

Clearing in Financial Networks¹

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1. El Bitar K., Kabanov Yu., Mokbel R. Clearing in Financial Networks. Theory of Probability and Its Applications, 62 (2017), 2, 311-344 (in Russian).

Eisenberg–Noe–Suzuki model^{2, 3}

System: N banks, $e^i \geq 0$ is the cash disposed by the i th banks, $L^{ij} \geq 0$ is the liability of the bank i to the bank j , $\tilde{L}^i := \sum_j L^{ij}$. Clearing is a procedure of repaying debts in full if possible or to the complete exhausting of resources. The repayment is proportional to the volume of borrowing.

Let $\Pi^{ij} := L^{ij}/\tilde{L}^i$, if $\tilde{L}^i \neq 0$, and $\Pi^{ij} := \delta^{ij}$ otherwise, where the Kronecker symbol $\delta^{ij} = 0$ for $i \neq j$ and $\delta^{ii} = 1$.

For the i th bank the repayment $p^i \geq 0$ is split between creditors: the j th creditors received the $\Pi^{ij}p^i$ unit.

The problem: find a (column) vector $p \in \mathbb{R}^N$ such that

$$p^i = (e^i + \sum_j \Pi^{ji} p^j) \wedge \tilde{L}^i.$$

2. Eisenberg L., Noe T.H. Systemic risk in financial systems. *Management Science*, 2001.

3. Suzuki T. Valuing corporate debt: the effect of cross- holdings and debts. *J. Oper. Res. Soc. of Japan*, 2002.

Existence of clearing vectors via fixpoint theorems

Consider the mapping $f : [0, \tilde{L}] \rightarrow [0, \tilde{L}]$ with

$$f(p) = (e + \Pi'p) \wedge \tilde{L}.$$

Notations correspond to the pathwise ordering generated by \mathbb{R}_+^N . The problem is to find its fixed points, i.e. solutions of the equation $f(p) = p$. Apparently, f is a continuous mapping of the compact convex set $[0, \tilde{L}]$ into itself. The Brouwer theorem ensures that such a point does exist.

Since $[0, \tilde{L}]$ is a complete lattice and f is a order preserving mapping, one can use the Knaster–Tarski theorem. It is much simpler and provides more information.

Knaster-Tarski fixpoint theorem⁴

A complete lattice is a poset where each subset $A \neq \emptyset$ has the supremum and infimum. By definition, $\sup A$ is an element \bar{x} such that $\bar{x} \geq x$ for all $x \in A$ and if $y \geq x$ for all $x \in A$ then $y \geq \bar{x}$.

Theorem

Let X be a complete lattice and $f : X \mapsto X$ be an order-preserving mapping, $L = \{x : f(x) \leq x\}$, $U = \{x : f(x) \geq x\}$. The set $L \cap U$ of fixed points of f is non-empty and has the smallest and the largest elements which are, respectively, $\underline{x} := \inf L$ and $\bar{x} := \sup U$.

Proof. Note: $\sup X \in L$. Let $x \in L$. Then $\underline{x} \leq x$. By monotonicity $f(\underline{x}) \leq f(x) \leq x$. Thus, $f(\underline{x}) \leq \underline{x} := \inf L$. So, $\underline{x} \in L$. Since $f(L) \subseteq L$, also $f(\underline{x}) \in L$, hence, $\underline{x} \leq f(\underline{x})$, i.e. $\underline{x} = f(\underline{x})$. All fixed points belong to L . Hence, \underline{x} is the smallest one.

The proof of the statement for the largest fixed point is analogous.

4. Tarski A. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.*, 1955.

Monotonicity with respect to a parameter⁵

Remark. Let f_1, f_2 be two order-preserving mappings of a complete lattice (X, \geq) into itself, $f_2 \geq f_1$. Then

$$\begin{aligned}\inf\{x: f_1(x) \leq x\} = \underline{p}_1 &\leq \underline{p}_2 = \inf\{x: f_2(x) \leq x\}, \\ \sup\{x: f_1(x) \geq x\} = \bar{p}_1 &\leq \bar{p}_2 = \sup\{x: f_2(x) \geq x\}.\end{aligned}$$

5. Milgrom J., Roberts J. Comparing equilibria. *Amer. Econ. Rev.*, 1994.

The equity $C(p)$ does not depend on the clearing vector.

