Monotone comparative statics on semilattices

Nikolai S. Kukushkin
Former Dorodnicyn Computing Centre
ququns@inbox.ru

CEMI Seminar
2021 March 23

Abstract

This paper strives to find out to what extent monotone comparative statics analysis in the style of Milgrom and Shannon (1994) can be extended from lattices to *semilattices* (e.g., budget sets).

The results obtained are somewhat ambivalent.
On the one hand, a condition on preference orderings is formulated which can claim a role similar to that of quasisupermodularity on lattices.

On the other hand, it remains unclear whether preferences satisfying that condition can be found in interesting and important models.

Lattice Programming

Arthur F. Veinott, Jr. (1934–2012).

Stanford University, Department of Operations Research

Delivered lectures, was writing a book, but never finished it.

Monotone comparative statics

"… nontrivial conclusion about the direction of change
of the endogenous choice variables
in response to changes in exogenous parameters."

Paul R. Milgrom (b. 1948)
Shirley and Leonard Ely Professor of Humanities and Sciences
at Stanford University

Key contributions

Milgrom and Shannon (Econometrica, 1994) vs

LiCalzi and Veinott (1992, unpublished, available on RePEc)

The central result of Milgrom and Shannon (1994), Theorem 4, established conditions for a monotone response of the optimal choice from a lattice to a perturbation of both the preferences and the feasible set.

John Quah's taxonomy

Quah (Econometrica, 2007)

Response of the optimal choice
to a perturbation of the feasible set:
"**type B**" problem.

LiCalzi and Veinott (1992), an exhaustive analysis.

Milgrom and Shannon (1994, Corollary 1).

Response of the optimal choice
to a perturbation of the preferences:
"**type A**" problem.

Kukushkin (Economic Theory, 2013), an exhaustive analysis.

Basic notions

Preferences

An agent has preferences over alternatives from a poset A.

Those preferences are described by an *ordering* $\succ$,
i.e., an irreflexive, transitive and *negatively transitive*,
$z \not\succ y \not\succ x \Rightarrow z \not\succ x$, binary relation on A.

Then the "non-strict preference" relation $\succeq$
defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total.

Utility functions

A *utility function* $u\colon A \to \mathbb{R}$ defines an ordering:

$$y \succ x \rightleftharpoons u(y) > u(x);$$

to make the converse statement true,
one would have to allow an arbitrary chain instead of $\mathbb{R}$
in the definition of a utility function.

Maximizers

The set of all subsets of A is denoted $\mathfrak{B}_A$.

Given $X \in \mathfrak{B}_A$, we define

$$M(X, \succ) := \{x \in X \mid \nexists\, y \in X\, [y \succ x]\} = \{x \in X \mid \forall\, y \in X\, [x \succeq y]\},$$

the set of *maximizers* of $\succ$ on X.

The agent has preferences over the whole A,
but may be faced with the necessity to choose from $X \in \mathfrak{B}_A$,
in which case any alternative from $M(X, \succ)$ will do.

Monotone comparative statics problems

Quah (Econometrica, 2007):

"type A" problem: compare $M(X, \succ)$ with $M(X, \succ')$;

"type B" problem: compare $M(X, \succ)$ with $M(Y, \succ)$.

In either case, one has to extend the order from A to $\mathfrak{B}_A$.
There are several ways to do so on a lattice.

Comparing subsets of a lattice

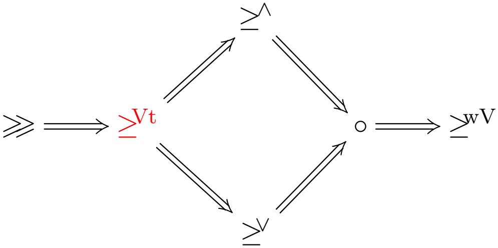
$$Y \gg X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \geq x];$$
$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \wedge x \in X];$$
$$Y \geq^{\vee} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y];$$

$$Y \geq^{\mathrm{Vt}} X \rightleftharpoons [Y \geq^{\vee} X \;\&\; Y \geq^{\wedge} X];$$

$$Y \geq^{\mathrm{wV}} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y \;\text{ or }\; y \wedge x \in X].$$

!!! Separation of existence and monotonicity problems !!!

