Decision-Making under Non-classical Uncertainty

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Abstract

In this paper we extend Savage's theory of decision-making under uncertainty from a classical environment into a non-classical one. The corresponding notions and axioms are formulated for general ortholattices and representation theorem for expected utility is provided. A simple example demonstrates differences between the classical and non-classical models of uncertainty.

Introduction

In this paper we propose an extension of the standard approach to decision-making under uncertainty in Savage's style from the classical model into the more general model of non-classical measurement theory. Formally, this means that we substitute the Boolean algebra model with a more general ortholattice structure.

In order to provide a first line of motivation for our approach we turn back to Savage's theory in a very simplified version. In (Savage 1954) the issue is about a valuation of "acts" with uncertain consequences or results. For simplicity we shall assume that the results can be evaluated in utils. Acts lead to results (measurable in utils), but the results are uncertain (they depend on a state of Nature).

The classical approach to a formalization of acts with uncertain outcomes amounts to the following. There is a set X of states of nature, which may in principle occur. (For simplicity, we assume that the set X is finite.) An act is a function $f : X \to \mathbb{R}$. If the state $s \in X$ is realized, our agent receives $f(s)$ utils. But before hand it is not possible to say which state s is going to be realized. To put it differently, the agent has to choose among acts *before* he learns about the state s. This is the heart of the problem.

Among possible acts there are constant acts, i.e. acts with a result known before hand, independently of the state of nature s. The constant act is described by a (real) number $c \in \mathbb{R}$. It is therefore natural to link an arbitrary act f with its "utility equivalent" $CE(f) \in \mathbb{R}$ (such that our decision-maker is indifferent between the act f and the constant act $CE(f)$). The first postulate of our simplified Savage model asserts the existence of the *certainty equivalent*:

- *S1.* There exists a certainty equivalent $CE : \mathbb{R}^X \to \mathbb{R}$ and for the constant act 1_X we have $CE(1_X) = 1$.

It is rather natural to require monotonicity of the mapping CE:

- *S2.* If $f \le g$ then $CE(f) \le CE(g)$.

The main property we impose on CE is linearity:

- *S3.* $CE(f + g) = CE(f) + CE(g)$ for any acts f and $g \in \mathbb{R}^X$.

As a linear functional on the vector space $\mathbb{R}^X$, CE can be written in the form $CE(f) = \sum_x f(x)\mu(x)$. By axiom $S2$, $\mu \ge 0$; since $CE(1_X) = 1$ we have $\sum_x \mu(x) = 1$. Therefore $\mu(x)$ can be interpreted as the probability for realization of the state x. (Sometimes this probability is called subjective or personal, because it only expresses the likelihood that a specific decision-maker assigns to event x.) With such an interpretation, $CE(f)$ becomes the "expected" utility of the act f.

In this paper we propose to substitute the Boolean lattice of events with a more general ortholattice. The move in that direction was initiated long ago, in fact with the creation of Quantum Mechanics. The Hilbert space entered into the theory immediately, beginning with von Neumann (von Newmann 1932), who proposes to use the lattice of projectors in a Hilbert space as a suitable model instead of the classical (Boolean) logic. In their seminal paper (Birkhoff and von Neumann 1936) have investigated the necessary properties of such a non-distributive logic. The necessity to use more general ortho-lattices than the Boolean one, arises as soon as the measurements (i.e. an activity directed at obtaining information about the object that interests us) affect the measured object and change its state. If our measurements do not change the state of the object, one can use Savage's classical paradigm. But if the measurements significantly affect the object, one must turn to more general ortho-lattices. This is particularly important when one does not limit attention to a single measurement, but is interested in a sequence of measurements or decision problems.