Since $(x - y)^+ = x - x \wedge y$,

$$p = (e + \Pi'p) \wedge \tilde{L} \Leftrightarrow C(p) := (e + \Pi'p - \tilde{L})^+ = e + \Pi'p - p.$$

Multiplying from the left by $\mathbf{1}' = (1, \dots, 1)'$ we get that

$$\mathbf{1}'(e + \Pi'p - \tilde{L})^+ = \mathbf{1}'e$$

The total of equities does not depend on the clearing vector.

Since $C(p) \leq C(\bar{p})$, this implies that, $C(p) = C(\bar{p})$.

Graph structure is introduced as in Markov chains.

Let $o(i)$ be *the orbit of i* , i.e. the set of $j \neq i$ for which there is a path $i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow j$, where $i \rightarrow i_1$ means that $\Pi^{ij} > 0$. If $o(i) \neq \emptyset$, it is the set of all direct or indirect creditors of i .

Since $\Pi \mathbf{1}_{o(i)} \geq \mathbf{1}_{o(i)}$, we have

$$\mathbf{1}'_{o(i)} C = \mathbf{1}'_{o(i)} (e + \Pi'p - p) \geq \mathbf{1}'_{o(i)} e.$$

Uniqueness. If $\mathbf{1}'_{o(i)} e > 0$ for all $o(i) \neq \emptyset$, then $\underline{p} = \bar{p}$.

Proof. Note that $\mathbf{1}'_{o(i)} C = \mathbf{1}'_{o(i)} (e + \Pi' p - p) \geq \mathbf{1}'_{o(i)} e > 0$.

Suppose that $\underline{p}^i < \bar{p}^i$, hence, $o(i) \neq \emptyset$ contains a node m with $C^m > 0$ and there is a path $i \rightarrow i_1 \rightarrow \dots \rightarrow m$; assume wlg that m is the 1st node with strictly positive equity value. If $i_1 = m$, then there is an immediate contradiction: since

$$e^m + \sum_j \Pi^{jm} \underline{p}^j - \tilde{L}^m = C^m = e^m + \sum_j \Pi^{jm} \bar{p}^j - \tilde{L}^m,$$

we get the equality $\sum_j \Pi^{jm} (\bar{p}^j - \underline{p}^j) = 0$, impossible because **the i th term of the sum is strictly positive**. If $C^{i_1} = 0$, then

$$e^{i_1} + \sum_j \Pi^{ji_1} \underline{p}^j - \underline{p}^{i_1} = 0 = e^{i_1} + \sum_j \Pi^{ji_1} \bar{p}^j - \bar{p}^{i_1} = 0,$$

implying that $\bar{p}^{i_1} - \underline{p}^{i_1} = \sum_j \Pi^{ji_1} (\bar{p}^j - \underline{p}^j) > 0$.

That is, the strict inequality $\underline{p}^i < \bar{p}^i$ propagates along the path.

Computing the clearing vectors

We know how to solve the linear equation $p = e + \Pi'p$ by the Gauss elimination variable algorithm. To solve the non-linear equation $p = (e + \Pi'p) \wedge \tilde{L}$ we proceed as follows. Let us consider the set of indices $D := \{i: e^i + (\Pi'\tilde{L})^i < \tilde{L}^i\}$. If $D = \emptyset$, then $p = \tilde{L}$ is the solution. Let $D \neq \emptyset$. We can assume wlg that the index $1 \in D$. The first equation is linear :

$$p^1 = e^1 + \sum_j \Pi^{j1} p^j.$$

If $\Pi^{11} \neq 1$ we solve the equation and substitute the expression for p^1 into all other equations thus reducing the problem exactly as in the Gauss algorithm. In the case $\Pi^{11} = 1$ one can take p^1 arbitrarily from $[0, \tilde{L}^1]$; moreover, we obtain that $p^j = 0$ for $\Pi^{j1} > 0$. In this case, again the problem is reduced to the same one but in lower dimension.