Comparing comparisons of subsets of a lattice



Quasisupermodularity

Studying "type B" problems on a lattice,
LiCalzi and Veinott (1992) formulated these conditions:

$$\forall x, y \in A\left[[x > y \wedge x < y \;\&\; x \succeq y \wedge x] \Rightarrow y \vee x \succ y\right]; \qquad (1a)$$

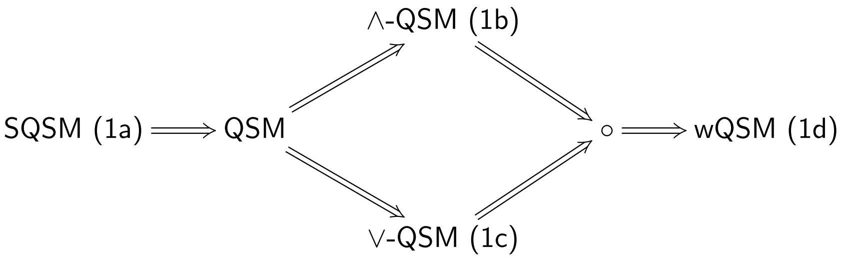
$$\forall x, y \in A\left[x \succ y \wedge x \Rightarrow y \vee x \succ y\right]; \qquad (1b)$$

$$\forall x, y \in A\left[y \succ y \vee x \Rightarrow y \wedge x \succ x\right]; \qquad (1c)$$

$$\forall x, y \in A\left[x \succ y \wedge x \Rightarrow y \vee x \succeq y\right]. \qquad (1d)$$

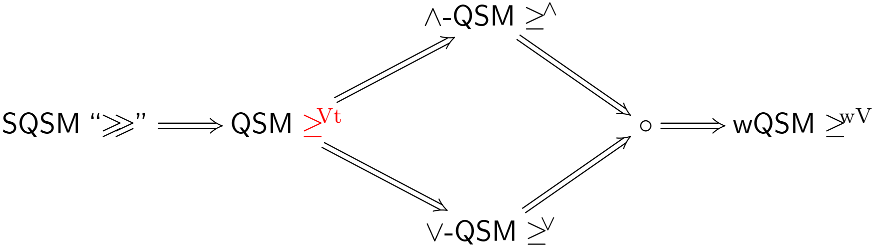
Milgrom and Shannon (1994) called the conjunction
of (1b) and (1c) *quasisupermodularity* (*QSM*).

Comparing versions of QSM



"Type B" problems on a lattice

$Y \succeq^{\mathrm{Vt}} X \Rightarrow M(Y,\succ) \succeq^{*} M(X,\succ)$
whenever Y and X are sublattices of A.



Single crossing conditions

Given two orderings, $\succ$ and $\succeq'$, we consider conditions:

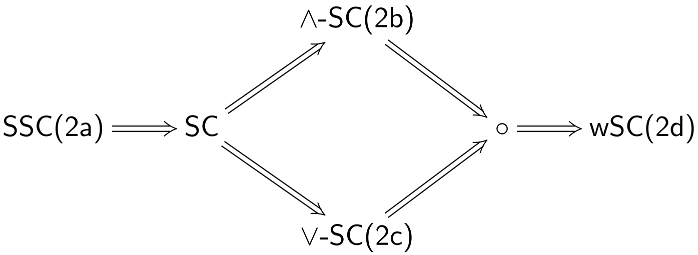
$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succ' x \right]; \tag{2a}$$
$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succ' x \right]; \tag{2b}$$
$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succeq' x \right]; \tag{2c}$$
$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succeq' x \right]. \tag{2d}$$

"One relation is closer to the basic order than the other is."

