Gale-Nikaido-Debreu and Milgrom-Shannon: Communal interactions with Endogenous Community Structures*

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Abstract

This paper examines Nash jurisdictional stability in a model with a continuum of agents whose characteristics are distributed over a unidimensional interval. Communal benefits and costs of each individual depend on her identity and the composition of the community which she belongs to. Since the framework is too general to yield an existence of Nash equilibrium, we introduce the essentiality of membership in one of the communities for all individuals. We highlight the Border Indifference Property (BIP), when all individuals located on a border between two adjacent jurisdictions are indifferent about joining either of them and show that BIP is a necessary condition for yielding a Nash equilibrium. We invoke the celebrated Gale-Nikaido-Debreu Lemma to guarantee the existence of a partition that satisfies BIP. We then proceed to demonstrate that BIP is not sufficient to yield a Nash equilibrium. The equilibrium existence under BIP is rescued when we use the Milgrom-Shannon monotone comparative statics conditions.

Keywords: Nash stability, continuity, increasing differences, monotonicity, group formation, market interactions.

JEL Classification Numbers: D70, H20, H73.

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1 Introduction

In this paper we consider a model of bilateral interactions among a population of agents characterized by a unidimensional parameter, such as geographical location, income level, political preferences, taste for redistribution, etc. Within a society, several disjoint communities (clubs, districts, jurisdictions, peer groups) are formed. Individuals derive benefits of the local public good type from membership in communities which they belong to. To provide for those benefits, community members are engaged in bilateral market interactions, such as trade links or business ventures, and social interactions and actions that can be driven by envy, conformism or social exclusion. Individuals maximize their net benefits across the existing communities and by exercising the option to refrain from joining any community. The value of communal benefits solely depends on the individual's identity and the composition of her community. At the same time, the structure of communal costs is affected by a degree of bilateral influence. Some individuals are more influential and relevant than others. In the context of social interactions, the impact of role models within a community could be crucial. Alternatively, the intensity level of market interactions of individuals close to the market center could be higher than of those located at the community periphery. In models of social exclusion, the income of a higher-income individual has a stronger impact on a person's well-being the smaller their income difference. In addition, every community faces community-specific formation costs.

Our main objective is to examine Nash stable structures where no member of existing community could raise her net benefit by switching to another community or staying alone. At the same time, none of the unattached individuals can benefit by joining one of the existing communities. In analyzing community configurations we restrict our attention to stratified communities represented by connected intervals, where communities consist of agents with somewhat similar characteristics.[1] Our first result, Proposition 1, states that every Nash stable partition satisfies the *border indifference property — BIP*, which implies that every individual located at the border between two communities would incur the same cost in either of them. Moreover, every individual located at the border between a formed community and the universe of unattached agents, receives her stand-alone

[1] The property is called *consecutiveness* in Greenberg and Weber, 1986.

utility level.

Our results show that even in simple cases, an equilibrium, and even a partition satisfying BIP, does not necessarily exist. Indeed, Examples 1 and 2 exhibit a society where no single community satisfies BIP. While there is no hope for existence results in our general setting, we rule out the stand-alone option and consider the case where all individuals join one of the existing communities to attain a certain standard of living, health care, education services, or, simply, to acquire a citizenship. By using the celebrated Gale-Nikaido-Debreu Lemma (Debreu, 1959, and Mas-Colell et al., 1995), rooted in the techniques of general equilibrium theory, we demonstrate (Proposition 2), that for every positive integer n there is a partition of the entire society into n communities that satisfies BIP. The importance of BIP is highlighted by the fact that in the most of the literature on public projects (e.g., Alesina and Spolaore, 1997, Jéhiel and Scotchmer, 2001), BIP is a necessary and sufficient condition for a partition to be Nash stable. It is, however, not the case in our general framework, where Example 4 exhibits a society that satisfies BIP but does not admit a Nash stable partition. Together with Proposition 1, this implies that while BIP is necessary for Nash stability, it is by no means sufficient.

To obtain sufficient conditions for the existence of Nash equilibrium, we utilize the monotone comparative statics approach of Milgrom and Shannon, 1994. We introduce an ordering (called a *right side ordering — RO*) on the set of connected jurisdictions, denoted by $S \succ_{RO} T$, when there exists an individual u such that $s \geq u \geq t$ for every agent s in S and every t in T. In other words, the entire interval S is located to the right of interval T. We impose the monotonicity of the cost function c with respect to this ordering, and since RO is stronger than the well-known Veinott ordering (Veinott, 1989), our monotonicity requirement is weaker than Veinott monotonicity. However, it is sufficient to ensure that every partition satisfying BIP is Nash stable. Thus, under RO-monotonicity, existence of partitions satisfying BIP, which is guaranteed by the Gale-Nikaido-Debreu Lemma, yields the existence of Nash stable partitions.

To summarize, our main existence result relies on both continuous and monotone comparative statics. It also highlights the contribution of each of comparative statics facets to the existence of a Nash equilibrium. The continuous aspect is reflected by the Gale-Nikaido-Debreu Lemma that yields the existence of a partition satisfying BIP, while the monotonicity ensures that every such partition is Nash stable.

The paper is organized as follows: in the next section, we present the model. In Section 3, we examine the concept of BIP and show, by the means of Examples 1 and 2, that a nontrivial Nash stable structure may fail to exist and, moreover, there may be no single community that satisfies BIP. In Section 4, we introduce the essentiality of membership and prove that for every integer n there is a partition that satisfies BIP. We also show (Example 3) that the boundedness away from zero for community formation costs is indispensable for obtaining this result. Moreover, we demonstrate (Example 4) that while BIP is necessary, it is not sufficient for the existence of a Nash equilibrium. In Section 5, we examine the monotone comparative statics conditions (Propositions 3 and 4) that yield a Nash equilibrium. In Section 6, we offer concluding remarks. Finally, the Appendix contains detailed proofs of technical results and a tedious analysis of Examples 2 and 4.

Relation to the existing literature

Our model covers various types of group organization, including the Tiebout model (Tiebout, 1956), and the public project framework of Mas-Colell, 1980, where a community chooses the location of its public project, and each member incurs the cost of transportation to the chosen location. We also offer a general framework for stable community configurations with public projects in various settings, examined in Cremer et al., 1985, Alesina and Spolaore 1997, Bolton and Roland, 1997, Bogomolnaia et al., 2008a,b, Westhoff, 1977, Jéhiel and Scotchmer, 2001, Le Breton and Weber, 2003, 2004, Drèze et al., 2007, and others. An important feature of our model is that community structures are formed endogenously on the basis of the characteristics of the entire population. It is in line with Alesina and Spolaore, 1997, where a community (region, country) chooses the location of its public project at its median and each member incurs the cost of transportation to the selected location (capital or market place) and an appropriately chosen fraction of the project cost. We allow for a general type of cost-sharing scheme that includes the equal-share, where all individuals contribute the same amount towards the cost of the public project (Alesina and Spolaore, 1997), and "Rawlsian" (Drèze et al., 2007), where the total cost, including both the transportation and the contribution towards the public project, is the same for all individuals in the society.

Our social interaction features are related to the vast literature on the subject (see Akerlof, 1997, Axelrod and Bennett, 1993, Becker, 1974, Blume and Durlauf, 2003, Brock and Durlauf, 2001, Chwe, 1999, Durlauf, 2004, Le Breton and Weber, 2011, Manski, 2000, Schelling, 1973, among others.). Of particular importance are social interaction models with neighborhood effects (Durlauf, 2004). Hussar and Sonnenberg, 2001, point to large differences in per pupil expenditures that persist across districts in the United States. Those costs are taken into consideration by those who contemplate a possible relocation. The costs, however, are not always of a monetary nature. The issues of residential crime could be crucial (Glaeser et al., 1996) for relocation decisions. Considerations of status and prestige do play their role and impact even the pairwise interactions (and their cost) for members of the community. One has also mentioned peer group effects that produce some sort of imitative behavior either contemporaneous or across age cohorts (Durlauf, 2004). The cost of conformity through various interactions could also be meaningful (Jones, 1984, Bernheim, 1994, Chwe, 1999). Note also that while most of the literature on communities and neighborhoods is focused on the geographical interpretation, it is not always the case, and the social space of Akerlof, 1997 could be an appropriate setting for our applications. Finally, in models of deprivation and social exclusion, the presence of higher-income individuals in the group impacts a persons deprivation and well-being (Bossert and D'Ambrosio, 2014). The strength of the influence is impacted by the difference between higher incomes in the group and the income of the individual whose well-being is assessed.

2 The model

We consider a population of agents distinguished on the basis of a single parameter, which could represent income, policy preferences or location. The distribution of agents' characteristics is given by the probability distribution F on the interval $I = [0, 1]$ defined via continuous and positive density f with $F(S) = \int_S f(r)dr$. We identify every individual with the value of her characteristic t, and, for simplicity, label her t henceforth.

In our setup, there could be several disjoint communities, each providing its members with various benefits. However, the membership and community interactions are costly. The difference between our treatment of benefits and costs is based on the assumption that benefits are expanded for the entire community and are, in fact, a local public good, while the communal costs are presented through bilateral interactions and fixed costs.

The primitive data for our model consists of four functions: benefits of membership, community formation costs, bilateral community costs, and distribution of influence, which all depend on the composition of the community.

- **Benefits of membership.** By joining a community S, an individual t will derive the membership benefit, denoted by $v(t, S)$. In fact, the benefit offered by S is a club good, the evaluation of which is individual-dependent. An agent may also choose the stay-alone option and obtain the utility level $v^0(t)$, which is set at zero for each t.

- **Community formation cost.** In every group S, each member bears per capita formation cost $g(S)$. The total community formation cost is represented by the term $g(S)F(S)$, where $F(S)$ is the mass of individuals in S.