Relations with optimal stopping

There is a striking resemblance of the equation

$$p = (e + \Pi' p) \wedge \tilde{L} \quad (1)$$

with the Bellman equation

$$v = (e + \Pi v) \vee g \quad (2)$$

whose minimal solution (the smallest $(1, -e)$ -excessive majorant in the terminology of Shiryaev) is the payoff function

$$v(x) = \sup_{\tau} E^x \left(g(X_{\tau}) + \sum_{s=0}^{\tau-1} e(X_s) \right)$$

in the optimal stopping problem with cost of observations where X is the discrete-time Markov process with transition matrix Π and initial state x . The modification of the Gauss elimination algorithm described above was suggested by Sonin in 2005 in the context of optimal stopping. Another algorithm is due to Presman (2011).

Rogers–Veraart model⁶

An extension of the EN model where the clearing vectors are solutions of the non-linear equation :

$$p = (I - \Lambda)\tilde{L} + \Lambda(\alpha e + \beta \Pi' p) =: f(p),$$

where $\Lambda = \Lambda(p) := \text{diag } D$ with $D := \{i: e^i + (\Pi' p)^i < \tilde{L}^i\}$. The parameters $\alpha, \beta \in]0, 1]$ serves to express the default losses. If the i th bank fails the amount $(1 - \alpha)e^i + (1 - \beta)(\Pi' p)^i$ is used to cover the liquidation expenditures. The EN model corresponds to the case $\alpha = \beta = 1$.

The function $f : [0, \tilde{L}] \rightarrow [0, \tilde{L}]$ is monotone the Knaster–Tarski theorem ensures the existence of the clearing vectors \underline{p} and \bar{p} . The model is used to study effects of merging and rescue consortium.

6. Rogers L.C.G., Veraart L.A.M. Failure and rescue in an interbank network. *Management Science*, 2013.

Greatest Clearing Vector Algorithm

This is a recursively defined sequence $p_0 := \tilde{L}$,

$$p_{n+1} := (I - \Lambda_n)\tilde{L} + \Lambda_n\hat{p}_{n+1}, \quad n \geq 0,$$

where $\Lambda_n := \text{diag } \mathbf{1}_{D_n}$ with $D_n := \{i \leq N: e^i + (\Pi' p_n)^i < \tilde{L}^i\}$, and \hat{p}_{n+1} is the maximal solution in $[0, \Lambda_n p_n]$ of the linear equation

$$p = \Lambda_n(\alpha e + \beta \Pi'(I - \Lambda_n)\tilde{L} + \beta \Pi' \Lambda_n p) =: I_n(p).$$

This sequence is well-defined and decreasing. The proof uses the Knaster–Tarski theorem.

Proposition

There exists $n_0 \leq N + 1$ such that $p_n = \bar{p}$ for all $n \geq n_0$.

But the Gauss elimination algorithm also works...

The Suzuki–Elsinger model with crossholdings⁷

Crossholdings are given by a substochastic matrix $\Theta = (\theta^{ij})$ where $\theta^{ij} \in [0, 1]$ is a share of the bank i held by the bank j . Assume that

H. There is no subset $A \subseteq \{1, \dots, N\}$ such that $\mathbf{1}'_A \Theta = \mathbf{1}'_A$.

Equivalently : 1 is not an eigenvalue of Θ .

The problem is to find the set of solutions $\Gamma_1 \subseteq [0, \tilde{L}] \times \mathbf{R}_+^N$ of

$$\begin{aligned} p &= (e + \Pi' p + \Theta' V)^+ \wedge \tilde{L}, \\ V &= (e + \Pi' p - p + \Theta' V)^+. \end{aligned}$$

For $(p, V) \in \Gamma_1$ the components p and V are called, respectively, *clearing vector* and *equity*.