(2b) and (2c) define preorders on the set of orderings on A.
Relation (2a) is transitive, but generally not reflexive.
Relation (2d) is reflexive, but need not be transitive.

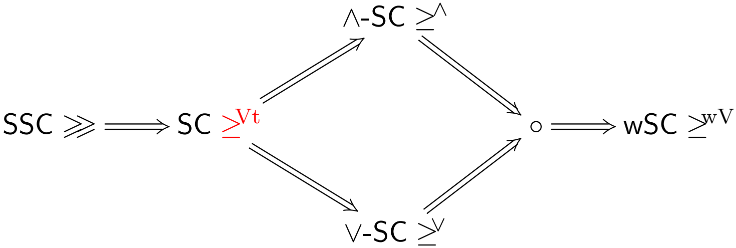
Comparing single crossing conditions

Milgrom and Shannon (1994) called the conjunction of (2b) and (2c) *single crossing* (*SC*).



“Type A” problems for subchains

$\ldots \Rightarrow M(X, \succ') \geq^* M(X, \succ)$ whenever $X \subseteq A$ is a chain.



"Type A" problems for sublattices

If $X \subseteq A$ is *not* a chain, then no SC-condition would ensure that $M(X, \succ')$ and $M(X, \succ)$ are comparable in any sense without some conditions on $\succ$ (or $\succ'$).

Appropriate conditions are obtained by partitioning each condition (1) into two "halves," each of the eight being necessary and sufficient for a kind of monotonicity.

(... but the margins here are too narrow ...)

Comparing subsets of a semilattice

On a *semilattice* A, neither Veinott's order $\geq^{\mathrm{Vt}}$, nor any of conditions (1) could even be formulated.

We consider just two relations on $\mathfrak{B}_A$:

$$Y \ggg X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \geq x];$$

$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \wedge x \in X].$$

Only "type A" problems

Neither $\geq^{\wedge}$, nor $\ggg$ could serve as a base for non-trivial "type B" monotone comparative statics theorems, e.g.,

if $Y \ggg X$, then $M(Y, \succ) \ggg M(X, \succ)$ regardless of what $\succ$ is.

Therefore, we have to restrict ourselves to "type A" problems.

Single crossing "backwards"

Of all single crossing conditions (2), only two are of use here:

$$\forall x, y \in A\, \big[y > x \;\&\; y \succeq x \Rightarrow y \succ' x\big]; \tag{2a}$$

$$\forall x, y \in A\, \big[y > x \;\&\; y \succ x \Rightarrow \; y \succ' x\big]. \tag{2b}$$

And they are more convenient when written "backwards":

$$\forall x, y \in A\, \big[y > x \;\&\; y \succeq' x \Rightarrow y \succ x\big]; \tag{3a}$$

$$\forall x, y \in A\, \big[y > x \;\&\; y \succ' x \Rightarrow \; y \succ x\big]. \tag{3b}$$

Clearly, $\succ$ and $\succ'$ satisfy either one of conditions (3)
if and only if $\succ'$ and $\succ$ satisfy the corresponding condition (2).

Results

"Semiquasisupermodularity"

From the viewpoint of "type A" monotone comparative statics, the following condition upon an ordering $\succ$ on a semilattice A may pretend to the role of quasisupermodularity:

$$\forall x, y \in A \left[y \wedge x \succeq x \ \text{ or } \ y \wedge x \succeq y \right]. \tag{4}$$

Main result

Theorem.
Let A be a semilattice and $\succ$ be an ordering on A.
Then the following statements are equivalent.

- $\succ$ satisfies (4).

Main result

Theorem.
Let A be a semilattice and $\succ$ be an ordering on A.
Then the following statements are equivalent.

- $\succ$ satisfies (4).
- There holds $M(X, \succ) \geq^{\wedge} M(X, \not\preceq)$ whenever X is a subsemilattice of A and $\not\preceq$ is an ordering on A satisfying (3b).

