Smooth Feasible Solutions to a Dual Monge—Kantorovich Problem with Applications to Best Approximation and Utility Theory in Mathematical Economics

Vladimir L. Levin*

Central Economics and Mathematics Institute of the Russian Academy of Sciences, Nakhimovskii Prospect 47, 117418 Moscow, Russia (e-mail:vl_levin@cemi.rssi.ru)

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Abstract. Given a (closed or open) subset X in $\mathbb{R}^n$, which is stable with respect to shifts in positive directions, we consider inequalities $u(x) - u(y) \leq c(x, y)$, $x, y \in X$, and for a wide class of functions c on $X \times X$, derive a smooth solution to these inequalities from a Lebesgue measurable one. Applications are given to a best approximation problem and to several problems of mathematical economics relating to preferences that admit smooth (or Lipschitz continuous) utility functions, smooth-utility-rational choice, and smooth representations of interval orders.

Keywords: dual Monge—Kantorovich problem, exact solutions to a best approximation problem, preference, strengthened acyclicity assumption, Lipschitz continuous and smooth utility functions, axioms of revealed preference, smooth-utility-rational choice, representations of interval orders.

1 Introduction

This paper is concerned with the existence of smooth feasible solutions to dual Monge—Kantorovich problem with given marginal difference on some particular

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subsets X in $\mathbb{R}^n$. We develop a method that enables us to derive smooth solutions from Lebesgue measurable ones. This method was already used in [35, 36, 37] in connection with some problems of best approximation theory and of utility theory. In the present paper, we consider it in general setting and give further applications[1].

Let X be a separable metrizable topological space. Given a cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ and two positive finite Borel measures σ_1 and σ_2 on X with $\sigma_1 X = \sigma_2 X$, the Monge—Kantorovich mass transportation problem (MKP) with a cost function c and a marginal difference $\rho = \sigma_1 - \sigma_2$ is to minimize the functional

$$c(\mu) := \int_{X \times X} c(x, y)\, \mu(d(x, y))$$

over the set of positive Borel measures μ on $X \times X$ satisfying $\pi_1\mu - \pi_2\mu = \sigma_1 - \sigma_2$ where $\pi_1\mu$ and $\pi_2\mu$ are the marginals of μ: for every Borel set $B \subset X$, $\pi_1\mu(B) = \mu(B \times X)$, $\pi_2\mu(B) = \mu(X \times B)$. It is an infinite linear program, and the dual problem is to maximize the functional

$$\rho(u) := \int_X u(x)\, \rho(dx)$$

over the constraint set $Q(c)$ of bounded continuous functions $u : X \to \mathbb{R}$ satisfying

$$u(x) - u(y) \le c(x, y) \tag{1}$$

for all $x, y \in X$. Every $u \in Q(c)$ may be considered thus as a feasible solution to this dual linear program.

For the first time, MKP was posed by Kantorovich, who studied duality of two problems in the case that (X, d) is a metric compact space and $c = d$; see [11, 12, 14, 15]. In that case, $Q(c)$ is the set $Lip_1(X, d)$ of Lipschitz continuous functions on (X, d) with the Lipschitz constant $L \le 1$, and the above-posed mass transportation problem is equivalent to a different variant of MKP, with given marginals [14, 15]. The latter consists in minimizing $c(\mu)$ subject to $\pi_1\mu = \sigma_1$, $\pi_2\mu = \sigma_2$, and it is a relaxation of the Monge 'excavation and embankments' problem [42].

Duality for both variants of MKP with arbitrary continuous cost functions on (not necessarily metrizable) compact topological spaces is studied since 1974; see [17, 18, 20].[2] Notice that the term itself, *Monge—Kantorovich problem*, has appeared in seventies (see [17, 46]) in connection with both variants of MKP[3]. In

[1]Some results of this paper are announced without proofs in [38].

[2]In general case, two variants of MKP, with a given marginal difference and with given marginals, are not equivalent.

[3]For the first time, this word combination as a title of a paper has appeared in [19, 20].

[39, 22, 25, 26, 31],[4] more general (topological and non-topological) spaces X were considered and duality theory for both variants of MKP with a general (discontinuous) cost function was developed.