No monotonicity, but Brouwer is OK. What is the equity?

7. Elsinger H. Financial networks, cross holdings, and limited liability. Working paper from Oesterreichische Nationalbank, 2009.

We introduce the systems

$$\begin{aligned} p &= (e + \Pi' p + \Theta' U)^+ \wedge \tilde{L}, \\ U &= (e + \Pi' p - \tilde{L} + \Theta' U)^+. \end{aligned}$$

with the set of solutions $\Gamma_2 \subseteq [0, \tilde{L}] \times \mathbf{R}_+^N$ and the system

$$\begin{aligned} p &= (e + \Pi' p + \Theta' W^+)^+ \wedge \tilde{L}, \\ W &= e + \Pi' p - \tilde{L} + \Theta' W^+. \end{aligned}$$

with the set of solutions $\Gamma_3 \subseteq [0, \tilde{L}] \times \mathbf{R}^N$.

Lemma

$\Gamma_1 = \Gamma_2 = \varphi(\Gamma_3)$ where $\varphi(x, y) := (x, y^+)$.

Lemma

For any $x \in \mathbf{R}^N$ the equations $v = (x + \Theta' v)^+$, and $w = x + \Theta' w^+$, have unique solutions $v = v(x) \in \mathbf{R}_+^N$ and $w = w(x) \in \mathbf{R}^N$.

The mappings $x \mapsto v(x)$ and $x \mapsto w(x)$ are monotone, positive homogeneous, convex, and Lipschitz.

Theorem

Suppose that for any subset of indices A such that for all $i \in A$

$$\sum_{j \in A} \Theta^{ij} = 1 \quad \text{or} \quad \sum_{j \in A} \Pi^{ij} = 1$$

it holds that

$$\sum_{i \in A} e^i > \sum_{i \in A} \left(1 - \sum_{j \in A} \Pi^{ij}\right) \tilde{L}^i.$$

Then the clearing vector is unique. In particular, for the Eisenberg–Noe model where $\Theta = 0$, if any subset of indices A such that $\sum_{j \in A} \Pi^{ij} = 1$ for all $i \in A$ we have that $\sum_{i \in A} e^i > 0$, then the clearing vector is unique.

The Elsinger model with debts of different seniority

The debt structure is defined matrices $L_1 = (L_1^{ij}), \dots, L_M = (L_M^{ij})$ representing liabilities with decreasing seniority.

The relative liabilities for the seniority S are defined by the matrix

$$\Pi_S^{ij} = L_S^{ij} / \tilde{L}_S^i, \quad \text{if } \tilde{L}_S^i \neq 0, \text{ and } \Pi_S^{ij} = \delta^{ij} \text{ otherwise.}$$

The clearing requires full reimbursement of debts starting from the highest priority and, for each seniority, the distribution is proportional to the volume of debts of this seniority. For the bank i we denote by p_S^i the value distributed to cover debts of seniority S . The clearing is described by vectors $p_S^i, S \leq M$, which can be considered as a “long” vector in $(\mathbb{R}^N)^M$ such that

$$p_1^i = \min \left\{ e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j, \tilde{L}_1^i \right\},$$

$$p_S^i = \min \left\{ \left(e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j - \sum_{r < S} \tilde{L}_r^i \right)^+, \tilde{L}_S^i \right\}, \quad 1 < S \leq M.$$

Existence of fixed points

In a vector form these equations can be written as follows :

$$p_S = \left(e + \sum_S \Pi'_S p_S - \sum_{r < S} \tilde{L}_r \right)^+ \wedge \tilde{L}_S, \quad S = 1, \dots, M.$$