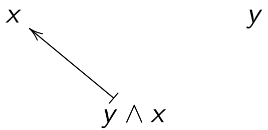
Main result

Theorem.
Let A be a semilattice and $\succ$ be an ordering on A. Then the following statements are equivalent.

- $\succ$ satisfies (4).
- There holds $M(X, \succ) \geq^{\wedge} M(X, \not\succeq')$ whenever X is a subsemilattice of A and $\not\succeq'$ is an ordering on A satisfying (3b).
- There holds $M(X, \succ) \ggg M(X, \not\succeq')$ whenever X is a subsemilattice of A and $\not\succeq'$ is an ordering on A satisfying (3a).

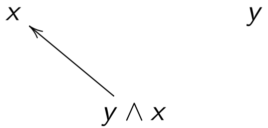
Sufficiency proof ($\succeq^{\wedge}$)

Let (4) and (3b) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$.
Suppose, to the contrary, that $y \wedge x \notin M(X, \not\succ')$.



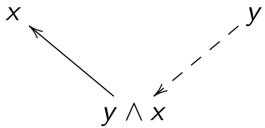
Sufficiency proof ($\geq^{\wedge}$)

Let (4) and (3b) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$.
Suppose, to the contrary, that $y \wedge x \notin M(X, \not\succ')$.



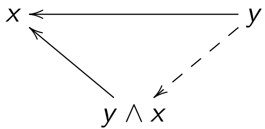
Sufficiency proof ($\succeq^{\wedge}$)

Let (4) and (3b) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$.
Suppose, to the contrary, that $y \wedge x \notin M(X, \not\succ')$.



Sufficiency proof ($\succeq^{\wedge}$)

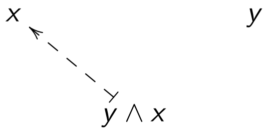
Let (4) and (3b) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$.
Suppose, to the contrary, that $y \wedge x \notin M(X, \not\succ')$.



Contradiction!

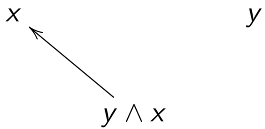
Sufficiency proof ($\gg$)

Let (4) and (3a) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$. Suppose, to the contrary, that $y \geq x$ does not take place; then $x > y \wedge x$ and $x \not\succeq' y \wedge x$.



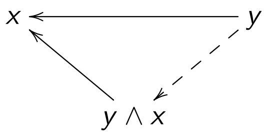
Sufficiency proof ($\gg$)

Let (4) and (3a) hold, $y \in M(X, \succ)$ and $x \in M(X, \succ')$.
Suppose, to the contrary, that $y \geq x$ does not take place;
then $x > y \wedge x$ and $x \succeq' y \wedge x$.



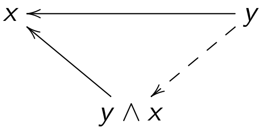
Sufficiency proof ($\gg$)

Let (4) and (3a) hold, $y \in M(X, \succ)$ and $x \in M(X, \not\succ')$.
Suppose, to the contrary, that $y \geq x$ does not take place;
then $x > y \wedge x$ and $x \not\succeq' y \wedge x$.



Sufficiency proof ($\gg$)

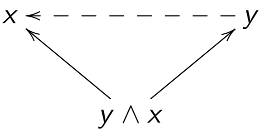
Let (4) and (3a) hold, $y \in M(X, \succ)$ and $x \in M(X, \succ')$. Suppose, to the contrary, that $y \geq x$ does not take place; then $x > y \wedge x$ and $x \succeq' y \wedge x$.



Contradiction again!

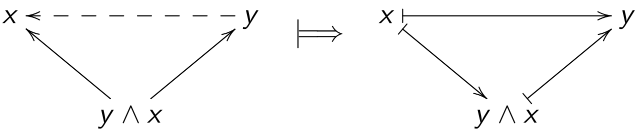
Sketch of necessity proof

Let (4) be violated: $x \succ y \wedge x$ and $y \succ y \wedge x$; w.r.g., $x \succeq y$. Denoting $X := \{x, y, y \wedge x\}$, we have $y \wedge x \notin M(X, \succ) \ni x$.