In this paper, we don't deal with the Monge—Kantorovich duality as such. Properties of the dual constraint set $Q(c)$ and especially conditions for it to be non-empty is the only thing that will be needed to us. Also we will consider the set $Q_0(c)$ of all functions $u : X \to \mathbb{R}$ satisfying (1) and the set $Q(c; \mathcal{L}^\infty(X))$ of bounded Lebesgue measurable functions[5] satisfying (1). Properties of these and similar sets and conditions for them to be non-empty were studied earlier in connection with various generalizations and applications of MKP (see [26, 27, 28, 29, 35, 36]).

An important role in study of such sets is played by reduced cost functions [19, 20, 39]. Given a cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$, the *reduced cost function* c_* is defined as follows:

$$c_*(x, y) := \min \left(c(x, y), \inf_m \inf_{x^1,\dots,x^m} \sum_{i=1}^{m+1} c(x^{i-1}, x^i) \right), \tag{2}$$

where $x^0 = x, x^{m+1} = y$. Clearly, it turns into

$$\begin{aligned} c_*(x, y) : &= \inf_m \inf \left\{ \sum_{i=1}^{m+1} c(x^{i-1}, x^i) : x^i \in X, x^0 = x, x^{m+1} = y \right\} \\ &= \lim_{m\to\infty} \inf \left\{ \sum_{i=1}^{m+1} c(x^{i-1}, x^i) : x^i \in X, x^0 = x, x^{m+1} = y \right\} \end{aligned} \tag{3}$$

when c vanishes on the diagonal: $c(x, x) = 0 \quad \forall x \in X$. Clearly, $c_* \le c$ and c_* satisfies the triangle inequality: for every $x, y, z \in X$,

$$c_*(x, y) + c_*(y, z) \ge c_*(x, z)$$

(we assume, by definition, that $+\infty + (-\infty) = +\infty$). Also, it follows from the above definition that: (i) $Q(c) = Q(c_*)$, (ii) if c_* is bounded continuous and vanishes on the diagonal then $Q(c)$ is non-empty and $c_*(x, y) = \sup_{u\in Q(c)}(u(x) - u(y))$. Similar facts hold for $Q_0(c)$ and $Q(c; \mathcal{L}^\infty(X))$; see [35, 36] for details.

The paper consists of five sections. In Section 2 we suppose that $X \subset \mathbb{R}^n$ is closed or open and stable with respect to shifts in positive directions (i.e. $x \in$

[4] Also see [44, Chapter 4] where some results of [39, 22, 25, 26] are presented.

[5] Here, $\mathcal{L}^\infty(X)$ stands for the Banach space of bounded Lebesgue measurable functions (Lebesgue equivalent functions are not identified) with the uniform norm $\|u\|_{\mathcal{L}^\infty(X)} = \sup_{x\in X} |u(x)|$. The classical Lebesgue space $L^\infty(X)$ is naturally isometric to its factor-space: $L^\infty(X) = \mathcal{L}^\infty(X)/\mathcal{N}_0$ where $\mathcal{N}_0$ is a subspace in $\mathcal{L}^\infty(X)$ consisting of Lebesgue negligible functions.

$X,\ z \in \mathbb{R}^n_+ \Rightarrow x + z \in X$). We prove a theorem on the existence of smooth functions in $Q(c)$ for some particular class of cost functions c. The rest of the paper is devoted to applications of this existence theorem in approximation theory and in utility theory.

In Section 3, we deal with the following best approximation problem [35, 36]. Given a function $f \in C_b(X \times X)$, one has to approximate it by functions h_u from the subspace

$$H = \{h_u(x, y) = u(x) - u(y) : u \in C_b(X)\} \subset C_b(X \times X),$$

where C_b stands for the Banach space of bounded continuous real-valued functions on the corresponding topological space. The optimal value of this problem is thus

$$m(f; H) := \inf_{h_u \in H} \|f - h_u\| = \inf_{u \in C_b(X)} \sup_{x,y \in X} |f(x, y) - u(x) + u(y)|,$$

and u proves to be an exact solution to the problem if and only if $u \in Q(\varphi)$ where $\varphi = \min(f(x, y), -f(y, x)) + m(f; h)$ [36, Lemma 1]. Basing on this fact, along with relation $Q(\varphi; \mathcal{L}^\infty(X)) \neq \varnothing$ (see [36, Lemma 2]), and applying the theorem of Section 2, we generalize our previous results ([36, Theorems 3 and 4]) on the existence of smooth exact solutions to the best approximation problem.