Most closely related to our work are two decision-theoretical papers (Gyntelberg and Hansen 2004, Lehrer and Shmaya 2006) in which the standard expected utility theory was transposed into Hilbert space model. Our first aim is to show that there is no need for a Hilbert space, that the Savage approach can just as well (and even easier) be developed within the frame of general ortholattices. Beside the formal arguments, a motivation for this research is that a more general description of the world allows to explain some behavioral paradoxes e.g., the Eldsberg paradox (see for example La Mura 2005, Gyntelberg and Hansen 2004). In this respect our paper belongs to a very recent and rapidly growing literature where formal tools of Quantum Mechanics are proposed to explain a variety of behavioral anomalies in social sciences and psychology (see e.g., Lambert-Mogiliansky et al. 2003, Pitowsky 2003, Khrennikov 2007, Busemeyer and Wang 2007, Franco 2007). In Example we show that the results in this paper are relevant to modelling interaction in simple games when a decision-maker faces a type indeterminate opponent, i.e. an agent whose type changes under the impact of decision-making as proposed in (Lambert-Mogiliansky et al. 2003).

For the sake of comparison with the Savage setup, we develop the theory in a static context. But non-classical measurement theory (see Danilov and Lambert-Mogiliansky 2007) was originally developed to deal with situations when measurements impact on the states of the measured system (we understand here acts as measurements). Therefore, a genuine theory of non-classical expected utility should apply to sequences of acts or measurements.

Non-classical utility theory

First of all we need to modify the Savagian concept of act for general ortho-lattices. Recall that an *ortholattice* is a lattice $\mathcal{L}$ (with operations join $\vee$ and meet $\wedge$) equipped with an operation of *ortho-complementation* $\perp$: $\mathcal{L} \to \mathcal{L}$. This operation is assumed to be involutive ($a^{\perp\perp} = a$), to reverse the order ($a \leq b$ if and only if $b^{\perp} \leq a^{\perp}$) and to satisfy the following property $a \vee a^{\perp} = \mathbf{1}$ (or, equivalently, $a \wedge a^{\perp} = \mathbf{0}$).

Definition. An *Orthogonal Decomposition of the Unit* (ODU) in an ortholattice $\mathcal{L}$ is a (finite) family of $\alpha = (a(i),\ i \in I(\alpha)\ )$ of elements of $\mathcal{L}$ satisfying the following condition: for any $i \in I(\alpha)$ $a(i)^{\perp} = \vee_{j \neq i} a(j)$.

The justification for this terminology is provided by that $a(i) \perp a(j)$ for $i \neq j$ and $\vee_i a(i) = \mathbf{1}$.

We understand an ODU as a measurement with the set of outcomes $I(\alpha)$. In order to justify such a understanding let us introduce the notion of (potential) state.

Definition. A *state* (on an ortholattice $\mathcal{L}$) is a monotone mapping $\sigma : \mathcal{L} \to \mathbb{R}_+$ such that for any ODU $\alpha = (a(i),\ i \in I(\alpha)$ there holds

$$\sum_{i \in I(\alpha)} \sigma(a(i)) = 1.$$

The number $\sigma(a(i))$ we understand as a probability to obtain an outcome $i \in I(\alpha)$ at an execution of the measurement-ODU α if our system was in the state σ.

Roughly speaking an act is a bet on the result of some measurement.

Definition. An *act* is a pair (α, f), where $\alpha = (a(i), i \in I(\alpha))$ is some ODU, and $f : I(\alpha) \to \mathbb{R}$ is a payoff function.

We call the measurement α the *basis* of our act. Intuitively, if an outcome $i \in I(\alpha)$ is realized as a result of measurement α, then our agent receives $f(i)$ utils.

In such a way the set of acts with basis α can be identified with the vector space $F(\alpha) = \mathbb{R}^{I(\alpha)}$. The set of all acts F is the disjoint union of $F(\alpha)$ taken over all ODUs α.

We are concerned with the comparison of acts with respect to their attractiveness for our decision-maker. We start with an explicit formula for such a comparison. Assume that the agent knows (or she thinks she knows) the state β of the system. Then, for any act f on the basis of a measurement $\alpha = (a(i), i \in I(\alpha))$, he can compute the following number (expected value of the act f)

$$CE_\beta(f) = \sum_i \beta(a(i)) f(i).$$

Using those numbers our agent can compare different acts.

We now shall (following Savage) go the other way around. We begin with a preference relation $\preceq$ on the set F of all acts, thereafter we impose conditions and arrive at the conclusion that the preferences are explained by some state-belief β on $\mathcal{L}$.