For the componentwise partial ordering in $(\mathbb{R}^N)^M$ the function

$$(p_1, \dots, p_M) \mapsto \left(\left(e + \sum_S \Pi'_S p_S^* \right)^+ \wedge \tilde{L}_1, \dots, \left(e + \sum_S \Pi'_S p_S^* - \sum_{r < M} \tilde{L}_r \right)^+ \wedge L_M \right)$$

is a monotone mapping of the order interval $[0, \tilde{L}_1] \times \dots \times [0, \tilde{L}_M] \subset (\mathbb{R}^N)^M$ into itself. By the Knaster–Tarski theorem the set of fixed points of this mapping, i.e. the solutions of the above equation, is non-empty and has the maximal and the minimal elements.

For maximal clearing vector \bar{p} we define the *default index* d^i of the node i as the smallest r such that

$$\bar{p}_r^i = e^i + \sum_S \sum_j \Pi_S^{ji} \bar{p}_S^j - \sum_{r' < r} \tilde{L}_{r'}^i.$$

That is, d^i is the lowest seniority for which the i th bank equity after clearing is equal to zero. Define the matrix $\Delta = \Delta(p)$ by putting $\Delta^{ij} = 1$ if $\Pi_{d(i)}^{ij} > 0$, and $\Delta^{ij} = 0$ otherwise. We use the notation $i \rightsquigarrow j$ if $\Delta^{ij} = 1$ and denote by $O(i)$ the Δ -orbit of i , that is the set of all j for which there is a directed path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \dots \rightsquigarrow j$.

Theorem

Suppose that for the clearing vector \bar{p} any Δ -orbit is a surplus set. Then the clearing vector is unique.

Proof. Recall that the default index

$$d^i := \min \left\{ r : \bar{p}_r^i = e^i + \sum_S \sum_j \Pi_S^{ji} \bar{p}_S^j - \sum_{r' < r} \tilde{L}_{r'}^i \right\}.$$

It follows that $\bar{p}_r^i = 0$, hence, $\underline{p}_r^i = 0$ for every $r > d^i$. Suppose that $\underline{p}_r^i < \bar{p}_r^i$ and consider a path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \dots \rightsquigarrow m$ ending up at the node with strictly positive equity value.

First, we show that at least for one seniority $\underline{p}_S^{i_1} < \bar{p}_S^{i_1}$.

Let $r' := d^{i_1}$. By definition, $\bar{p}_r^{i_1} = \tilde{L}_r^{i_1}$, $r \leq r'$, and $\bar{p}_r^{i_1} = \underline{p}_r^{i_1} = 0$, $r > r'$. The claim holds, if $\underline{p}_r^{i_1} < \tilde{L}_r^{i_1}$ for some $r < r'$. Consider the case where $\underline{p}_r^{i_1} = \bar{p}_r^{i_1} = \tilde{L}_r^{i_1}$ for all $r < r'$ and prove that $\underline{p}_{r'}^{i_1} < \bar{p}_{r'}^{i_1}$.

Either $\underline{p}_{r'}^{i_1} < \bar{p}_{r'}^{i_1} \leq \tilde{L}_{r'}^{i_1}$ (what we need), or $\underline{p}_{r'}^{i_1} = \bar{p}_{r'}^{i_1} \leq \tilde{L}_{r'}^{i_1}$. The 2nd case is impossible, since the equalities

$$\begin{aligned} \bar{p}_{r'}^{i_1} &= e^{i_1} + \sum_S \sum_j \Pi_S^{j i_1} \bar{p}_S^j - \sum_{r < r'} \tilde{L}_r^{i_1}, \\ \underline{p}_{r'}^{i_1} &= e^{i_1} + \sum_S \sum_j \Pi_S^{j i_1} \underline{p}_S^j - \sum_{r < r'} \tilde{L}_r^{i_1}. \end{aligned}$$

leads to a contradiction

$$\bar{p}_{r'}^{i_1} - \underline{p}_{r'}^{i_1} = \sum_S \sum_j \Pi_S^{j i_1} (\bar{p}_S^j - \underline{p}_S^j) \geq \Pi_{r'}^{i_1} (\bar{p}_r^{i_1} - \underline{p}_r^{i_1}) > 0.$$