Section 4 is devoted to several aspects of utility theory. Given a separable metric space (X, d), by a preference on it we mean any binary relation $\preceq$ (in general, neither transitivity nor reflexivity of $\preceq$ is required). According with the Debreu theorem [5, 6], any closed total preorder $\preceq$ is generated by a continuous utility function. In [21, 24] the existence of continuous utility functions was established for several classes of closed non-total preorders. More general (non-total and non-transitive) preferences that admit d-Lipschitz continuous utility functions were completely characterized in [28] in terms of some strengthened acyclicity assumption. In subsection 4.1 we obtain further results in that direction including a jointly Lipschitz continuous utility theorem for a varying preference $\preceq_\omega$ depending on a parameter ω.

In subsection 4.2, we deal with preferences on subsets of $\mathbb{R}^n$. Conditions for the existence of a smooth utility function based on manifold theory may be found in [41][6] in the case that X is open in $\mathbb{R}^n$ while $\preceq$ is a locally non-satiated closed total preorder. In [37] we have proved the existence of smooth utility functions for a class of closed (non-total) preorders on subsets $X \subset \mathbb{R}^n$ that are stable with respect to shifts in positive directions. This class of preorders is given by the condition that adding one and the same positive vector to each of two comparable alternatives cannot affect the preference relation between them. For $X = \mathbb{R}^n$ a close class of

[6] Also see [2, 7, 8].

total preorders was considered earlier in connection with social choice problems; see [9, 43] and references therein. Also, in [37] we have proved the existence of universal smooth utilities for preferences depending on a parameter. Basing on the existence theorem from Section 2, we now extend the approach of [37] to a broader class of preferences and prove the existence of smooth utility functions (and smooth joint utilities) u satisfying $\|\nabla u(x)\| \leq 1$, $x \in X$.

In subsection 4.3, we deal with rationalizable choice functions. Suppose that $\mathcal{M}$ is a family of non-empty subsets in a separable metric space (X, d), and in each $M \in \mathcal{M}$ a certain subset $\gamma(M) \subset M$ is chosen. The problem is to find conditions on a choice function γ that ensure the existence of utility function $u : X \to \mathbb{R}$ that has nice properties (such as continuity, smoothness, monotonicity, concavity and so on) and rationalizes the choice:

$$\gamma(M) = \{x \in M : u(x) = \max_{y \in M} u(y)\},\ M \in \mathcal{M}.$$

Originally, the rational choice problem appeared in demand theory in form of consumer's choice [10, 4]. In this setting, $X \subset \mathbb{R}^n_+$ is a commodity space, and $\mathcal{M} = \{M_{p,I} : p \in \mathbb{R}^n_+,\ I \in \mathbb{R}_+\}$, where p is the price vector, I is the consumer's wealth, and $M_{p,I} = \{x \in X : px \leq I\}$ is the consumer's budget set. Later, various aspects of general rational choice theory were developed by many authors; see e.g. [45, 16]. An important role here is played by intensifications of the Houthakker strong axiom of revealed preference[7]. Following [28, Corollary 2], we give a criterion for a choice function to be d-Lipschitz-utility-rational. We show further that:

(a) if X is closed or open in $\mathbb{R}^n$, γ satisfies the strengthened axiom of revealed preference, and X, $\mathcal{M}$, γ are stable with respect to shifts in positive directions[8], then γ is smooth-utility-rational;

(b) if, in addition, $X = \mathbb{R}^n$, $\mathcal{M}$, and γ are stable with respect to shifts by any $z \in \mathbb{R}^n$, and there is a closed convex set $M \in \mathcal{M}$ with non-empty interior and smooth boundary, then γ can be rationalized by a linear function.

Subsection 4.4 is devoted to representations of interval orders on a (closed or open) set $X \subset \mathbb{R}^n$ satisfying $X = X + \mathbb{R}^n_+$. We consider an interval order $\succ$ satisfying

$$x \succ y \Leftrightarrow (x + z) \succ (y + z)$$

for each $z \in \mathbb{R}^n_+$, and show that for this class of interval orders, the existence of Lebesgue measurable representations and of smooth ones are equivalent.