More precisely, instead of a preference relation $\preceq$ on the set F of acts, we at once assume the existence of a certainty equivalent $CE(f)$ for every act $f \in F$ (such that $CE(1) = 1$). (Of course that does simplify the task a little. But this step is unrelated to the issue of classicality or non-classicality of the "world"; it is only the assertion of the existence of a utility on the set of acts. It would have been possible to obtain the existence of CE from yet other axioms. We chose a more direct and shorter way).

Given that we impose two requirements on CE. The first one relates to acts defined on a fixed basis α. Such acts are identified with elements of the vector space $F(\alpha) = \mathbb{R}^{\alpha}$.

Linearity axiom. For any measurement α the restriction of CE on the vector space $F(\alpha)$ is a linear functional.

The second axiom links acts defined on different but in some sense comparable bases. Let $f : I(\alpha) \to \mathbb{R}$ and $g : I(\beta) \to \mathbb{R}$ be two acts on the basis of measurements α and β respectively. We say that g *dominates* f (and write $g \succcurlyeq f$) if inequality $f(i) > g(j)$ implies $a(i) \perp b(j)$. Intuitively the dominance $g \succcurlyeq f$ means that the act g always gives no less of utils than the act f. It is natural to require that the certainty equivalent of g is not less than that of f.

Dominance axiom. If $f \preccurlyeq g$ then $CE(f) \leq CE(g)$.

Theorem. *Assume that the axioms of linearity and dominance are satisfied. Then CE is an expected utility for some state β on $\mathcal{L}$.*

Proof. First of all we assign some "probability" $\beta(a)$ to every $a \in \mathcal{L}$. Suppose that $\alpha = (a(i), i \in I(\alpha))$ is a measurement such that $a = a(i_0)$ for some $i_0 \in I(\alpha)$. Let the function $1_a : I(\alpha) \to \mathbb{R}$ be equal to 1 for the element i_0 and to 0 for all other elements of $I(\alpha)$. We pose $\beta(a) = CE(1_a)$.

Now we have to check that this definition is correct, that is independent of the choice of the measurement α. For this, we consider a special measurement $(a, a^{\perp})$. It is easy to see that the acts $(\alpha, 1_a)$ and $((a, a^{\perp}), 1_a)$ dominate each other. Therefore, by the dominance axiom, they have the same certainty equivalent.

Let now (α, f) be an arbitrary act. Since $f = \sum_{i \in I(\alpha)} f(i) 1_{a(i)}$, the linearity axiom implies the equality

$$CE(f) = \sum_i f(i) CE(1_{a(i)}) = \sum_i f(i) \beta(a(i)).$$

Thus CE is an expected utility.

Applying this equality to the constant function $1 : I \to \mathbb{R}$, we obtain that

$$1 = \sum_{i \in I} \beta(a(i))$$

for any ODU $(a(i), i \in I)$. That is β is ortho-additive.

In order to show that β is a state, we have to check that β is a monotone function on $\mathcal{L}$. Suppose $a \leq b$ and consider two measurements $(a, a^{\perp})$ and $(b, b^{\perp})$. Let 1_a be a bet on the event a - the agent receives one util if measurement α reveals the property a, and receives nothing in the opposite case. We define 1_b similarly on the $(b, b^{\perp})$ basis. Clearly $1_a \preccurlyeq 1_b$. In fact if the first measurement reveals the property a then b is true for sure since $a \leq b$. Therefore 1_b gives the agent one utils when a occurs, and ≥ 0 utils when $a^{\perp}$ occurs, which is not worth less than 1_a. By the dominance axiom $CE(1_a) \leq CE(1_b)$. The first term is equal to $\beta(a)$ and the second to $\beta(b)$. *QED*

Example

Suppose our decision-maker is confronted with the following situation. He may propose to his opponent one of two decision problems. For the sake of concreteness we can call them *PD* (for Prisoners' Dilemma) and *UG* (for Ultimatum Game). When confronted with the *PD* problem, the opponent may choose between action *C* and *D*. It is difficult to say beforehand what he will actually choose because it depends on his state or "type". This choice can be viewed as a "measurement" of the opponent; if the opponent chooses *C* we interpret that as the opponent being (after the choice) in state *C*. Of course we implicitly assume that this measurement is of the first-kind (i.e. if we immediately after repeat the *PD* the opponent chooses C again). We proceed similarly with the decision problem *UG* which also has two outcomes denoted *G* and *E*.