The Fisher model: clearing with derivatives⁸

It is a generalization of the Elsinger–Suzuki model covering systems where banks, besides of straight debts, may have liabilities in terms of derivatives having different seniorities. This means that matrices L_S may depend on the clearing vectors. The equations are:

$$p_S = \left(e + \Theta' V + \sum_{r \leq M} \Pi'_r p_r - \sum_{r < S} \tilde{L}_r(p) \right)^+ \wedge \tilde{L}_S(p), \quad S = 1, \dots, M,$$
$$V = \left(e + \Theta' V + \sum_{r \leq M} \Pi'_r p_r - \sum_S p_S \right)^+.$$

Now the matrices Π_S become input parameters of the model.

Theorem

Suppose that the functions $p \mapsto L_S(p)$ are bounded and continuous, $|\Theta| < 1$. Then the system has a solution.

8. Fischer T. No arbitrage pricing under systemic risk : accounting for cross-ownership. *Math. Finance*, 2014.

Theorem

Suppose that $e \geq 0$, the functions $p \mapsto L_S(p)$ are continuous, and $|\Theta| < 1$, $|\Pi_S| < 1$ for all S . Then the system has a solution.

Theorem

In addition to the assumptions of preceding theorem suppose that

$$\tilde{L}_r^i(p) = \psi_r^i \left(\sum_{r \leq M+1} (\Pi_r' p_r)^i \right)$$

where $\psi_r^i : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are increasing functions such that for any $u, v \in \mathbf{R}_+$ such that $v \geq u$ we have the bound

$$v - u \geq \sum_{r \leq M} (\psi_r^i(v) - \psi_r^i(u)), \quad i = 1, \dots, N.$$

Then the system has a unique solution.

Models with illiquid assets and a price impact⁹

The bank i owns cash e^i and K illiquid assets, in quantities y^{i1}, \dots, y^{iK} represented in the model by the row i of the matrix $Y = (y^{im})$, $i \leq N$, $m \leq K$. The nominal prices per unit are $Q^1, \dots, Q^K > 0$. The clearing may require sales. If $u^{im} \in [0, y^{im}]$ units of the m -th assets for the price q_m are sold, the increase in cash is $(Uq)^i = \sum_{m=1}^K u^{im} q^m$.

The price formation is modeled by the inverse demand function $F_0 : \mathbb{R}^K \rightarrow \mathbb{R}^K$, continuous and monotone decreasing ($F_0(z) \leq F_0(x)$ when $z \geq x$ in the sense of partial ordering in \mathbb{R}_+^K) and such that $F_0(0) = Q$ and $F_0^m(Y' \mathbf{1}) > 0$ for $m = 1, \dots, K$.

The clearing rules : each bank pays debts in accordance to the matrix of liabilities and sells illiquid assets if it is needed. All debts should be covered or bank's equity falls down to zero.

9. El Bitar K., Kabanov Yu., Mokbel R. On uniqueness of clearing vectors reducing the systemic risk. Informatics and Applications, 11 (2017), 1, 109-118.

Equilibrium

The important question is what are strategies for the banks? We suppose that all assets are sold in equal proportions. More precisely, the i th bank sells u^{im} units of the m th asset where

$$u^{im} := u^{im}(p, q) := \frac{y^{im} \left(\tilde{L}^i - e^i - \sum_j \Pi^{ij} p^j \right)^+}{\sum_k y^{ik} q^k} \wedge y^{im}.$$

The total supply of the illiquid assets is the vector $U'(p, q)\mathbf{1}$ where $U(p, q) = (u^{im})$.