- **Bilateral community costs.** If both t and r would join a community S, the agent t would bear bilateral cost $\rho(t, r, S)$ while interacting with r. Note that these costs are not necessarily symmetric.

- **Distribution of influence within a community.** The impact of various individuals on internal community costs is not uniform. For every potential community S, there is a distribution P^S on I which satisfies $\int_S f(r)dP^S(r) = 1$ and $P^S \equiv 0$ outside S.[2,3]

[2]While the distribution P^S is concentrated on S, we define it on the entire set I in order to invoke the Kantorovich continuity in the space of distributions on I.

[3]If applicable, we denote by p^S the density function of P^S, where the value $p^S(t)$ represents the relative importance of agent t within S. If the distribution P^S has atoms, then p^S is a generalized function. If, for example, all communal interactions are conducted at point t^* ("center"), then P^S is a distribution degenerated to t^*, and its "density" is the value of the Dirac function at t^* divided by $f(t^*)$.

Note that we cover the possibility that an individual, while not a member of community S, may contemplate joining S. In this case, the functions above would indicate their values after joining the community. In this vein, for every individual t, the total cost $c(t, S)$ of being in S is given as follows:

$$c(t, S) = \int_S \rho(t, r, S) f(r) dP^S(r) + g(S).$$

The net utility of an individual t joining community S is, as usual, determined by the difference between the benefits and the costs she incurs:

$$u(t, S) = v(t, S) - c(t, S).$$

In the case where an individual stays alone, she incurs no costs, and her utility is zero.

The technical assumptions we impose further, continuity and boundedness, are standard and do not require an extensive discussion. The one exception is the Kantorovich continuity (1942),[4] the definition of which is presented in the Appendix. We need this notion to compare influence distributions in different communities.

We restrict our attention to stratified communities, i.e., to nonempty subintervals $[a, b]$ of I, the set of which is denoted by $\mathcal{I}$. This set can be viewed as a subset of the two-dimensional Euclidean space $\mathbb{R}^2$:

$$\mathcal{I} = \{x = (a, b) \in \mathbb{R}^2 : 0 \leq a < b \leq 1\}.$$

We will treat the intervals as closed, so that border individuals may associate themselves with either of the two adjacent intervals with no impact on our stability analysis. This simplification allows us to define the continuity of functions ρ and g with respect to intervals, or, in fact, their endpoints. Our assumptions are summarized as follows.

Assumption A**:**

(i) **Density function.** The population density function f is continuous and positive-valued on I.

[4] It is defined with respect to the Kantorovich metric, also known as the Kantorovich-Monge metric, based on the old contribution of Monge, 1781, or the Vasserstein metric (Vasserstein, 1969) defined for pairs of distributions on I.

(ii) **Benefit function.** The benefit function $v\colon I \times \mathcal{I} \to \mathbb{R}_+$ is continuous and bounded. Specifically, there are constants M and M' such that $M \leq v(t, S) \leq M'$ for all t and S.

(iii) **Community formation costs.** The function $g : \mathcal{I} \to \mathbb{R}_+$ is continuous, and the total formation cost is bounded. Specifically, there are positive constants K and K', such that $K \leq g(S)F(S) \leq K'$ for all $S \in \mathcal{I}$.

(iv) **Bilateral community costs.** The function $\rho : I^2 \times \mathcal{I} \to \mathbb{R}_+$ is continuous, and, moreover, positive-valued for all $t \neq r$.[5]

(v) **Distribution of influence.** The family of distributions P^S is Kantorovich-continuous in $S = [a, b]$ over the whole range $\mathcal{I}$.

We impose assumption A throughout the rest of the paper. It immediately implies the continuity and boundedness of the function $c(t, S)$, and, hence, of $u(t, S)$. The proof of the following lemma is relegated to the Appendix.

Lemma 1: Let assumption A hold. Then, the function c is continuous on $I \times \mathcal{I}$. Moreover, there exist positive constants L and L', such that $L \leq c(t, S)F(S) \leq L'$ for all $t \in I$ and all $S \in \mathcal{I}$.

In some settings, the value of $\rho(t, r, S)$ depends only on the distance between individuals t and r regardless of S. In order to account for this important special case we consider stronger assumption:

Assumption A': Everything is the same as under assumption A, except that the value of $\rho(t, r, S)$ is given by $q(|t-r|)$, where q is continuous and increasing with $q(x) > 0$ for $x > 0$ and $q(0) = 0$.

In our model, the society consists of stratified communities and the universe of unaffiliated individuals who opt to stay alone. Our purpose is to examine the societal Nash stability, when no member of an existing community would benefit by switching to another community or staying alone, while no unaffiliated individual would gain by joining any of the existing communities.

More formally, define a partial ordering — called the right side ordering (RO) — on $\mathcal{I}$. For jurisdictions $S = [a, b]$ and $T = [c, d]$ we say that $S \prec_{RO} T$ if $b \leq c$. In other words, S lies

[5] Note that we do not suppose that it is symmetric in t and r.

entirely to the left of T. A *collection* π is a set of pairwise disjoint jurisdictions $\{S_1, \ldots, S_n\}$ where $S_1 \prec_{RO} S_2 \prec_{RO} \cdots \prec_{RO} S_n$. Denote by Π_n the set of all collections that consist of n communities.

Next, for each $\pi \in \Pi_n$ denote by $M(\pi)$ the set of all agents that belong to some community: $M(\pi) = \cup_{k=1}^{n} S_k$. For every $t \in M(\pi)$ denote by $S^t \in \pi$ the community in π which she belongs to. Now we can present the notion of Nash stability.

Definition 1 — Nash Stability: A collection $\pi \in \Pi_n$ is *Nash stable* if the following inequalities hold:

(i) **Individual rationality:** $u(t, S^t) \geq 0$ for all $t \in M(\pi)$.

(ii) **Stability condition for community members:** $u(t, S_k) \leq u(t, S^t)$ for every $S_k \in \pi$ and $t \in M(\pi)$.

(iii) **Stability condition for unaffiliated agents:** $u(t, S_k) \leq 0$ for every $S_k \in \pi$ and $t \notin M(\pi)$.

3 Border Indifference Property and non-existence results

In this section, we derive the necessary condition for Nash stability and show that it can be violated even in relatively simple cases, leading to non-existence of Nash equilibria.

Consider a collection $\pi = \{S_1, \ldots, S_k, \ldots, S_n\} \in \Pi_n$, with $S_k = [a_k, b_k]$ for every $k = 1, \ldots, n$.

Definition 2 — Border Indifference Property: A collection π satisfies the border indifference property (BIP) if all border agents are indifferent between their two adjacent options. These two options may be either joining one of the two adjacent communities or joining the only adjacent community and opting out. Formally, for all $k = 1, \ldots, n-1$, if $b_k = a_{k+1}$ then $u(b_k, S_k) = u(a_{k+1}, S_{k+1})$, and if $b_k < a_{k+1}$ then both $u(b_k, S_k) = u(a_{k+1}, S_{k+1}) = 0$. In addition, if $a_1 > 0$ then $u(a_1, S_1) = 0$ and if $b_n < 1$ then $u(b_n, S_n) = 0$.

Assume that an n-collection $\pi = \{S_1, \ldots, S_n\}$ is Nash stable. Consider two intervals, $S_{k-1} = [a_{k-1}, b_{k-1}]$ and $S_k = [a_k, b_k]$. If $a_k = b_{k-1}$, then the agent at this point must be indifferent between the two. Indeed, if in negation, she strictly prefers one community (say, S_{k-1}), then the same is true for individuals who are "slightly" to the right of a_k, and those individuals would switch to S_{k-1}, a violation of Nash stability. Thus, in a stable partition, the border individual between two adjacent communities is indifferent between them. If $a_k > b_{k-1}$ then $u(b_{k-1}, S_{k-1}) = 0$. Indeed, by individual rationality, $u(b_{k-1}, S_{k-1}) \geq 0$. Also, the latter expression cannot be positive; otherwise, all individuals who are slightly to the right of b_{k-1} would prefer to join S_{k-1} rather than staying alone. Similarly, $u(a_k, S_k) = 0$. Those arguments above show that BIP is a necessary condition for Nash stability. Thus, we have shown that

Proposition 1: Under Assumption A, every Nash stable n-collection $\pi \in \Pi_n$ satisfies BIP.

We shall now demonstrate that our framework is too general to yield Nash existence results even in simple cases. In fact, even if the benefit function $v(t, S)$ is a constant independent of S, a Nash equilibrium and even a collection satisfying BIP may fail to exist. This could be the case in the presence of one community in a collection only!

Example 1: Let the benefit function v be independent of communities and equal to 4 for all individuals. The bilateral community costs ρ are given by:

$$\rho(t, r, S) = \begin{cases} 7(t - r), & t > r; \\ 8(r - t), & t < r \end{cases}$$

and the density function f by[6]

$$f(t) = \begin{cases} 1, & t \in [0, 0.9] = I_1; \\ 0.5 & t \in (0.9, 1] = I_2. \end{cases}$$

Let the influence within a community be given by the Dirac function, where the entire influence mass of S is concentrated at its median $m(S)$. Finally, let per capita community formation cost for S be $\frac{1}{F(S)}$.

Note that there could be no more than one community in equilibrium. Indeed, consider community S of the length $s + \lambda$, where s is the length of $S \cap I_1$ and λ is the length of $S \cap I_2$. For the leftmost individual in S, the costs are given by $1/(s + \lambda/2) + 8(s/2 + \lambda/2)$. The minimum of that

[6]Note that both examples in this section could be easily modified to include a continuous population density function.

function is 4, which is equal to the value of the benefit, and is attained when $s = 1/2$ and $\lambda = 0$. Thus if $s < 1/2$, then the net benefit of the leftmost agent would be negative, and since $|I_1| < 1$, there cannot be two communities with $s = 1/2$.