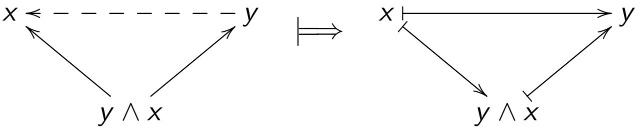
Sketch of necessity proof

Let (4) be violated: $x \succ y \wedge x$ and $y \succ y \wedge x$; w.r.g., $x \succeq y$.
Denoting $X := \{x, y, y \wedge x\}$, we have $y \wedge x \notin M(X, \succ) \ni x$.



Sketch of necessity proof

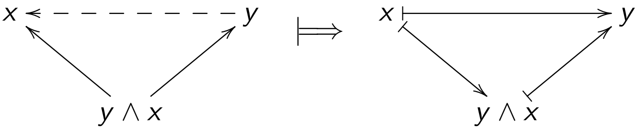
Let (4) be violated: $x \succ y \wedge x$ and $y \succ y \wedge x$; w.r.g., $x \succeq y$.
Denoting $X := \{x, y, y \wedge x\}$, we have $y \wedge x \notin M(X, \succ) \ni x$.



We have $y \wedge x \notin M(X, \not\succ') = \{y\}$, whereas (3a) holds.

Sketch of necessity proof

Let (4) be violated: $x \succ y \wedge x$ and $y \succ y \wedge x$; w.r.g., $x \succeq y$.
Denoting $X := \{x, y, y \wedge x\}$, we have $y \wedge x \notin M(X, \succ) \ni x$.



We have $y \wedge x \notin M(X, \not\succ') = \{y\}$, whereas (3a) holds.
(Then $\not\succ'$ should be extended to the whole A!)

“Upward-looking” conditions

In the terminology of Kukushkin (2013, p. 1048), condition (4) is “downward-looking.”
It seems natural to ask about “upward-looking” conditions on semilattices.
The answer is rather disheartening, these conditions emerge:

$$\forall x, y \in A \left[[x > y \wedge x < y] \Rightarrow y \wedge x \succeq x \right]; \qquad (5a)$$
$$\forall x, y \in A \left[[x > y \wedge x < y] \Rightarrow y \wedge x \succ x \right]. \qquad (5b)$$

Obviously, (5b) $\Rightarrow$ (5a) $\Rightarrow$ (4).

Characterizations

Proposition 1.
Let A be a semilattice and $\succ$ be an ordering on A.
Then $\succ$ satisfies (5a) if and only if $M(X, \succeq') \geq^{\wedge} M(X, \succ)$ whenever X is a subsemilattice of A and $\succeq'$ is an ordering on A satisfying (2b).

Characterizations

Proposition 1.
Let A be a semilattice and $\succ$ be an ordering on A.
Then $\succ$ satisfies (5a) if and only if $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (2b).

Proposition 2.
Let A be a semilattice and $\succ$ be an ordering on A.
Then $\succ$ satisfies (5b) if and only if $M(X, \succ') \ggg M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (2a).

Proposition 3.
Let $\succ$ be an ordering on a semilattice
$A := \{\langle x^1, \ldots, x^m \rangle \in \mathbb{R}^m_+ \mid \sum_{k=1}^m p_k x^k \leq B\}$
$(\forall k \in \{1, \ldots, m\}\,[p_k > 0]\,)$ with the order
$y \geq x \rightleftharpoons \forall k \in \{1, \ldots, m\}\,[y^k \geq x^k]$.

Proposition 3.
Let $\succ$ be an ordering on a semilattice
$A := \{\langle x^1, \ldots, x^m\rangle \in \mathbb{R}^m_+ \mid \sum_{k=1}^m p_k x^k \leq B\}$
$(\forall k \in \{1, \ldots, m\} \, [p_k > 0] \,)$ with the order
$y \geq x \rightleftharpoons \forall k \in \{1, \ldots, m\} \, [y^k \geq x^k]$.