Define the equilibrium vector $(p^*, q^*) \in [0, \tilde{L}] \times [F_0(Y'\mathbf{1}), Q]$ as the solution of the system of $N + K$ equations

$$p = (e + U(p, q)q + \Pi'p) \wedge \tilde{L} =: F(p, q), \quad (3)$$

$$q = F_0(U'(p, q)\mathbf{1}). \quad (4)$$

The existence follows because $(p, q) \mapsto (F(p, q), F_0(U'(p, q)\mathbf{1}))$ is a monotone mapping of the interval $[0, \tilde{L}] \times [F_0(Y'\mathbf{1}), Q]$ into itself.

The set of its fixed points contains the minimal and maximal elements $(\underline{p}^*, \underline{q}^*)$ and (\bar{p}^*, \bar{q}^*) .

For a fixed q the function $p \rightarrow F(p, q)$ is monotone and the set of solutions (3) contains the maximal element $\bar{p}(q)$.

For a fixed $q \in [F_0(Y'\mathbf{1}), Q]$ the largest solution $\bar{p} = \bar{p}(q)$ of (3) is

$$\bar{p} = \sup\{p \in [0, \tilde{L}] : p \leq (e + U(p, q)q + \Pi'p) \wedge \tilde{L}\}$$

implying that $q \mapsto \bar{p}(q)$ is an increasing (and continuous) function on $[F_0(Y), Q]$. It follows that the supply function

$$q \mapsto \zeta(q) := U'(\bar{p}(q), q)\mathbf{1}$$

is decreasing and, therefore, the $q \mapsto F_0(\zeta(q))$ is an increasing (and continuous) mapping of the interval $[F_0(Y'\mathbf{1}), Q]$ into itself and, therefore, it has the minimal and maximal fixed points q_1 and q_2 .

Lemma

If the function $x \rightarrow x'F_0(x)$ is strictly increasing on $[F_0(Y'\mathbf{1}), Q]$, then the solution of $q = F_0(\zeta(q))$ is unique, i.e. $q_1 = q_2$.

Theorem

Suppose that the scalar function $x \rightarrow x'F_0(x)$ is strictly increasing on $[F_0(Y'\mathbf{1}), Q]$. Then there is q^ such that the set of solutions of the system (3), (4) is contained in the interval with the extremities $(\underline{p}(q^*), q^*)$ and $(\bar{p}(q^*), q^*)$. In particular, if for each q the solution of (3) is a unique, then the solution of the system is also unique.*

Proof. Let Γ be the set of q for which (p, q) is a solution of the system (3), (4). If $q^* \in \Gamma$, then $(\bar{p}(q^*), q^*)$ is the solution of (3), (4). Accordingly to the above lemma the point q^* is uniquely defined. This implies the result.

The Feinstein model, 1

The row $u_i = (u^{i1}, \dots, u^{iK})$ of the strategy matrix $U = U(p, q)$ is composed by functions $u^{im} = u^{im}(p, q)$ on $[0, l] \times [F_0(Y'\mathbf{1}), Q]$ with values in $[0, y^{im}]$ such that $Uq = (\tilde{L} - e - \Pi'p)^+$, i.e. each bank sells the illiquid assets to cover, without excess, the shortfall.

For a strategy U the equilibrium $(p^*, q^*) \in [0, l] \times [F_0(Y'\mathbf{1}), Q]$ is defined as the solution of the system

$$\begin{aligned} p &= (e + Yq + \Pi'p) \wedge \tilde{L} =: F(p, q), \\ q &= F_0(U'(p, q)\mathbf{1}). \end{aligned}$$

It exists if $(p, q) \mapsto U(p, q)$ is continuous (Brouwer).

For given U and i define the maximization problem

$$\Phi^i(v, U) \rightarrow \max,$$

$$v \in \Gamma^i(p, q) := \{v \in \mathbb{R}^K : v'q = (y_i q) \wedge (\tilde{L}^i - e^i - (\Pi'p)^i)^+, v^m \in [0, y^{im}]\}.$$

Functions Φ^i are continuous and do not depend on the row u_i of U , e.g., Φ^i depends only on the vector $U'\mathbf{1} - u_i'$. Let $G^i(p, q, U)$ be the set of solutions of the above problem and by $V^i(p, q, U)$ its optimal value.