The latter argument implies that the only possible community S in equilibrium is the one of length $1/2$ that does not intersect with I_2. But then, the benefit of the rightmost individual in S is $4-(2+7\cdot 1/4) > 0$, violating BIP. Thus, there is no equilibrium and no partition satisfying BIP.

The relatively simple structure of Example 1 could be explained by the fact that the community cost structure is asymmetric. In fact, an asymmetry is not an unreasonable assumption. For example, for a community located along the river, the travel costs downstream and upstream could differ quite substantially. Also, the feelings of envy may vary in the case of poor towards rich and vice versa.

Nevertheless, our next example shows that restoring the symmetry of the cost function does not yield a community structure satisfying BIP:

Example 2: Let population density be given by:

$$f(x) = \begin{cases} 45, & x \in \left[\frac{3}{9}, \frac{4}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right]; \\ 36, & x \in \left(\frac{4}{9}, \frac{5}{9}\right]; \\ 27, & \text{otherwise.} \end{cases}$$

Let benefit be $v = 0.1115$ for all agents, and the bilateral community cost be $\rho(t, r, S) = \frac{2}{9}|t - r|$. As before, let the community S formation cost be $\frac{1}{F(S)}$, and the entire community influence is concentrated at the median $m(S)$.

Notice that, like in the previous example, there are potential communities that generate a positive benefit for all agents. Consider, e.g., the interval $S = \left[\frac{3}{9}, \frac{7}{9}\right]$. Its mass is 17, the median $m(S) = \frac{39}{72}$, and the maximal cost in S is $\frac{1}{17} + \frac{2}{9}\left(\frac{7}{9} - \frac{39}{72}\right) = \frac{613}{5508} \approx 0.1113 < 0.1115$. (The difference is less than 0.0002!). However, neither S nor any other community would satisfy BIP. The argument is that BIP requires the median to coincide with the middle point of the *inner* community, whose endpoints do not coincide with either 0 or 1. If this community S is perturbed a little, its median would still differ from its middle point. If S undergoes a substantial shift, the tedious analysis of this example, relegated to the Appendix, shows that the cost of at least one of the border agents would exceed v.

4 Essential Membership

We now consider the case where membership in one of the communities becomes essential. Individuals must possess citizenship, or have access to health services and education that are unavailable to non-members. Thus, the stand-alone option is simply ruled out. There will be no intervals of unattached individuals. Then a collection $\pi = \{S_1, \ldots, S_n\} \in \Pi_n$ becomes a partition, where $a_1 = 0$, $b_n = 1$ and $a_k = b_{k-1}$ for all $k = 2, \ldots, n$. Denote the set of all n-partitions by $\tilde{\Pi}_n \subset \Pi_n$. Formally,

Assumption — Essential Membership: No agent stays alone. Note that in this case, a partition $\pi = \{S_1, \ldots, S_n\} \in \tilde{\Pi}_n$ is Nash stable if $u(t, S_k) \leq u(t, S^t)$ for every $S_k \in \pi$ and $t \in I$.

The next proposition removes the first roadblock to identifying stable partitions. It shows that there is a partition of the society into n communities that satisfies BIP, i.e., all border individuals are indifferent between joining two adjacent communities. The proof of the proposition relies on what we believe is a novel application of the Gale-Nikaido-Debreu Lemma (Gale, 1955, Debreu, 1959, Mas-Colell et al., 1995).

Proposition 2: Let assumptions A and EM hold. For every positive integer n there exists a partition $\pi \in \tilde{\Pi}_n$ that satisfies BIP.

Proof of Proposition 2: For a positive integer n, denote by Δ_n and Δ_n^o the unit simplex in the non-negative orthant of $\mathbb{R}^n$ and its relative interior, respectively:

$$\Delta_n = \{s \in \mathbb{R}_+^n : \sum_{k=1}^{n} s_k = 1\} \quad \text{and} \quad \Delta_n^o = \{s \in \Delta_n : s_k > 0 \ \text{ for all } \ k = 1, \ldots, n\}.$$

The vector $s = (s_1, \ldots, s_n) \in \Delta_n^o$ generates an n-partition of I, defined by the following border individuals:

$$b_k = b_k(s) = F^{-1}\Big(\sum_{i=1}^{k} s_i\Big), k = 1, \ldots, n-1.$$

The agents $b_1, \ldots, b_{n-1}$ generate the partition $\pi = \{S_1, \ldots, S_n\} \in \Pi_n$, where $S_1 = [0, b_1), \ldots, S_k = [b_{k-1}, b_k), \ldots, S_n = [b_{n-1}, 1]$.[7] The mass of each community is determined by the corresponding coordinate of vector s, i.e., $F(S_k) = s_k$ for every $k = 1, \ldots, n$.

For every $k = 1, \ldots, n$, denote by

$$u_k^L = u_k^L(s) = u(b_{k-1}, S_k) \quad \text{and} \quad u_k^R = u_k^R(s) = u(b_k, S_k),$$

the utility that the leftmost and the rightmost *border* individuals would get in community S_k. (Since we deal with border individuals only, we will not utilize the values u_1^L and u_n^R.)

Let us construct a mapping φ by assigning the following n-dimensional vector $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))$ to any vector $s = (s_1, \ldots, s_n) \in \Delta_n^o$:

$$\varphi_k(s) = \begin{cases} s_2(u_2^L - u_1^R), & \text{if} \quad k = 1; \\ s_{k-1}(u_{k-1}^R - u_k^L) + s_{k+1}(u_{k+1}^L - u_k^R), & \text{if} \quad k = 2, \ldots, n-1; \\ s_{n-1}(u_{n-1}^R - u_n^L), & \text{if} \quad k = n. \end{cases}$$

For $k = 1$, the value $\varphi_k(s)$ is the total surplus of the border individual b_1 in community S_2 as compared with S_1 (and weighted by s_2, the size of community S_2);
for $k = 2, \ldots, n-1$, the value $\varphi_k(s)$ is the weighted sum of two total surpluses of two border individuals in S_k: for b_{k-1} it is the surplus in S_{k-1} as compared with S_k and for b_k it is the surplus in S_{k+1} as compared with S_k;
for $k = n$, $\varphi_n(s)$ is the weighted surplus of b_{n-1} in community S_{n-1} as compared with S_n.

Now extend the mapping φ to the entire positive orthant $\mathbb{R}^n_{++}$ of $\mathbb{R}^n$ by imposing the homogeneity of degree 0 condition on φ. Let us now state a variant of the Gale-Nikaido-Debreu lemma and verify that function φ satisfies all the lemma's properties.

The Gale-Nikaido-Debreu Lemma (Mas-Colell et al., 1995, Propositions 17C1 and 17B2):

Assume that function z defined on $\mathbb{R}^n_{++}$ satisfies the following five properties:

(i) z is continuous;

(ii) z is homogeneous of degree 0;

[7] Again, the community affiliation of border individuals does not matter.

(iii) The product $s \cdot z(s)$ equals zero for all $s \in \mathbb{R}^n_{++}$ (Walras' law);

(iv) There is a positive constant Q such that $z_k(s) > -Q$ for all $k = 1, \ldots, n$ and all $s \in \mathbb{R}^n_{++}$ (Uniform boundedness from below);

(v) If the sequence $s^i \in \mathbb{R}^n_{++}$ converges to s, where $s \neq 0$ and $s_k = 0$ for some k, then

$$\lim_{i\to\infty} \max_{k=1,\ldots,n} \{z_1(s^i), \ldots, z_n(s^i)\} = \infty.$$

Then there exists $s^* \in \mathbb{R}^n_{++}$ such that $z(s^*) = 0$.

Now we verify that φ satisfies all the conditions of the lemma:

(i) Continuity. By Lemma 1, the functions $u^L_k(s)$ and $u^R_k(s)$ are continuous in s. Thus φ is also continuous.

(ii) Homogeneity of degree 0: follows from the construction.

(iii) Walras' Law. Indeed, by utilizing the expression for φ_k, we obtain:

$$\sum_{k=1}^{n} s_k\varphi_k(s_1, \ldots, s_n) = s_1 s_2(u^L_2 - u^R_1) + \sum_{k=2}^{n-1} s_k s_{k-1}(u^R_{k-1} - u^L_k) +$$

$$\sum_{k=2}^{n-1} s_k s_{k+1}(u^L_{k+1} - u^R_k) + s_n s_{n-1}(u^R_{n-1} - u^L_n).$$

Since every parenthetical term in the last expression appears twice, once with a positive and once with a negative sign, everything cancels out, and $\sum_{k=1}^{n} s_k\varphi_k(s)$ is equal to zero for all vectors s in Δ^o_n.

(iv) Uniform boundedness from below: this follows from Lemma 1 and the boundedness of benefits. Recall that utility is the difference between benefits and costs. Denote $u^L_k = v^L_k - c^L_k$ and similarly $u^R_k = v^R_k - c^R_k$. There are four types of negative summands in the definition of φ: $-s_k c^L_k$, $-s_k c^R_k$, $-s_{k-1} v^L_k$ and $-s_{k+1} v^R_k$. The first two of them are bounded by $-L'$ by lemma 1, and the last two are bounded by $-M'$ by assumption A, so $\varphi_k(s)$ is greater than $-2M' - 2L'$.