Then:

- $\succ$ satisfies (5a) if and only if $\forall x, y \in A \, [x > y \Rightarrow y \succeq x]$;
- $\succ$ satisfies (5b) if and only if $\forall x, y \in A \, [x > y \Rightarrow y \succ x]$.

Neither equivalence holds on *all* semilattices,
but this is a very weak solace.

"Semiquasisupermodularity" once again

$$\forall x, y \in A \left[y \wedge x \succeq x \quad \text{or} \quad y \wedge x \succeq y \right] \qquad (4).$$

So, could this condition be satisfied by a preference relation in a nontrivial way?

A strategic game model

Strategies

Each agent i from a finite set N spends her budget B_i on several goods, some of them consumed privately, some openly.

$$X_i := \{\langle x_i^0, x_i^1, \ldots, x_i^m \rangle \in \mathbb{R}_+^{m+1} \mid \sum_{k=0}^{m} x_i^k = B_i\},$$

where x_i^0 is money spent on private consumption and x_i^k ($k \in \{1, \ldots m\}$) spent on the consumption of positional good k.

We denote $X_N := \prod_{i \in N} X_i$ and,

for each $i \in N$, $X_{-i} := \prod_{j \in N \setminus \{i\}} X_j$.

Satisfaction measures

We denote $K := \{0, 1, \ldots m\}$ and $K^* := \{1, \ldots m\}$.

For each $i \in N$ and $k \in K$, $v_i^k(x_i^k)$ expresses the satisfaction obtained from the consumption of the corresponding good.

The consumption levels of other agents exert a negative influence, $\bar{v}_i(x_{-i})$, "poisoning" satisfaction derived from the consumption of positional goods ($k \in K^*$).

Each v_i^k ($k \in K$) is increasing and upper semicontinuous in x_i; each $\bar{v}_i(x_{-i})$ is increasing and lower semicontinuous in x_{-i} (in both cases, not necessarily *strictly* increasing).

Utilities

All kinds of satisfaction are strictly complementary to one another; thus, the overall utility function of agent i on X_N is

$$u_i(x_N) = \min\{v_i^0(x_i^0), \min_{k \in K^*} v_i^k(x_i^k) - \bar{v}_i(x_{-i})\}. \qquad (6)$$

Semilattice order

We define a partial order on each X_i by

$$y_i \geq x_i \rightleftharpoons \forall k \in \{1, \ldots m\} \, [y_i^k \geq x_i^k];$$

In this order, each X_i becomes a complete semilattice.

X_N and each X_{-i} $(i \in N)$, also become complete semilattices in the Cartesian product of those orders.

Strategic complementarities

Lemma 1.
For every $i \in N$ and $x_{-i} \in X_{-i}$,
the ordering on X_i defined by $u_i(\cdot, x_{-i})$ satisfies (4).

Strategic complementarities

Lemma 1.
For every $i \in N$ and $x_{-i} \in X_{-i}$,
the ordering on X_i defined by $u_i(\cdot, x_{-i})$ satisfies (4).

Lemma 2.
Let $i \in N$ and let $x'_{-i}, x_{-i} \in X_{-i}$ be such that $x'_{-i} > x_{-i}$.
Let $\succ'$ and $\succ$ be orderings on X_i
defined by $u_i(\cdot, x'_{-i})$ and $u_i(\cdot, x_{-i})$, respectively.
Then (2b) holds on X_i.

Nash equilibrium

Proposition 4.
Every strategic game satisfying above assumptions possesses a Nash equilibrium.

Nash equilibrium

Proposition 4.
Every strategic game satisfying above assumptions possesses a Nash equilibrium.

Sketch of a proof
$R_i(x_{-i}) := \mathrm{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$; $r_i(x_{-i}) := \bigwedge R_i(x_{-i})$;
$R_i(y_{-i}) \geq^{\wedge} R_i(x_{-i})$ whenever $y_{-i} \geq x_{-i}$.

