Ordinal status games on networks

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A technical viewpoint

This paper investigates a class of strategic games with discontinuous utilities.

From a purely technical viewpoint, those games are interesting because no general conditions for equilibrium existence, see, e.g., Reny (1999, 2016), McLennan et al. (2011), Prokopovych (2013), or Kukushkin (2018), are applicable to them, even though they display features such as aggregation, monotonicity, and strict quasiconcavity.

An economic theory viewpoint

Those games are intended to model how people's decisions are influenced by concerns for relative social status.

This topic has been present in the literature since, at least, Veblen (1899), and has attracted ever growing attention (Frank, 1985; Akerlof, 1997; Clark and Oswald, 1998; Becker et al., 2005; Arrow and Dasgupta, 2009).

Ordinal and cardinal approaches (Bilancini and Boncinelli, 2008): in the first case, the status of a player is determined by the *comparisons* with other players' choices; in the second case, by the *differences* between them.

"HvM status games"

The starting point for this paper is
the model of Haagsma and von Mouche (2010).

Principal features:
finite set of players; ordinal approach ("status"="order rank").

The discrete nature of status in such games generates
unpleasant discontinuities of the utility functions.

The weakest point:
no general result on the existence of Nash equilibrium.

The only exception is the two-person case,
where it is geometrically obvious that
the graphs of the best responses must intersect.

Binary status games

Kukushkin and von Mouche (2018):
the existence of Nash equilibrium and
convergence of (consecutive) Cournot tâtonnement.

The number of players may be arbitrary,
but there are only two status levels:

a player belongs to the top tier
if her choice is the maximal of all,
and belongs to the bottom tier otherwise.

Kukushkin (2019)

A review of Kukushkin (2019) is given further on.

The objective of the present paper is to study
the implications of a *network* of players,

i.e., an assumption that some players
do not care at all about the choices of some others.

Strategic game

A strategic game:

a finite set of players N (we assume $n := \#N \geq 2$);

strategy sets X_i;

utility functions $u_i \colon X_N \to \mathbb{R}$, where $X_N := \prod_{i \in N} X_i$.

Status game: strategy sets

$X_i = [a_i, b_i] \cap X \subset \mathbb{R}$,
where X is a closed subset of $\mathbb{R}$ ("conceivable strategies");

Status game: utility function (aggregation)

There are

a finite chain S (of potential "status levels"),

a function $U_i\colon X_i \times S \to \mathbb{R}$ and

a mapping $\sigma_i\colon X_N \to S$ for each $i \in N$,

such that $u_i(x_N) = U_i(x_i, \sigma_i(x_N))$
for all $i \in N$ and $x_N \in X_N$.

Definitions of each player's status

Given $x_N \in X_N$, the *order rank* of player $i \in N$ is
$\rho_i(x_N) := \#\{j \in N \mid x_i \geq x_j\}$.

"HvM status" $\sigma_i(x_N) := \rho_i(x_N)$.

"Kukushkin (2019) status" $\sigma_i(x_N) := q \circ \rho_i(x_N)$
where $q \colon \{1, \ldots, n\} \to S$ is increasing
(not necessarily strictly increasing).

Assumptions on utility functions

Each $U_i(x_i, s)$ $(i \in N)$ is:

- strictly increasing in s;
- upper semicontinuous in x_i;
- single-peaked in x_i,
  i.e., strictly increases in x_i when $x_i \leq \hat{x}_i^s$
  and strictly decreases in x_i when $x_i \geq \hat{x}_i^s$.

The best responses

$\mathcal{R}_i(x_{-i}) := \mathrm{Argmax}_{x_i \in X_i}\, u_i(x_i, x_{-i})$.
X_i is compact; U_i, and hence u_i too, is upper semicontinuous.

A strategy profile $x^0_N \in X_N$ is a *Nash equilibrium*
iff $x^0_i \in \mathcal{R}_i(x^0_{-i})$ for all $i \in N$.

Cournot tâtonnement

A *Cournot path*: $\langle x_N^k \rangle_{k=0,1,\ldots}$ such that, whenever x_N^{k+1} is defined, there is an $i \in N$ for which $x_{-i}^k = x_{-i}^{k+1}$ and $x_i^k \notin \mathcal{R}_i(x_{-i}^k) \ni x_i^{k+1}$.