(v) Let sequence $(s^i_1, \ldots, s^i_n)$ approach the point $(s_1, \ldots, s_n)$ on the boundary of the simplex Δ_n as $i \to +\infty$. This means that $s^i_k \to 0$ for some (but not all) k. To prove that $\max_{k=1,\ldots,n}\{\varphi_k(s^i_1, \ldots, s^i_n)\}$ tends to $+\infty$ it is sufficient to show that $\sum_{k=1}^{n} \varphi_k(s^i_1, \ldots, s^i_n) \to +\infty$. The latter sum is equal to $\sum_{k=1}^{n-1} s^i_{k+1}(u^{iL}_{k+1} - u^{iR}_k) + \sum_{k=2}^{n} s^i_{k-1}(u^{iR}_{k-1} - u^{iL}_k) = \sum_{k=2}^{n} s^i_k u^{iL}_k +$

$\sum_{k=1}^{n-1} s_k^i u_k^{iR} - \sum_{k=1}^{n-1} s_{k+1}^i u_k^{iR} - \sum_{k=2}^{n} s_{k-1}^i u_k^{iL}$. By Lemma 1 and assumptions A and EM, the sum of the first two summands is bounded by $-2(n-1)L'$ from below. The benefit part of the last two summands is also bounded by $2(n-1)M'$. It remains to be proven that the expression $\sum_{k=1}^{n-1} s_{k+1}^i c_k^{iR} + \sum_{k=2}^{n} s_{k-1}^i c_k^{iL}$ tends to $+\infty$.

Let j be the value such that $s_j = 0$ and either $s_{j+1} > 0$ or $s_{j-1} > 0$. This means that $\frac{s_{j+1}^i}{s_j^i} + \frac{s_{j-1}^i}{s_j^i} \to +\infty$. By applying Lemma 1 in the last inequality, we obtain $\sum_{k=1}^{n-1} s_{k+1}^i c_k^{iR} + \sum_{k=2}^{n} s_{k-1}^i c_k^{iL} \geq (s_{j+1}^i + s_{j-1}^i) \min\{c_j^{iL}, c_j^{iR}\} = \left(\frac{s_{j+1}^i}{s_j^i} + \frac{s_{j-1}^i}{s_j^i}\right) s_j^i \min\{c_j^{iL}, c_j^{iR}\} \geq \left(\frac{s_{j+1}^i}{s_j^i} + \frac{s_{j-1}^i}{s_j^i}\right) L \to +\infty$, as stated. Thus, $\max_{k=1,\dots,n}\{\varphi_k(s_1^i, \dots, s_n^i)\}$ approaches infinity.

Now, by the Gale-Nikaido-Debreu Lemma, there exists a vector $s^* = (s_1^*, \dots, s_n^*) \in \Delta_n^o$ such that $\varphi_k(s^*) = 0$ for all $k = 1, \dots, n$. Since s_k^* is positive for every k, $\varphi_1(s^*) = 0$ implies that $u_1^R = u_2^L$. From this we have, $u_k^R = u_{k+1}^L$ for all $k = 2, \dots, n-1$. Thus, the border indifference property BIP holds for the n-partition generated by the vector s^*. □

Before proceeding with our next result, we show that the boundedness of community formation cost g from below, stated as a part of Assumption A, is necessary for Proposition 2 to hold. Consider an example of a society with a capital at point 0, which considers a partition into several communities, each with its own marketplace. In every community, the location of the marketplace coincides with the location of its median voter. The total community cost to an individual t in community S consists of two parts: *external transportation cost* to the capital, represented by the distance between the median of S, $m(S)$ and the capital, and *internal community cost* represented by the distance between t and the median $m(S)$. Formally,

Example 3: Consider a society with the community formation cost $g(S) = m(S)$, where the distribution of influence P^S is entirely concentrated at the median $m(S)$ and the bilateral community cost function is $\rho(t, r) = |t - r|$. That is, $c(t, S) = m(S) + |t - m(S)|$ for every S and every $t \in S$. The benefit function is flat, the stay-alone option is unachievable. Thus, the society satisfies all assertions of Assumptions A' and EM, except the boundedness away from zero of the function g. However, there is no partition with at least two communities satisfying BIP.

Proof: The observation that $g(S)F(S)$ tends to zero for communities $[0, a]$, when a approaches zero, is obvious. The verification of all other assertions of Assumption A', except the Kantorovich continuity of P^S, is straightforward. Notice that for our specification, the Kantorovich continuity is a consequence of the existence and continuity of the density function $f(\cdot)$ as well as its boundedness away from zero! Only under those conditions will $m(S)$, the point of the Dirac concentration for the influence function, vary continuously in a and b, which implies the Kantorovich continuity of the Dirac measures.

Consider a stratified partition of I, $[0, a), [a, b), \ldots$, where $0 < a < b \leq 1$. Then the total cost of citizen a at $[0, a)$ is $\frac{a}{2} + \frac{a}{2} = a$, while her cost at $[a, b)$ is $\frac{a+b}{2} + \frac{b-a}{2} = b > a$. Since the benefit is constant, she would prefer the adjacent community closer to the capital, and BIP is violated.□

Proposition 2, however, does not resolve the issue of existence of stable partitions. It merely guarantees the existence of partitions that satisfy the border indifference, and we indeed show that the condition BIP, while necessary, is not sufficient to guarantee the existence of a Nash stable partition. We show that non-existence cannot be rectified even if instead of A we use a more restrictive assumption A'. That is, for every integer $n > 1$ there is a society that satisfies A' and EM, for which the set of stable n-partitions is empty.

Example 4: We shall limit ourselves to describing a society for which no 2-partition $\pi \in \tilde{\Pi}_2$ is stable.[8]

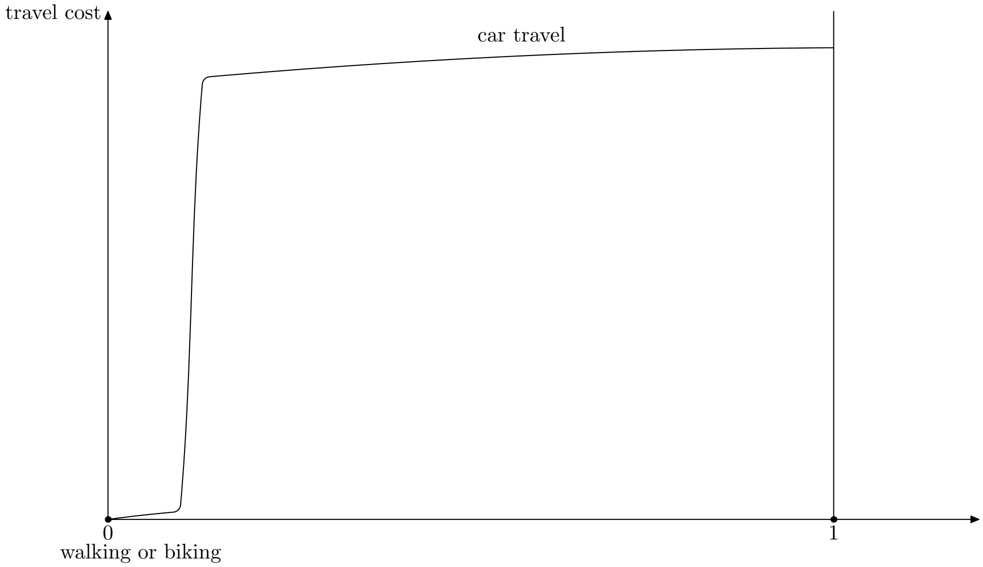
The influence distribution is again concentrated at the median that is interpreted as the location of the marketplace. The community cost function $\rho(t, r, S)$ satisfies Assumption A'. More specifically, $\rho(t, r, S) = q(|t - r|)$ where $q(\cdot)$ is a continuous piecewise linear function given by (see Figure 1):

$$q(x) = \begin{cases} 0.001 \cdot t, & \text{if} \quad x \leq 0.01; \\ 10^{-5} + 1000(t - 0.01), & \text{if} \quad 0.01 \leq x \leq 0.02; \\ 10 + 0.001(t - 0.01), & \text{if} \quad x \geq 0.02. \end{cases}$$

The cost function, which in our setting represents the cost of transportation to the marketplace, indicates the usage of a bike for short distances, and a car for long distances. Obviously

[8] This construction could be easily extended to point out societies that, for a given integer n, do not admit a stable partition.

Figure 1: Transportation cost function and its interpretation



the latter would icur substantial fixed costs.

We assume that the group formation function $g(S)$ is the inverse of its mass: $g(S) = \frac{1}{F(S)}$, i.e., the total cost of setting up a marketplace is constant. Thus, the cost function $c(t, S)$ is defined by

$$c(t, S) = q(|t - m(S)|) + \frac{1}{F(S)}.$$

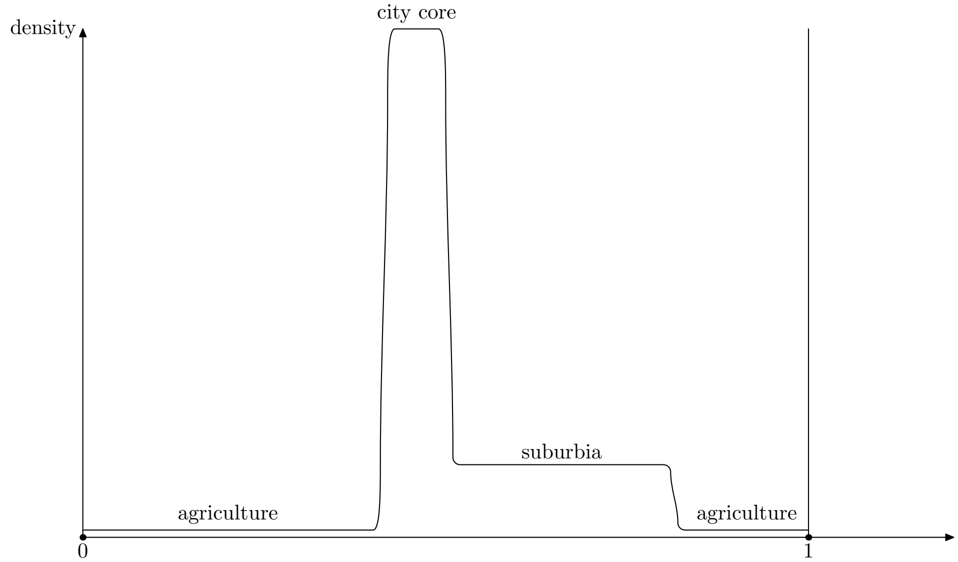
To complete the description of our example, we must still introduce the distribution of traders across the interval I and to specify the bilateral community cost function q.