Therefore, each $r_i \colon X_{-i} \to X_i$ is increasing;
hence their Cartesian product $r \colon X_N \to X_N$ is increasing too.

Now a fixed point theorem due to Abian and Brown (1961) implies the existence of a fixed point of r, i.e., a Nash equilibrium.

Discrete version

It is worth noting that the above propositions do not need the concavity of the utility functions.
Moreover, everything remains valid
for a discrete version of the model, e.g., if

$$X_i := \{\langle x_i^0, x_i^1, \ldots, x_i^m \rangle \in \mathbb{N}^{m+1} \mid \sum_{k=0}^{m} x_i^k = B_i\}.$$

Minimum aggregation

The minimum aggregation in a utility function, i.e., the "absolute complementarity" of components, is not met in economic models very often; however, it is not exceptionally rare either.

Galbraith (The Affluent Society, 1958, Chapter XVIII) effectively viewed it as most natural in the evaluation of tradeoffs between public and private consumption ("social balance").

Germeier and Vatel (1974)

The minimum aggregation of local utilities was employed in the model of Germeier and Vatel' (1974), as well as in a wide class of related models;

It ensures even the existence of a strong Nash equilibrium, see Harks et al. (2013) or Kukushkin (2017).

However, the general context there is different from ours (closer to Galbraith's):
"fellow travelers" rather than consumers of positional goods.

Summary (+)

A condition on preference orderings is found
which is necessary and sufficient for
a monotone response of the optimal choice
from every subsemilattice
to a perturbation of the preferences
satisfying an appropriate single crossing condition.

Summary (+)

A strategic game model related to
the consumption of positional goods is formulated
where Nash equilibrium exists because
the players' preferences satisfy our condition.

Summary (−)

Admittedly, that game model is rather artificial.

The suitability of our condition
for any interesting application
remains unclear.

The Last Word

If a reader will interpret the findings of this paper

as showing that any attempt to extend monotone comparative statics analysis beyond lattices

is essentially hopeless,

well, so be it.

Thanks to everybody

for your attention and patience

References

Abian, S., and A.B. Brown, 1961. A theorem on partially ordered sets with applications to fixed point theorems.
Canadian Journal of Mathematics 13, 78–83.

Dushnik, B., and E.W. Miller, 1941. Partially ordered sets.
American Journal of Mathematics 63, 600–610.

Galbraith, G.K., 1958. The Affluent Society.
The New American Library, New York and Toronto.

Germeier, Yu.B., and I.A. Vatel', 1974.
On games with a hierarchical vector of interests.
Izvestiya Akademii Nauk SSSR, Tekhnicheskaya Kibernetika, 3, 54–69 [in Russian; English translation in
Engineering Cybernetics 12(3), 25–40 (1974)].

Harks, T., M. Klimm, and R.H. Möhring, 2013. Strong equilibria in games with the lexicographical improvement property. International Journal of Game Theory 42, 461–482.

Kukushkin, N.S., 2013. Monotone comparative statics: Changes in preferences versus changes in the feasible set. Economic Theory, 2013, 52, 1039–1060.

Kukushkin, N.S., 2017. Strong Nash equilibrium in games with common and complementary local utilities. Journal of Mathematical Economics 68, 1–12.

LiCalzi, M., and A.F. Veinott, Jr., 1992. Subextremal functions and lattice programming. EconWPA Working Paper 0509001. Available at https://ideas.repec.org/p/wpa/wuwpge/0509001.html

Milgrom, P., and C. Shannon, 1994.
Monotone comparative statics. Econometrica 62, 157–180.

Quah, J., 2007. The comparative statics of constrained optimization problems. Econometrica 75, 401–431.

Shannon, C., 1995. Weak and strong monotone comparative statics. Economic Theory 5, 209–227.

Topkis, D.M., 1978. Minimizing a submodular function on a lattice. Operations Research 26, 305–321.

Veinott, A.F., Jr., 1989.
Lattice Programming. Unpublished lectures.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.