The *finite best response improvement property* (*FBRP*): there is no infinite Cournot path (Milchtaich, 1996).

Then every Cournot path, if extended whenever possible, ends at a Nash equilibrium.

"keeping up with the Joneses"

Let $x_N \in X_N$, $i \in N$, $x_i \in \mathcal{R}_i(x_{-i})$ and $s = \sigma_i(x_N)$.
Then either $x_i = \hat{x}_i^s$, or $x_i = x_j > \hat{x}_i^s$ with $j \neq i$.

FBRP $\iff$ no Cournot cycles

Strong equilibrium

Given $x_N \in X_N$ and $\emptyset \neq I \subseteq N$,

$y_I \in X_I$ $[:= \prod_{i \in I} X_i]$ is

a *weak coalitional improvement* at x_N iff

- $u_i(y_I, x_{-I}) \geq u_i(x_N)$ for all $i \in I$;
- $u_i(y_I, x_{-I}) > u_i(x_N)$ for at least one $i \in I$.

$x_N^0 \in X_N$ is a *very strong equilibrium* iff

there is no weak coalitional improvement at x_N^0.

Kukushkin (2019): Main results

Theorem 1.
If $\#S = 2$, then there exists a very strong equilibrium, which weakly Pareto dominates all other Nash equilibria.

Proposition 2.
If $\#S = 2$, then FBRP

(Cournot tâtonnement finds a Nash equilibrium in a finite number of steps).

Theorem 3.
If $S = \{1, 2, 3\}$ and $\hat{x}_i^1 \geq \hat{x}_i^2$ for all $i \in N$, then there exists a Nash equilibrium.

Simultaneous Cournot tâtonnement

Let $N = \{1, 2\}$, $X_i = \{0, 1\}$ for both $i \in N$, and

$$u_i(x_i, x_j) = \begin{cases} 3 - x_i, & \text{if } x_i \geq x_j, \\ 1 - x_i, & \text{otherwise.} \end{cases}$$

Two Nash equilibria: $(0, 0)$ and $(1, 1)$.
The former is a strong equilibrium; the latter is not.

Simultaneous Cournot tâtonnement path starting from $x_N^0 := (1, 0)$ never reaches either equilibrium: $x_N^{2k} = (1, 0)$ and $x_N^{2k+1} = (0, 1)$ for all $k \in \mathbb{N}$.

Kukushkin (2019): "Counterexamples"

- Under assumptions of Theorem 3, there may be no strong equilibrium.
- Under assumptions of Theorem 3, the set of equilibrium utility profiles may have a non-trivial Pareto border.
- If $\#S = 3$, there may be no Nash equilibrium without the additional assumption in Theorem 3.
- If $\#S > 3$, there may be no Nash equilibrium, even when no $\hat{x}_i^s$ depends on s.

Players' network: The simplest approach

$\emptyset \neq G(i) \subseteq N$ for each $i \in N$ such that:

- $i \in G(i)$;
- $i \in G(j) \iff j \in G(i)$.

$g_i := \#G(i)$ $[\leq n]$.

$\rho_i(x_N) := \#\{j \in G(i) \mid x_i \geq x_j\}$.

$\sigma_i(x_N) := q_i \circ \rho_i(x_N)$, where
$q_i \colon \{1, \ldots, g_i\} \to S$ is increasing
(not necessarily strictly increasing).

Kukushkin (2020?): Main results

Theorem 1.
If $\#S = 2$, then there exists a very strong equilibrium.

Theorem 2.
If $\#S = 2$, then FBRP

(Cournot tâtonnement finds a Nash equilibrium
in a finite number of steps).

Kukushkin (2020?): "Counterexamples"

- Even when $\#S = 2$,
  the set of equilibrium utility profiles
  may have a non-trivial Pareto border.
- If $\#S > 2$, there may be no Nash equilibrium,
  even under the additional assumption
  from Theorem 3 of Kukushkin (2019).