The population basically consists of two parts (see Figure 2): city core — the center at $[0.49, 0.5]$ and a relatively populated suburbia $[0.51, 0.61]$, with sparse population elsewhere (we want to preserve the continuity of density). More specifically, the density function f is piecewise linear, which is flat on four segments and linear on three others:

$$f(x) = \begin{cases} 0.01, & \text{if} \quad x \in [0, 0.48]; \\ 0.01 + 3300(x - 0.48), & \text{if} \quad x \in [0.48, 0.49]; \\ 33.01, & \text{if} \quad x \in [0.49, 0.5]; \\ 3.01 + 3000(0.51 - x), & \text{if} \quad x \in [0.5, 0.51]; \\ 3.01, & \text{if} \quad x \in [0.51, 0.61]; \\ 0.01 + 300(0.62 - x), & \text{if} \quad x \in [0.61, 0.62]; \\ 0.01, & \text{if} \quad x \in [0.62, 1]. \end{cases}$$

The examination of the example is relegated to the Appendix. We would like to point out the non-convex cost structure of our example, which is relevant to the discussion in the following section.

Figure 2: Population density and its interpretation



5 Monotonicity

We turn now to identifying additional *monotonicity* conditions to guarantee Nash stability. In order to do so, recall the strict partial ordering RO on the set of closed intervals $S = [a, b] \in \mathcal{I}$. We say that the interval $S = [a, b]$ is *to the right of* $T = [c, d]$ or $S \succ_{RO} T$, if $a \geq d$. Under RO, every resident of S is located to the right (in a weak sense) of every resident of T.

Since the monotonicity discussed here is defined with respect to some ordering, the stronger is the ordering, the weaker the monotonicity requirement. In particular, RO is stronger than the well-known Veinott ordering (Veinott, 1989), under which the interval $S = [a, b] \succeq_v T = [c, d]$ if $a \geq c$ and $b \geq d$.

Assumption ID — Increasing Differences Property and Unimodality: The utility function $u(t, S)$ satisfies the *increasing differences* property[9] with respect to RO over the space of all intervals $S \subset I$ if the difference $u(t, S) - u(t, T)$ is nondecreasing in t when $S \succ_{RO} T$. We also assume that $u(t, S)$ is *unimodal*: it increases for $t < t_0$ and decreases for $t > t_0$, for some $t_0 \in S$.

We immediately obtain the following result:

[9]See Milgrom and Shannon, 1994, for detailed treatment of all those matters.

Proposition 3: Under assumptions A and ID, every community structure which satisfies BIP is Nash stable.

Proof: Consider a stratified n-collection $\pi = \{S_1, \ldots, S_n\}$, which satisfies BIP.

We utilize the following

Statement SP – Staying Put: If $S_k = [a_k, b_k)$ and $S_l = [a_l, b_l)$ are two formed communities then $u(b_k, S_l) \leq u(b_k, S_k)$ and $u(a_l, S_k) \leq u(a_l, S_l)$. That is, the border agents would not prefer to change their communities.

The proof proceeds through induction on $|l-k|$. We start with the case $l = k+1$. If $a_l = b_k$ then, by BIP, $u(b_k, S_l) = u(b_k, S_k)$ and $u(a_l, S_k) = u(a_l, S_l)$. If $a_l > b_k$ then $u(b_k, S_k) = u(a_l, S_l) = 0$. By unimodality, $u(b_k, S_l) \leq u(a_l, S_l) = u(b_k, S_k)$ and, similarly, $u(a_l, S_k) \leq u(b_k, S_k) = u(a_l, S_l)$. We proceed to the case when $l = k-1$. If $a_k = b_l$ then, by ID and BIP, $u(b_k, S_k) - u(b_k, S_l) \geq u(a_k, S_k) - u(a_k, S_l) = 0$. Symmetrically, $u(a_l, S_l) - u(a_l, S_k) \geq u(a_k, S_l) - u(a_k, S_k) = 0$. Finally, if $a_k > b_l$ then $u(b_k, S_k) - u(b_k, S_l) \geq u(a_k, S_k) - u(a_k, S_l)$, again by ID, and, by unimodality, $u(a_k, S_l) \leq u(b_l, S_l)$. Since $u(a_k, S_k) = u(b_l, S_l) = 0$ by BIP, we get $u(b_k, S_k) - u(b_k, S_l) \geq 0 - u(a_k, S_l) \geq -u(b_l, S_l) = 0$ and hence $u(b_k, S_l) \leq u(b_k, S_k)$, as stated. The inequality $u(a_l, S_k) \leq u(a_l, S_l)$ is proven in a similar manner.

The induction step proceeds in the following way: let S_m be a community between S_k and S_l. Let, without loss of generality, $k < m < l$. Then ID implies that $u(b_k, S_l) - u(b_k, S_m) \leq u(b_m, S_l) - u(b_m, S_m)$. By the induction hypothesis, the last term is non-positive, hence $u(b_k, S_l) \leq u(b_k, S_m)$. By invoking the induction hypothesis again, we obtain $u(b_k, S_m) \leq u(b_k, S_k)$, hence $u(b_k, S_l) \leq u(b_k, S_k)$, as stated. Similarly, we obtain $u(a_l, S_k) \leq u(a_l, S_l)$. □

Now we verify that the collection π is indeed an equilibrium. Firstly, consider an unaffiliated individual t and show that she would not benefit by joining any existing community. Indeed, consider communities S_k and S_{k+1} such that $b_k < t < a_{k+1}$ (in case $t < a_1$ or $t > b_n$ we consider only one of them). By unimodality and BIP, $u(t, S_k) \leq u(b_k, S_k) = 0$ and $u(t, S_{k+1}) \leq u(a_{k+1}, S_{k+1}) = 0$. Thus, t would not join the two closest communities. Let us show that she would not join more distant communities either. If $l < k-1$, then by unimodality, $u(t, S_l) \leq u(b_k, S_l)$, by Statement SP,

$u(b_k, S_l) \leq u(b_k, S_k)$, and by BIP, $u(b_k, S_k) = 0$, yielding $u(t, S_l) \leq 0$. If $l > k + 1$, then, similarly, $u(t, S_l) \leq u(a_{k+1}, S_l) \leq u(a_{k+1}, S_{k+1}) = 0$. Thus, t would stay alone.

Now assume that t is a member of community S_k. Let $l < k$. By assumption ID we have $u(t, S_k) - u(t, S_l) \geq u(a_k, S_k) - u(a_k, S_l)$. The latter expression is non-negative by Statement SP, hence $u(t, S_k) \geq u(t, S_l)$. Symmetrically, for $l > k$ we get $u(t, S_k) - u(t, S_l) \geq u(b_k, S_k) - u(b_k, S_l) \geq 0$, and $u(t, S_k) \geq u(t, S_l)$ again. Finally, the unimodality assumption implies that $u(t, S_k) \geq \min\{u(a_k, S_k), u(b_k, S_k)\} \geq 0$, so agent t would not prefer to leave S_k to stay alone. This yields the Nash stability of π. □

Note that under assumption EM, Proposition 2 yields the existence of a partition satisfying BIP. Thus, by Proposition 3, Assumptions A, ID and EM yield the existence of a Nash stable partition. The nonexistence of a Nash equilibrium in Example 4 under A and EM, indicates that the assumption ID does not hold in this case. One of the reasons could be the non-convexity of the bilateral community function q used there. Since convexity is a standard assumption in this type of transportation cost model (see, e.g., d'Aspremont et al., 1979), the question arises whether, under assumption A', the convexity of the cost function q guarantees the existence of a stable partition. The following proposition offers an affirmative answer to this question:

Proposition 4: Let assumptions A' and EM hold. If the benefits are flat and cost function q is convex, then for every positive integer n, there exists a Nash stable n-partition.

Proof: It suffices to show that the convexity of the community cost function yields assumption ID. Consider two communities, S and T, such that $S \succ_{RO} T$, which means that $s \geq t$ for every $s \in S$ and every $t \in T$. Since $v(x, S) = v(x, T) = v$, these terms cancel each other out and we must show that $c(x, T) - c(x, S)$ is increasing. In turn, the costs consist of formation costs independent of x and bilateral community costs. Thus, we need to show that the difference in communty costs is increasing. That is, the expression

$$D(x) = \int_T q(|x - r|) f(r) dP^T(r) - \int_S q(|x - r|) f(r) dP^S(r)$$

should weakly increase for any convex function q, density function f and influence distributions P^S

and P^T which satisfy assumption A'.

Take two individuals x, x' with $x' > x$. The convexity of the function q implies that the difference $q(|x'-r|) - q(|x-r|)$ weakly decreases in r. Since for any $s \in S$ and $t \in T$ we have $s \geq t$, it follows that $q(|x'-t|) - q(|x-t|) \geq q(|x'-s|) - q(|x-s|)$. Hence, since P^S and P^T are distributions concentrated on S and T respectively,

$$\int_T (q(|x'-r|) - q(|x-r|))f(r)dP^T(r) \geq \int_S (q(|x'-r|) - q(|x-r|))f(r)dP^S(r).$$

By reordering the terms in the last inequality, we obtain

$$\int_T q(|x'-r|)f(r)dP^T(r) - \int_S q(|x'-r|)f(r)dP^S(r) \geq$$

$$\geq \int_T q(|x-r|)f(r)dP^T(r) - \int_S q(|x-r|)f(r)dP^S(r).$$

That is, $D(x') \geq D(x)$, as desired. To complete the proof, we point out that the function $u(x, S)$ is unimodal in x, which is a simple implication of the convexity of q. □

6 Concluding Remarks

In this paper we show that in our model of endogenous community formation, there is little hope to obtain a general existence result. However, by imposing the essentiality of membership, we offer a combination of the continuous and monotone comparative statics in order to examine the existence of Nash equilibria in a model of market interactions within a unidimensional set of agents. Our approach allows us to separate the roles of continuity and monotonicity in ensuring the existence of Nash equilibria. While the assumption of the Kantorovich continuity yields the existence of a partition satisfying border indifference, it still leaves the possibility that the resulting partition is not Nash stable. Only the imposition of the Milgrom-Shannon monotonicity conditions guarantees the existence of a Nash equilibrium.