[7] As is shown in [28, 34], various versions of the strong axiom of revealed preference mean the validity of some strengthened acyclicity assumption, which, in turn, may be formulated as the inequality $c_*(x, x) \geq 0$, $x \in X$, where c_* is the reduced cost function associated with an appropriate cost function c.

[8] That is, for each $z \in \mathbb{R}^n_+$, $x \in X \Rightarrow (x + z) \in X$ and $M \in \mathcal{M} \Rightarrow [M + z \in \mathcal{M},\ \gamma(M + z) \supseteq \gamma(M) + z]$.

Section 5 is a conclusion.

2 An existence theorem

In what follows, X is a closed (or open) subset in $\mathbb{R}^n$ satisfying $X = X + \mathbb{R}^n_+$ (that is $x \in X$, $z \in \mathbb{R}^n_+$ implies $x + z \in X$), $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a Lebesgue measurable bounded below function,

$$Q_0(c) = \{u \in \mathbb{R}^X : u(x) - u(y) \leq c(x, y) \quad \text{for all } x, y \in X\},$$

and

$$Q(c) = Q_0(c) \cap C_b(X),$$

where $C_b(X)$ is the space of all bounded continuous real-valued functions on X.

In this section, we give a sufficient condition for the existence of smooth functions in $Q(c)$.

Let $C^\infty(X) = \bigcap\limits_{r=1}^{\infty} C^r(X)$ where $C^r(X)$ is the class of all r times continuously differentiable real-valued functions on X: $u \in C^r(X)$ if and only if for every $x = (x_1, \ldots, x_n) \in int\, X$ all the partial derivatives $\frac{\partial^k u(x)}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$, $k \leq r$, $k_1 + \ldots + k_n = k$, exist and each of them is uniquely continued with preserving continuity to the whole of X.

As is established in [26] (see also [30, 34]), if $c \in C^1(X \times X)$ and $c(x, x) = 0$ for all $x \in X$ then either $Q_0(c)$ is empty or $Q_0(c) = \{u + \alpha : \alpha \in \mathbb{R}\}$, where $u \in C^1(X)$ is a unique, up to a constant term, function satisfying

$$\nabla u(x) = -\nabla_y c(x, y)|_{y=x}, \quad x \in X.$$

Also in these papers conditions on c are given, separately necessary ones and sufficient ones, for $Q_0(c)$ to be non-empty. In contrast to this, in the following theorem $Q_0(c)$ is supposed to be non-empty, and although it may be rather extensive, the existence of a smooth function in it is non-trivial.

Theorem 2.1 *Suppose there is a bounded Lebesgue measurable function u on X satisfying (1), and for every $x, y \in X$ the inequality holds true*

$$\int_0^\infty \ldots \int_0^\infty c(x + z, y + z)\eta(-z)\, dz_1 \ldots dz_n \leq c(x, y), \tag{4}$$

where

$$\eta(z) = a_n \prod_{i=1}^{n} h(z_i), \; z = (z_1, \ldots, z_n),$$

$$h(t) = \begin{cases} 0 & \text{for } t \geq 0, \\ \exp(t - \frac{1}{t^2}) & \text{for } t < 0, \end{cases}$$

$$a_n = \left(\int_0^\infty h(-t)\, dt \right)^{-n};$$

then there is a function $v \in Q(c) \cap C^\infty(X)$. Moreover, $u(X) \subset (0,1)$ implies $v(X) \subset (0,1)$, and if u is non-decreasing[9] *then v is non-decreasing, too. Also, if X is convex and u is concave (or convex) then one can find $v \in Q(c) \cap C^\infty(X)$ with the same properties.*

Proof. We extend c to the whole of $\mathbb{R}^n \times \mathbb{R}^n$ by setting $c(x,y) = +\infty$ for $(x,y) \notin X \times X$. Now, given a function $u \in \mathcal{L}^\infty(X) \cap Q_0(c)$, we extend it to the whole of $\mathbb{R}^n$ by setting $u(x) = 0$ for $x \notin X$ and observe that so extended function satisfies (1) whenever $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Let us define a function $\Phi(u)$ on $\mathbb{R}^n$ to be the convolution of u and η: for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\Phi(u)(x) := (u * \eta)(x) = \int_{\mathbb{R}^n} u(x-z)\eta(z)\, dz_1 \ldots dz_n \tag{5}$$

$$= \int_{\mathbb{R}^n} u(z)\eta(x-z)\, dz_1 \ldots dz_n.$$

Since $\eta \in C^\infty(\mathbb{R}^n)$, it follows from (5) that $\Phi(u) \in C^\infty(\mathbb{R}^n)$, and since $\eta(-z) = 0$ for $z \notin \mathbb{R}^n_+$, we get

$$\Phi(u)(x) = \int_{\mathbb{R}^n_+} u(x+z)\eta(-z)\, dz_1 \ldots dz_n. \tag{6}$$