The Feinstein model, 2

Define the set-valued mapping Ψ by putting

$$\Psi(p, q, U) := \{(e + Yq + \Pi'p) \wedge \tilde{L}\} \times \{F_0(U'\mathbf{1})\} \times \prod_{i \leq N} G^i(p, q, U).$$

Theorem

Suppose that for each U the functions $v \mapsto \Phi^i(v, U)$ are quasiconcave on $[0, Y'\mathbf{1}]$. Then there exists the triple $(p_, q_*, U_*) \in \Psi(p_*, q_*, U_*)$.*

Proof. The function $v \mapsto \Phi^i(v, U)$ is continuous and $\Gamma^i(p, q) \neq \emptyset$ is compact. Thus, the set $G^i(p, q, U)$ is nonempty. It is closed and convex as an intersection of all nonempty convex closed subsets $\{v \in \Gamma^i(p, q) : \Phi^i(v, U) \geq a\}$ over all $a < V^i(p, q, U)$. Hence, Ψ has nonempty compact convex values and the existence of the equilibrium (p_*, q_*, U_*) follows from the Kakutani fixpoint theorem. It remains to verify that the graph of G^i is closed (hence, Ψ is closed).

The Feinstein model, 3

Let $(p_n, q_n, U_n, v_n) \rightarrow (\tilde{p}, \tilde{q}, \tilde{U}, \tilde{v})$ where $v_n \in G^i(p_n, q_n, U_n)$. To prove that $\Phi^i(\tilde{v}, \tilde{U}) = V^i(\tilde{p}, \tilde{q}, \tilde{U})$, take $\varepsilon > 0$. Since a function on a compact is uniformly continuous, for all large n , uniformly in v with $v^m \in [0, y^{im}]$

$$\Phi^i(v, U_n) - \varepsilon \leq \Phi^i(v, \tilde{U}) \leq \Phi^i(v, U_n) + \varepsilon. \quad (5)$$

In particular, for $v = v_n$

$$V^i(p_n, q_n, U_n) - \varepsilon \leq \Phi^i(v_n, \tilde{U}) \leq V^i(p_n, q_n, U_n) + \varepsilon. \quad (6)$$

Taking in (5) supremum over v in $\Gamma^i(p, q)$ we get that

$$V^i(p_n, q_n, U_n) - \varepsilon \leq \sup_{v \in \Gamma^i(p_n, q_n)} \Phi^i(v, \tilde{U}) \leq V^i(p_n, q_n, U_n) + \varepsilon.$$

Since q^i are bounded away from zero, $\Gamma^i(p_n, q_n) \rightarrow \Gamma^i(\tilde{p}, \tilde{q})$ in the Hausdorff metric. It follows that for large n

$$V^i(p_n, q_n, U_n) - 2\varepsilon \leq \sup_{v \in \Gamma^i(\tilde{p}, \tilde{q})} \Phi^i(v, \tilde{U}) \leq V^i(p_n, q_n, U_n) + 2\varepsilon. \quad (7)$$

Taking in (6) and (7) liminf, we get that $|\Phi^i(\tilde{v}, \tilde{U}) - V^i(\tilde{p}, \tilde{q}, \tilde{U})| \leq 3\varepsilon$ implying the required property.

Remark

In the Feinstein paper¹⁰ the goal functionals are

$$\Phi^i(v, U) := y_i F_0(U' \mathbf{1} - u'_i + v).$$

This means that the bank i maximizes the total value of its available illiquid assets calculated in the prices q using the clearing vector p and knowing the total sell of each asset by other banks. A more natural choice of the goal functional could be

$$\Phi^i(v, U) := (y_i - v') F_0(U' \mathbf{1} - u'_i + v).$$

10. Feinstein Z. (2017) Financial contagion and asset liquidation strategies. *Oper. Res. Lett.*, 45, 2, 10–114.