Given the unidimensionality of our model, the immediate and important question is whether this approach could be extended to the multidimensional setting, similar to that in Drèze et al., 2009. This is left for future research.

Our Example 2 with no Nash equilibrium is very delicate, and the differences between net benefits under different options are minuscule. Thus, it would be worthwhile to examine the existence of an appoximate equilibrium similar to that in Drèze et al., 2009.

While the analysis of the number of communities has attracted much attention, the theoretical investigation of community sizes in Nash equilibrium has not advanced to the same degree. It would be very interesting and challenging to link our results to important empirical literature that addresses the issue of population distribution across cities, counties, metropolitan areas, or regions, and focuses on the implications of the Zipf's and Gibrat's laws (See e.g., Gabaix, 1999 and Eeckhout, 2004.)

7 Appendix

The Kantorovich continuity.

We first define the distance d between any two measurable subsets S and T of $[0,1]$ by setting $d(S,T) = F(S \triangle T) = F(S \setminus T) + F(T \setminus S)$, which represents the mass of all agents that belong to exactly one of two sets, S and T. For instance, if $S = [a,b]$ and $T = [c,d]$ for $a < c < b < d$ then $d(S,T) = \int_a^c f(r)dr + \int_b^d f(r)dr$.

For every two distributions P_1 and P_2 on I, define by $\mathcal{H} = \mathcal{H}(P_1, P_2)$ the set of all measures on I^2 with marginal distributions P_1 and P_2. That is, for all measurable $Q \subset I$ and all $H \in \mathcal{H}(P_1, P_2)$ we have $P_1(Q) = \int_{Q \times [0,1]} dH(x,y)$ and $P_2(Q) = \int_{[0,1] \times Q} dH(x,y)$.

Next, define the Kantorovich metric D_K: $D_K(P_1, P_2) = \min_{H \in \mathcal{H}} \int_{[0,1]^2} |x - y| dH(x,y)$. Its intuitive interpretation is the minimal cost required to "transform" P_1 into P_2. If each distribution is viewed as a unit amount of dirt piled on I, the metric is the minimum cost of turning one pile into the other. The cost is counted as the amount of dirt that needs to be shifted multiplied by the distance of the move. In fact, this interpretation is the reason that this metric is known in computer science as the earth mover's distance.[10]

Finally, the family $\{P^S\}_{S \subset I}$ is called *Kantorovich-continuous* if for any measurable set $\hat{S}$ and any sequence $S_1, S_2, \ldots, S_n, \ldots$ of measurable subsets of I with $\lim_{n \to \infty} d(S_n, \hat{S}) = 0$, we also have

[10]See, e.g., Rubner et al., 1998.

the Kantorovich distance $D_K(P^{S_n}, P^{\hat{S}})$ approaching zero as n tends to infinity. Informally, a small change of a set leads to a small change of the distribution on this set. An easy logical implication is that P^S is not impacted by null-set changes in S.

Proof of Lemma 1: Indeed, Kantorovich continuity of P^S implies continuity of the integral of any continuous function, including $\rho(t, r, S)$. (The idea is the following: if S is slightly changed then both ρ and P^S are also slightly changed. Then the integral is slightly changed as well.) Since g is also continuous, so is c. Moreover, $c(t, S)F(S) \geq g(S)F(S) \geq K$ and $c(t, S)F(S) \leq (g(S) + \max_{t,r} \rho(t, r, S))F(S) \leq K' + \max_{t,r,S} \rho(t, r, S)F(I)$. Continuity of ρ implies that the last maximum is finite, so we may take $L' = K' + \max_{t,r,S} \rho(t, r, S)F(I)$. □

Analysis of Example 2: Now we wish to demonstrate that there is no community $[a, b]$ satisfying BIP in Example 2. We provide the series of 19 observations that cover all possible communities $[a, b]$. The table below classifies the communities according to their endpoints, where rows indicate left endpoints and columns indicate right ones. A number in each cell identifies which observation (among 19) is applied to rule out BIP for the corresponding interval.

a \ b	$(0, \frac{3}{9}]$	$(\frac{3}{9}, \frac{4}{9}]$	$(\frac{4}{9}, \frac{5}{9}]$	$(\frac{5}{9}, \frac{6}{9}]$	$(\frac{6}{9}, \frac{7}{9}]$	$(\frac{7}{9}, 1)$	1
0	2	4	6	8	12	12	12
$(0, \frac{3}{9})$	2	3	5	7	11	16	19
$[\frac{3}{9}, \frac{4}{9})$	1	2	3	3	10	15	19
$[\frac{4}{9}, \frac{5}{9})$	1	1	2	3	9	14	19
$[\frac{5}{9}, \frac{6}{9})$	1	1	1	2	3	13	18
$[\frac{6}{9}, \frac{7}{9})$	1	1	1	1	2	3	17
$[\frac{7}{9}, 1)$	1	1	1	1	1	2	2

Notice that whenever $a > 0$ and $b < 1$, BIP implies that both border agents must bear the zero cost. Therefore, the middlepoint of the interval must coincide with its median. If $a = 0$ or $b = 1$, then it is not necessary, but in any case, the total costs of either border agent should not exceed

$v = 0.1115$. We show that it is not the case.

1. It is impossible because a does not exceed b.

2. If the population is at most 8, then the total cost of any agent is at least $\frac{1}{8} > 0.1115$. If the population is 9 (which may happen only on $\left[0, \frac{3}{9}\right]$), the length is greater than $\frac{2}{9}$, so the total cost of at least one of the two border agents is greater than $\frac{1}{9} + \frac{2}{81} = \frac{11}{81} \approx 0.135 > 0.1115$.

3. If $a \neq 0$ and $b \neq 1$, the middle point should coincide with the median. It cannot happen if the population density is monotone and non-constant on $[a, b]$.

4. $[0, b]$, where $\frac{3}{9} < b \leq \frac{4}{9}$. The population of $[a, b]$ is $9 + 45\left(b - \frac{3}{9}\right) = 45b - 6$. The median is located on $\left(0, \frac{3}{9}\right)$ and is determined by $m = \frac{5}{6}b - \frac{1}{9}$. The total cost of agent 0 is $\frac{1}{45b-6} + \frac{2}{9}\left(\frac{5}{6}b - \frac{1}{9}\right)$. The standard minimization shows that the minimal cost is attained at $b = \frac{4}{9}$ (corner solution) and is equal to $\frac{1}{14} + \frac{2}{9} \cdot \frac{7}{27} = \frac{439}{3402} \approx 0.129 > 0.1115$.

5. $[a, b]$, where $0 < a < \frac{3}{9}$, $\frac{4}{9} < b \leq \frac{5}{9}$. Since the median should coincide with the middle point, $\frac{a+b}{2}$ cannot belong to $\left(0, \frac{3}{9}\right)$, as in this case the left half would contain a smaller population mass. Also $\frac{a+b}{2}$ cannot belong to $\left(\frac{4}{9}, \frac{5}{9}\right)$, since it is incompatible with $a \in \left(0, \frac{3}{9}\right)$. Thus, $\frac{a+b}{2}$ belongs to $\left(\frac{4}{9}, \frac{5}{9}\right)$. We get an equation $27\left(\frac{3}{9} - a\right) + 45\left(\frac{a+b}{2} - \frac{3}{9}\right) = 45\left(\frac{4}{9} - \frac{a+b}{2}\right) + 36\left(b - \frac{4}{9}\right)$, the solution of which yields $a = \frac{5}{9} - \frac{b}{2}$. The straightforward calculations show that the population is $49.5b - 17$ and the length is equal to $\frac{3b}{2} - \frac{5}{9}$. So, the border individuals bear the cost $\frac{1}{49.5b-17} + \frac{b}{6} - \frac{5}{81}$. It is easy to see that the minimum cost is achieved at $b = \frac{5}{9}$, and is equal to $\frac{1}{10.5} + \frac{5}{54} - \frac{5}{81} = \frac{143}{1134} \approx 0.126 > 0.1115$.

6. $[0, b]$, where $\frac{4}{9} < b \leq \frac{5}{9}$. The population mass is not greater than 18. The median is at or to the right of $\frac{7}{27}$, so the total cost of agent 0 is greater than $\frac{1}{18} + \frac{14}{243} = \frac{55}{486} \approx 0.113 > 0.1115$.