An example of non-existence

Let $N = \{1, 2, 3, 4\}$; $X_i = [0, 2]$ for each $i \in N$;
$G(1) = \{1, 2\}$, $G(2) = \{1, 2, 3\}$, $G(3) = \{2, 3, 4\}$, $G(4) = \{3, 4\}$;
$S = \{0, 1, 2\}$ (with the natural order);
$q_1(1) = q_2(1) = q_3(1) = q_4(1) = 0$, $q_2(2) = q_3(2) = 1$,
and $q_1(2) = q_2(3) = q_3(3) = q_4(2) = 2$;
$U_1(x, s) = x + s$, $U_2(x, s) = -2x + 3s$,
$U_3(x, s) = -2x^2 + s^2$, and $U_4(x, s) = \min\{x, 2 - x\} + s$.

Players' network: A more sophisticated approach

There is a finite set C of "communities" (or reference groups); there are correspondences $\Phi\colon N \to C$ and $\Psi\colon C \to N$ such that $c \in \Phi(i) \iff i \in \Psi(c)$.

$\rho_i^c(x_N) := \#\{j \in \Psi(c) \mid x_i \geq x_j\}$ $(i \in \Psi(c))$.

$\sigma_i^c(x_N) := q^c \circ \rho_i^c(x_N)$,
with increasing $q^c\colon \{1, \ldots, \#\Psi(c)\} \to S^c$.

$\sigma_i(x_N) := q_i(\langle \sigma_i^c(x_N) \rangle_{c \in \Phi(i)})$,
with increasing $q_i\colon S^{\Phi(i)} \to S$.

Connection to the graph model

Given G, let $C := \{\{i, j\} \in N^2 \mid j \in G(i) \setminus \{i\}\}$,

$\Phi(i) := \{\{i, j\}\}_{j \in G(i) \setminus \{i\}}$, $\Psi(\{i, j\}) := \{i, j\}$

$\sigma_i^c(x_N) := \rho_i^c(x_N) - 1$ $(\forall i \in N\ \forall c \in \Phi(i))$.

Then

$$r_i(x_N) = \sum_{c \in \Phi(i)} \sigma_i^c(x_N).$$

"Results"

Plausible Conjecture.

If $\#S = 2 = \#S^c$ for all $c \in C$, then

there exists a very strong equilibrium.

Open Problem.

What additional assumptions are needed for the FBRP?

Theorem 1 (Strong equilibrium): Sketch of a proof

Let $S = \{\bot, \top\}$ $(\bot < \top)$.

$r_i(x_{-i}) := \min \mathcal{R}_i(x_{-i})$.

Then we recursively construct a sequence $\langle x_N^k \rangle_{k=0,1,\dots}$, starting with $x_i^0 := \hat{x}_i^\top$ for each $i \in N$.

Proof of Theorem 1: Recursion step

Having $x_N^k \in X_N$ already defined, we set
$I_k^{\uparrow} := \{i \in N \mid r_i(x_{-i}^k) > x_i^k\}$; $I_k^{\downarrow} := \{i \in N \mid r_i(x_{-i}^k) < x_i^k\}$;
$I_k^{=} := \{i \in N \mid r_i(x_{-i}^k) = x_i^k\}$.

If $N = I_k^{=}$, then STOP (x_N^k is, at least, a Nash equilibrium).

Otherwise, we define $h_i^k := r_i(x_{-i}^k)$ for $i \in I_k^{\uparrow}$, $h_i^k := x_i^k$ for $i \in I_k^{\downarrow}$, $h_i^k := -\infty$ for $i \in I_k^{=}$, and $H^k := \max_i h_i^k$.

If $I_k^{\downarrow} \cap \operatorname{Argmax}_i h_i^k \neq \emptyset$, then we pick $\mathfrak{i}(k) \in I_k^{\downarrow} \cap \operatorname{Argmax}_i h_i^k$; otherwise, we pick $\mathfrak{i}(k) \in \operatorname{Argmax}_i h_i^k \subseteq I_k^{\uparrow}$.

Finally, $x_{\mathfrak{i}(k)}^{k+1} := r_{\mathfrak{i}(k)}(x_{-\mathfrak{i}(k)}^k)$ and $x_i^{k+1} := x_i^k$ for $i \neq \mathfrak{i}(k)$.