This looks like a very classical situation. Our decision-maker must evaluate the probability $\mu(C)$ (in this case $\mu(D) = 1 - \mu(C)$) and she must evaluate $\mu(G)$. More precisely, it is natural to assume that the opponent maybe in one of four states: *CG* (choose *C* in *PD* and *G* in *EG*), *CE*, *DG*, and *DE*. And our decision-maker needs to evaluate the probability for each one of these states. Assume that our decision-maker receives 100 utils when the choice is *C*, 0 when it is *D* or *E* and 10 when it is *G*. Suppose that our decision-maker evaluate all four states as equally probable. Then, the expected utility of *PD* is larger than the expected utility associated with *UG* and she chooses *PD*.

But in order to really receive the payoff, our decision-maker must perform the *PD* measurement. Suppose that the opponent chooses the action *D*, what will our decision-maker do if she has a chance to repeat her choice of *PD* or *UG*? Again she has to evaluate the (mixed) state of her opponent. According to Bayes' rule she now assigns probability 1/2 to states DG and DE (and zero to CG and CE). Now with regard to her own choice *PD* has a zero value (expected utility or certainty equivalent), while *UG* has an expected value of 5 utils. And for the second try she selects *UG*. If she again is unlucky so *E* is the outcome, there is no point in interacting with this opponent anymore. The state of the opponent is *DE*, any repetition of the game will yield her a zero payoff.

The reasoning we just made is straightforward and unquestionable if the situation is classical and our measurements *PD* and *UG* commute. That is if performing one measurement does not affect the result of the other measurement. In particular, we may perform them in any order and even simultaneously. A very different situation arises if our measurements do not commute. Suppose as we did above that our decision-maker performed measurement *PD* and obtained *D*. If she repeats this measurement she will obtain *D* again. But if she, in between, performs the *UG* measurement, that may change the state of the opponent and in a subsequent performance of *PD* the opponent may choose *C*. And if this really does happen, we have to radically change our model of the opponent. It is now meaningless to speak about state *CG* and so on. It is more natural to view as states *C, D, G* and *E*. The states *C* and *D* have to be orthogonal so must *G* and *E*. Moreover it is now necessary to evaluate the transition probabilities $\tau(C, G)$ (the probability of transition from the state *C* to the state *G* at providing of the measurement *UG*) and $\tau(C, E) = 1 - \tau(C, G)$ as well as the transition probabilities $\tau(D, G)$ and $\tau(D, E) = 1 - \tau(D, G)$.

Let the payoffs be the ones given above (and the transition probabilities be all equal to 1/2). Our decision-maker will as a first step choose PD as earlier. Suppose she is unlucky and the D outcome is realized. As a second step she prefers UG. Suppose that she is again unlucky and the opponent's choice is E. Up to here the behavior is identical to the one described earlier. But now she has a strict incentive to return to PD, because with probability 1/2 she receives 100 utils. And if she is lucky that time she may in each subsequent step receive 100 utils.

What do we learn from this simple example? The main point is that there are different models of uncertainty. To solve a decision problem under uncertainty it is not sufficient to assign probabilities to different events. An important aspect is the choice of the model describing uncertainty. If our acts (or the corresponding measurements) do not impact on the state of Nature, then we can use the classical model. But if "Nature" reacts to the performed measurement (or, equivalently, measurements do not commute or are incompatible with each other), then a more suitable model is the one developed in this paper. This is particularly true if our decision-maker faces not a single choice but a sequences of choices between acts. In our example in a classical situation, our decision-maker wins 100 with probability 1/2. In the non-classical situation, she sooner or later necessarily obtains that payoff.

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