  $N$ . These partitions derive from the partitions of coordinates in non-negative and negative parts;  $W$  denotes the non-negative coordinates of vectors of a cone, whereas  $W'$  denotes the negative ones.

Now let  $f$  be a convex function on  $\mathbb{A}_N$  compatible with the fan  $(\Sigma(\mathbb{A}_N))$ . Consider the following two functions  $a$  and  $b$  on  $2^N$ :  $a(S) := -f(-\mathbf{1}_S)$  and  $b(S) := f(\mathbf{1}_S)$  for  $S \subset N$ . There are three kinds of relations between crossings:  $\mathbf{1}_S + \mathbf{1}_T = \mathbf{1}_{S \cup T} + \mathbf{1}_{S \cap T}$ ,  $-\mathbf{1}_S - \mathbf{1}_T = -\mathbf{1}_{S \cup T} - \mathbf{1}_{S \cap T}$ , and

$$\mathbf{1}_S + (-\mathbf{1}_T) = \mathbf{1}_{S-T} + (-\mathbf{1}_{T-S}). \tag{4}$$

The first two yield submodularity of  $b$  and supermodularity of  $a$ , respectively, while the third yields the following inequalities

$$\begin{aligned} b(S) - a(T) &= f(\mathbf{1}_S) + f(-\mathbf{1}_T) \\ &\geq f(\mathbf{1}_{S-T}) + f(-\mathbf{1}_{T-S}) = b(S-T) - a(T-S). \end{aligned} \tag{5}$$

Thus, the pair  $(b, a)$  is a strong pair in the sense of [9]. The corresponding polyhedron  $\partial f$  is given by the inequalities

$$a(S) \leq x(S) \leq b(S),$$

where  $S \subset N$  and, by definition,  $\partial f$  is a generalized polymatroid.

Conversely, we can extend any strong pair  $(b, a)$  to a convex function on  $\mathbb{R}^N$  compatible with the fan  $\Sigma(\mathbb{A}_N)$ . Thus, the class of (bounded) generalized polymatroids coincides with the class of  $\mathbb{A}_N$ -polytopes. Similarly, the class of all generalized polymatroids coincides with the class of  $\mathbb{A}_N$ -polyhedra, and the class of integer generalized polymatroids coincides with the class of integer  $\mathbb{A}_N$ -polytopes.

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