Proof of Theorem 1: Recursion process

Claim 1. $H^{k+1} \leq H^k$ whenever H^{k+1} is defined.

Claim 2. If $I_k^{\downarrow} \cap \text{Argmax}_i\, t_i^k = \emptyset$ and $H^{k+1} = H^k$, then $I_{k+1}^{\downarrow} \cap \text{Argmax}_i\, t_i^{k+1} = \emptyset$ too.

Claim 3. Let $i \in N$ and $i = \mathfrak{i}(k)$.
Then $x_i^{k+1} = r_i(x_{-i}^m) = x_i^m$ and $u_i(x_N^{k+1}) = u_i(x_N^m)$ for all $m > k$.

Corollary. The process stops at $K \leq n$.

Claim 4. If $u_i(x_N) > u_i(x_N^K)$ for some $i \in N$ and $x_N \in X_N$, then there is $j \in N$ such that $x_j \neq x_j^K$ and $u_j(x_N) < u_j(x_N^K)$.

Theorem 2 (FBRP): Sketch of a proof

Again, $S = \{\bot, \top\}$ $(\bot < \top)$.

Suppose, to the contrary, $\langle x_N^k \rangle_{k=0,1,\ldots,K}$ to be a Cournot cycle.

$\mathcal{K} := \{0, 1, \ldots, K\}$ and $\mathcal{K}(i) := \{k \in \mathcal{K} \mid x_{-i}^k = x_{-i}^{k+1}\}$ for $i \in N$.

$N^* := \{i \in N \mid \mathcal{K}(i) \neq \emptyset\}$, participants in the cycle;

$N^= := N \setminus N^* = \{i \in N \mid \forall k, h \in \mathcal{K}\, [x_i^k = x_i^h]\}$, everybody else.

$G(i) \cap (N^* \setminus \{i\}) \neq \emptyset$ whenever $i \in N^*$.

For each $k \in \mathcal{K}$, $\quad m_k := \max_{i \in N^*} \min\{y_i \in X \mid \sigma_i(y_i, x_{-i}^k) = \top\}$;

$M^+ := \max_{k \in \mathcal{K}} m_k$; $I^+ := \{i \in N^* \mid \exists k \in \mathcal{K}\, [x_i^k = M^+]\}$.

Proof of Theorem 2: Cournot cycle

Claim 1. $m_k = M^+$ for all $k \in \mathcal{K}$.

Claim 2. If $x_i^k > M^+$ for some $i \in N$ and $k \in \mathcal{K}$, then $i \in N^=$.

Claim 3. $I^+ \neq \emptyset$.

For each $i \in I^+$ and $k \in \mathcal{K}$: $\beta_i := \min q_i^{-1}(\top)$;
$\nu_i := \#\{j \in G(i) \cap N^= \mid \forall k \in \mathcal{K}\, [x_j^k \geq M^+]\}$;
$\chi_i^k := 0$ if $x_i^k < M^+$; $\chi_i^k := 1$ if $x_i^k = M^+$; $s_i^k := \sum_{j \in G(i) \cap I^+} \chi_j^k$.

Claim 4. Let $i \in I^+$ and $k \in \mathcal{K}(i)$.
If $x_i^{k+1} = M^+$, then $g_i - \nu_i - s_i^k < \beta_i$.
If $x_i^k = M^+$, then $g_i - \nu_i - s_i^k \geq \beta_i$.

Proof of Theorem 2: Finale, apotheosis

Finally, we define a function $H\colon \mathcal{K} \to \mathbb{R}$ by

$H(k) := \frac{1}{2} \sum_{i \in I^+} \sum_{j \in G^*(i) \cap I^+} \chi_i^k \cdot \chi_j^k +$
$\sum_{i \in I^+} \chi_i^k \cdot (\nu_i + \beta_i - g_i - 1/2)$.

Claim 5. For each $k \in \mathcal{K}$, there holds $H(k+1) \geq H(k)$.
If $k \in \mathcal{K}(i)$ with $i \in I^+$, and $x_i^k = M^+$ or $x_i^{k+1} = M^+$, then $H(k+1) > H(k)$.

Since $I^+ \neq \emptyset$ by Claim 3, we have a final contradiction.

Thanks to everybody

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