7. $[a, b]$, where $0 < a < \frac{3}{9}$, $\frac{5}{9} < b \leq \frac{6}{9}$. In this case, there are two subcases depending on whether $\frac{a+b}{2}$ lies on $\left[\frac{3}{9}, \frac{4}{9}\right]$ or on $\left[\frac{4}{9}, \frac{5}{9}\right]$. (It is easy to see that it cannot be less than $\frac{3}{9}$ or greater than $\frac{5}{9}$). In the first subcase, we obtain the equation $27\left(\frac{3}{9} - a\right) + 45\left(\frac{a+b}{2} - \frac{3}{9}\right) = 45\left(\frac{4}{9} - \frac{a+b}{2}\right) + 4 + 27\left(b - \frac{5}{9}\right)$. It yields $a = \frac{5}{6} - b$, hence $\frac{a+b}{2} = \frac{5}{12}$, that indeed lies on $\left[\frac{3}{9}, \frac{4}{9}\right]$. Next, the population is equal to $54b - 19.5$ and the length is equal to $2b - \frac{5}{6}$. Thus, the border

individuals bear the cost $\frac{1}{54b-19.5} + \frac{2}{9}\left(b - \frac{5}{12}\right)$.Standard minimization shows that minimum cost is achieved at $b = \frac{13}{36} + \frac{\sqrt{3}}{6}$. (This is a rare case with an interior solution!) Plugging this value of b back, yields the cost $\frac{1}{9\sqrt{3}} + \frac{2}{9} \cdot \frac{3\sqrt{3}-1}{18} \approx 0.115 > 0.1115$.

In the second subcase, we obtain the equation $27\left(\frac{3}{9} - a\right) + 5 + 36\left(\frac{a+b}{2} - \frac{4}{9}\right) = 36\left(\frac{5}{9} - \frac{a+b}{2}\right) + 27\left(b - \frac{5}{9}\right)$. Solving this leads to $a = \frac{7}{9} - b$, hence $\frac{a+b}{2} = \frac{7}{18}$, which does not lie on $\left[\frac{4}{9}, \frac{5}{9}\right]$. Thus, this subcase is ruled out.

8. $[0, b]$, where $\frac{5}{9} < b \leq \frac{6}{9}$. The population is not greater than 21. The median is at or to the right of $\frac{1}{3}$, so the total cost of agent 0 is greater than $\frac{1}{21} + \frac{2}{27} = \frac{23}{189} \approx 0.12 > 0.1115$.

9. $[a, b]$, where $\frac{4}{9} \leq a < \frac{5}{9}$, $\frac{6}{9} < b \leq \frac{7}{9}$. Since $a > 0, b < 1$, the median should be equal to $\frac{a+b}{2}$, and to belong to $\left[\frac{5}{9}, \frac{6}{9}\right]$. We have the equation $36\left(\frac{5}{9} - a\right) + 27\left(\frac{a+b}{2} - \frac{5}{9}\right) = 27\left(\frac{6}{9} - \frac{a+b}{2}\right) + 45\left(b - \frac{6}{9}\right)$. Its solution leads to $a = \frac{17}{9} - 2b$. This point indeed lies on $\left[\frac{4}{9}, \frac{5}{9}\right)$ if $b < \frac{13}{18}$, and therefore, the population is not greater than 9.5. Since the length is not less than $\frac{1}{9}$, the total cost is at least $\frac{1}{9.5} + \frac{2}{9} \cdot \frac{1}{18} = \frac{181}{1539} \approx 0.117 > 0.1115$.

10. $[a, b]$, where $\frac{3}{9} \leq a < \frac{4}{9}$, $\frac{6}{9} < b \leq \frac{7}{9}$. There are two subcases depending on whether $\frac{a+b}{2}$ lies on $\left[\frac{4}{9}, \frac{5}{9}\right]$ or on $\left[\frac{5}{9}, \frac{6}{9}\right]$. In the first subcase, we get an equation $45\left(\frac{4}{9} - a\right) + 36\left(\frac{a+b}{2} - \frac{4}{9}\right) = 36\left(\frac{5}{9} - \frac{a+b}{2}\right) + 3 + 45\left(b - \frac{6}{9}\right)$. Solving it leads to $a = \frac{11}{9} - b$, that lies outside of the interval $\left[\frac{3}{9}, \frac{4}{9}\right)$. Thus, this subcase is impossible.

    In the second subcase, we obtain the equation $45\left(\frac{4}{9} - a\right) + 4 + 27\left(\frac{a+b}{2} - \frac{5}{9}\right) = 27\left(\frac{6}{9} - \frac{a+b}{2}\right) + 45\left(b - \frac{6}{9}\right)$. Its solution yields $a = \frac{7}{6} - b$ that indeed lies on $\left[\frac{3}{9}, \frac{4}{9}\right)$ if $b > \frac{13}{18}$. The median equals $\frac{7}{12}$ and indeed lies on $\left[\frac{5}{9}, \frac{6}{9}\right]$. The population is equal to $90b - 55.5$, and the length is equal to $2b - \frac{7}{6}$, so that border individuals bear the cost $\frac{1}{90b-55.5} + \frac{2}{9}\left(b - \frac{7}{12}\right)$. Standard minimization shows that we have a corner solution again: the minimum cost is achieved at $b = \frac{7}{9}$ and is equal to $\frac{1}{14.5} + \frac{2}{9} \cdot \frac{7}{36} = \frac{527}{4698} \approx 0.1121 > 0.1115$.

11. $[a, b]$, where $0 < a \leq \frac{3}{9}$, $\frac{6}{9} < b \leq \frac{7}{9}$. There are two subcases depending on whether $\frac{a+b}{2}$ lies on $\left[\frac{3}{9}, \frac{4}{9}\right]$ or on $\left[\frac{4}{9}, \frac{5}{9}\right]$. (Other subcases are obviously impossible). In the first subcase, we obtain an equation $27\left(\frac{3}{9} - a\right) + 45\left(\frac{a+b}{2} - \frac{3}{9}\right) = 45\left(\frac{4}{9} - \frac{a+b}{2}\right) + 7 + 45\left(b - \frac{6}{9}\right)$. Solving it leads

to $a = \frac{1}{6}$. The median is $\frac{b}{2} + \frac{1}{12}$ and lies on $\left[\frac{3}{9}, \frac{4}{9}\right]$ if $b \le \frac{13}{18}$. The population is $45b - 13.5$ and the length is $b - \frac{1}{6}$. The border individuals bear the cost $\frac{1}{45b-13.5} + \frac{b}{9} - \frac{1}{54}$. Standard minimization shows that the minimum cost is attained at $b = \frac{13}{18}$ and equals $\frac{1}{19} + \frac{13}{162} - \frac{1}{54} = \frac{176}{1539} \approx 0.114 > 0.1115$.

In the second subcase, we obtain the equation $27\left(\frac{3}{9} - a\right) + 5 + 36\left(\frac{a+b}{2} - \frac{4}{9}\right) = 36\left(\frac{5}{9} - \frac{a+b}{2}\right) + 3 + 45\left(b - \frac{6}{9}\right)$. Solving it leads to $a = b - \frac{5}{9}$. The median is at $b - \frac{5}{18}$ and lies on $\left[\frac{4}{9}, \frac{5}{9}\right]$ for $b \ge \frac{13}{18}$. In this case, the population is at most 20 and the length is $\frac{5}{9}$, so the total cost to any of two border agents is at least $\frac{1}{20} + \frac{5}{81} = \frac{181}{1620} \approx 0.1117 > 0.1115$.

12. $[0, b]$, where $\frac{6}{9} < b \le 1$. The population is not greater than the maximal possible value, 32, and the median is not less than $\frac{11}{30}$ (which is the case for $b = \frac{6}{9}$). Hence, the total cost to the leftmost agent is not less than $\frac{1}{32} + \frac{2}{9} \cdot \frac{11}{30} = \frac{487}{4320} \approx 0.1127 > 0.1115$.

13. $[a, b]$, where $\frac{5}{9} \le a < \frac{6}{9}$, $\frac{7}{9} < b < 1$. Since the median should coincide with the middle point, a and b should be symmetric around $\frac{13}{18}$, hence, $a = \frac{13}{9} - b$ (and b must be not greater than $\frac{8}{9}$). The population is $54b - 37$ and the length is $2b - \frac{13}{9}$. The total cost to either of the border agents is $\frac{1}{54b-37} + \frac{2}{9}b - \frac{13}{81}$. The minimal cost is attained at $b = \frac{8}{9}$ and is equal to $\frac{1}{11} + \frac{1}{27} = \frac{38}{297} \approx 0.127 > 0.1115$.

14. $[a, b]$, where $\frac{4}{9} \le a < \frac{5}{9}$, $\frac{7}{9} < b < 1$. As before, we have two subcases depending on whether $\frac{a+b}{2}$ lies on $\left[\frac{5}{9}, \frac{6}{9}\right]$ or on $\left[\frac{6}{9}, \frac{7}{9}\right]$ (other options are obviously ruled out). In the first subcase we get $36\left(\frac{5}{9} - a\right) + 27\left(\frac{a+b}{2} - \frac{5}{9}\right) = 27\left(\frac{6}{9} - \frac{a+b}{2}\right) + 5 + 27\left(b - \frac{7}{9}\right)$. Solving it leads to $a = \frac{1}{3}$, so this subcase is impossible.

In the second subcase, we obtain the equation $36\left(\frac{5}{9} - a\right) + 3 + 45\left(\frac{a+b}{2} - \frac{6}{9}\right) = 45\left(\frac{7}{9} - \frac{a+b}{2}\right) + 27\left(b - \frac{7}{9}\right)$. Solving it leads to $a = \frac{7}{3} - 2b$, which lies on $\left[\frac{4}{9}, \frac{5}{9}\right]$ if $b \in \left[\frac{8}{9}, \frac{17}{18}\right]$. The population is equal to $99b - 77$ and the length is $3b - \frac{7}{3}$, so the total cost to any of the two border agents is $\frac{1}{99b-77} + \frac{2}{9}\left(\frac{3}{2}b - \frac{7}{6}\right) = \frac{1}{99b-77} + \frac{b}{3} - \frac{7}{27}$. It is easy to see that the last expression is minimal when $b = \frac{17}{18}$, and the minimum is $\frac{1}{16.5} + \frac{17}{54} - \frac{7}{27} = \frac{23}{198} \approx 0.116 > 0.1115$.

15. $[a, b]$, where $\frac{3}{9} \le a < \frac{4}{9}$, $\frac{7}{9} < b < 1$. Here we have three subcases depending on whether $\frac{a+b}{2}$

lies on $\left[\frac{4}{9},\frac{5}{9}\right]$, $\left[\frac{5}{9},\frac{6}{9}\right]$ or $\left[\frac{6}{9},\frac{7}{9}\right]$ (other cases are obviously impossible). In the first subcase, we obtain the equation $45\left(\frac{4}{9}-a\right)+36\left(\frac{a+b}{2}-\frac{4}{9}\right)=36\left(\frac{5}{9}-\frac{a+b}{2}\right)+8+27\left(b-\frac{7}{9}\right)$. This leads to $a=b-\frac{1}{3}$, which is inconsistent with $a\in\left[\frac{3}{9},\frac{4}{9}\right)$ and $b\in\left(\frac{7}{9},1\right)$. So, this subcase is impossible.

In the second subcase, we obtain the equation $45\left(\frac{4}{9}-a\right)+4+27\left(\frac{a+b}{2}-\frac{5}{9}\right)=27\left(\frac{6}{9}-\frac{a+b}{2}\right)+5+27\left(b-\frac{7}{9}\right)$. Its solution yields $a=\frac{7}{18}$. The median indeed lies on $\left[\frac{5}{9},\frac{6}{9}\right]$ if $b\in\left[\frac{7}{9},\frac{17}{18}\right]$. The population is $27b-6.5$, and the length is $b-\frac{7}{18}$, so the total cost of a border agent is $\frac{1}{27b-6.5}+\frac{b}{9}-\frac{7}{162}$. The minimum is achieved at $b=\frac{13}{54}+\frac{\sqrt{3}}{3}$ (an interior solution again!) and equals to $\frac{18\sqrt{3}-4}{243}\approx 0.1118>0.1115$.

In the third subcase, we obtain the equation $45\left(\frac{4}{9}-a\right)+7+45\left(\frac{a+b}{2}-\frac{6}{9}\right)=45\left(\frac{7}{9}-\frac{a+b}{2}\right)+27\left(b-\frac{7}{9}\right)$. Solving this, we obtain $b=\frac{17}{18}$. It is possible if $a\in\left[\frac{7}{18},\frac{4}{9}\right)$. The population is $36.5-45a$, and the length is $\frac{17}{18}-a$, so the total cost to any of the two border agents is $\frac{1}{36.5-45a}+\frac{17}{162}-\frac{1}{9}a$. The minimum is achieved at $a=\frac{7}{18}$ and is equal $\frac{1}{19}+\frac{5}{81}=\frac{176}{1539}\approx 0.114>0.1115$.

16. $[a,b]$, where $0<a<\frac{3}{9}$, $\frac{7}{9}<b<1$. Here we have three subcases again, depending on whether $\frac{a+b}{2}$ lies on $\left[\frac{3}{9},\frac{4}{9}\right]$, $\left[\frac{4}{9},\frac{5}{9}\right]$ or $\left[\frac{5}{9},\frac{6}{9}\right]$. (All the other cases are obviously impossible.) In the first subcase, we obtain the equation $27\left(\frac{3}{9}-a\right)+45\left(\frac{a+b}{2}-\frac{3}{9}\right)=45\left(\frac{4}{9}-\frac{a+b}{2}\right)+12+27\left(b-\frac{7}{9}\right)$. This equation yields $\frac{a+b}{2}=\frac{17}{36}>\frac{4}{9}$, thus, this subcase is impossible.

    In the second subcase, we obtain the equation $27\left(\frac{3}{9}-a\right)+5+36\left(\frac{a+b}{2}-\frac{4}{9}\right)=36\left(\frac{5}{9}-\frac{a+b}{2}\right)+8+27\left(b-\frac{7}{9}\right)$. Solving it, we get $a=1-b$. In this case, the population is $54b-22$ and the length is $2b-1$. The total cost is $\frac{1}{54b-22}+\frac{2}{9}b-\frac{1}{9}$. It is minimized when b approaches $\frac{7}{9}$, and the limit value is $\frac{1}{20}+\frac{5}{81}=\frac{181}{1620}>0.1115$.

    In the third subcase, we obtain the equation $27\left(\frac{3}{9}-a\right)+9+27\left(\frac{a+b}{2}-\frac{5}{9}\right)=27\left(\frac{6}{9}-\frac{a+b}{2}\right)+5+27\left(b-\frac{7}{9}\right)$. It can be simplified to show that the left half is more populated then the right half. Thus, this subcase is also impossible.

17. $[a,1]$, where $\frac{6}{9}\leq a<\frac{7}{9}$. If the population is T, then the median is at $1-\frac{T}{54}$. So, the agent 1 bears the cost $\frac{1}{T}+\frac{2}{9}\cdot\frac{T}{54}$, which is minimal at $T=11$ (or $a=\frac{6}{9}$) and is equal to $\frac{1}{11}+\frac{11}{243}\approx 0.136>0.1115$.

18. $[a,1]$, where $\frac{5}{9} \le a < \frac{6}{9}$. The population is at most 14 and the median is at or to the left of $\frac{43}{54}$. So, the agent at 1 bears the cost which is at least $\frac{1}{14} + \frac{11}{243} = \frac{397}{3402} \approx 0.116 > 0.1115$.

19. $[a,1]$, where $0 < a < \frac{5}{9}$. If the population T is between 14 and 22, the median m falls into the segment $\left[\frac{6}{9}, \frac{7}{9}\right)$ and can be determined by solving the equation $\frac{T}{2} = 6 + 45\left(\frac{7}{9} - m\right)$. Thus, m is located at $\frac{41}{45} - \frac{T}{90}$. For larger T, the median is to the left of that point. Thus, the agent at 1 bears the cost, which is at least $\frac{1}{T} + \frac{2}{9}\left(\frac{4}{45} + \frac{T}{90}\right)$. It is minimized at $T = 9\sqrt{5}$ and is equal to $\frac{2}{9\sqrt{5}} + \frac{8}{405} = \frac{18\sqrt{5}+8}{405} \approx 0.119 > 0.1115$.

□

The analysis of Example 4: Note that the median $m(I)$ is located at the center, 0.5, the median $m([0,0.5])$ lies to the right of 0.491 and $m([0.5,1])$ is located to the right of 0.53. (Since the masses of $[0,0.5]$ and $[0.5,1]$ are equal to 0.5, the mass of $[0.491,0.5]$ is equal to $0.29709 > 0.25$ and the mass of $[0.5,0.53]$ is equal to $0.2403 < 0.25$).

Since assumption A holds for the constructed society, Proposition 2 implies that there exists a 2-partition that satisfies BIP. By Proposition 1, it suffices to demonstrate that no such partition is Nash stable.

The proof proceeds as follows. We first show that BIP is satisfied only if the population sizes of two communities are sufficiently different. However, we demonstrate that in this case the population size advantage, represented by the function $g(S) = \frac{1}{F(S)}$, outweighs the benefits generated by a shorter distance to the median. Thus, the agent from the outer edge of the smaller community (located at either 0 or 1) would like to join the larger one, even though it is far away from her location.

Let $t^* \in I$ be such that the 2-partition $\pi = \{S_1, S_2\} = \{[0,t^*],(t^*,1]\}$ satisfies BIP. Let us first examine a possible location of t^*.

The point t^* must belong to the interval $[0.48, 0.62]$. Otherwise, the population of the smaller community is less than 0.01, yielding total cost greater than 100 to all its members. However, the cost of any member of the larger community is bounded by $\dfrac{1}{0.99} + (10 + 0.001(1 - 0.01)) < 12$, so that BIP does not hold.

t^* does not belong to the segment $[0.499, 0.501]$. Suppose, in negation, that t^* belongs to this interval. The distance between t^* and $m(S_1)$ would be less than 0.01 (since $m(S_1) > 0.491$), hence t^* would bear the comunal cost, which does not exceed .001 in S_1. At the same time, the distance between t^* and $m(S_2)$ is greater than 0.02 (since $m(S_2) > 0.53$), hence t^* would bear the community cost at least 10 in S_2. Since the population of each community is bounded by 0.46 from below and by 0.54 from above, the cost $c(t^*, S_2)$ incurred by t^* in S_2 exceeds $\frac{1}{0.54} + 10 > 11$, while at S_1 it is less than $\frac{1}{0.46} + 0.001 < 3$. Thus, BIP is violated.

Thus, t^* belongs to either segment $[0.48, 0.499]$ or $[0.501, 0.62]$. If $t^* \in [0.48, 0.499]$, then the population of S_1 is less than 0.47. Thus, for the agent located at 0, the cost at S_1 is greater than $\frac{1}{0.47} + 10 > 12.12$. However, at S_2, the cost of that agent would be less than $\frac{1}{0.53} + 10.001 < 11.89$. Hence, the agent 0 would benefit from moving to S_2, so that the partition π is not Nash stable.

The case $t^* \in [0.501, 0.62]$ is examined along the same lines: the agent located at 1 would be better off at S_1 rather than at S_2, and, again, the partition π cannot be Nash stable.

Thus, there are no Nash stable 2-partitions in our environment. □

We would like to add the comment about the relatively complicated construction of the functions f and q above. It is due to the fact that in some of the cases, we needed to generate a seemingly counterintuitive situation where individual 0 would prefer joining S_2 rather than S_1, whereas the individual t^* is indifferent about joining either of two communities. In terms of formalization, the equality $g(S_1) + q(|t^* - m(S_1)| = g(S_2) + q(|t^* - m(S_2)|)$ had to be consistent with the inequality $g(S_1) + q(m(S_1)) > g(S_2) + q(m(S_2))$, which required a somewhat delicate construction. According to Proposition 5, this would be impossible for a convex cost function.

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