Now, multiplying the inequality

$$u(x+z) - u(y+z) \leq c(x+z, y+z), \quad z \in \mathbb{R}^n_+$$

by $\eta(-z)$, along with subsequent integrating by the Lebesgue measure on $\mathbb{R}^n_+$, yields

$$\Phi(u)(x) - \Phi(u)(y) \leq \int_{\mathbb{R}^n_+} c(x+z, y+z)\eta(-z)\, dz_1 \ldots dz_n. \tag{7}$$

We define v to be the restriction of $\Phi(u)$ onto X. Taking into account (4), we derive from (7) that $v \in C^\infty(X) \cap Q_0(c)$.

If $u(X) \subset (0,1)$ then $v(X) = \Phi(u)(X) \subset (0,1)$ since $\int_{\mathbb{R}^n} \eta(z)\, dz_1 \ldots dz_n = \int_{\mathbb{R}^n_+} \eta(-z)\, dz_1 \ldots dz_n = 1$. If u is non-decreasing then $u(x+z) \leq u(y+z)$ whenever $x \leq y$, $z \in \mathbb{R}^n_+$, and $v(x) \leq v(y)$ by (6). Finally, if X is convex and u is concave (resp. convex) then, by (6), $\Phi(u)$ is concave (resp. convex), as well. $\square$

[9] That is, $x \leq y$ implies $u(x) \leq u(y)$, where $x \leq y$ means $x_i \leq y_i$, $i = 1, \ldots, n$.

Remark 2.1 Inequality (4) is obviously satisfied with the equality sign when $c(x,y) = g(x-y)$.

Remark 2.2 Clearly (4) is satisfied when $c(x+z, y+z) \le c(x,y)$ for all $x, y \in X$, $z \in \mathbb{R}^n_+$. In particular, it is a case where c is non-increasing in (x,y). Moreover, non-increase of the functions on $\mathbb{R}^n_+$, $z \mapsto c(x+z, y+z)$, whenever $x, y \in X$ is sufficient for (4) to be satisfied. Also, (4) is satisfied when $c(x,y) = g(x,y,x-y)$ where $g : X \times X \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and the function $z \mapsto g(x+z, y+z, x-y)$, is non-increasing on $\mathbb{R}^n_+$ whenever $x, y \in X$.

Remark 2.3 A bounded Lebesgue measurable function on X satisfying (1) exists in any of the following cases: (a)(cf. [26, Theorem 9.2]) c satisfies the triangle inequality, vanishes on the diagonal, and is Borel measurable (or its Lebesgue sublevel sets $L(c;\alpha) = \{(x,y) \in X \times X : c(x,y) \le \alpha\}$, $\alpha \in \mathbb{R}$, are Souslin); (b) (cf. [35, Proposition 11] or [27, Lemma 2]) c_* is bounded above, Borel measurable, and $c_*(x,x) > -\infty$ for all $x \in X$ (or, what is the same, for all positive integers l and all cycles $x^0, \dots, x^{l-1}, x^l = x_0$ in X, the inequality holds $\sum_{i=1}^{l} c(x^{i-1}, x^i) \ge 0$). Yet one (trivial) case is that c is non-negative.

Remark 2.4 Theorem 2.1 remains true with the same proof when η is an arbitrary $C^\infty(\mathbb{R}^n)$-smooth non-negative function that equals 0 outside $\mathbb{R}^n_-$ and satisfies $\int_{\mathbb{R}^n_+} \eta(-z)\, dz_1 \dots dz_n = 1$.

3 An application to approximation theory

In this section we deal with the following best approximation problem. Given a closed subset $X \subseteq \mathbb{R}^n$, a subspace H in $C_b(X \times X)$,

$$H := \{h_u(x,y) = u(x) - u(y) : u \in C_b(X)\},$$

and a function $f \in C_b(X \times X)$, one has to find the optimal value

$$m(f;H) := \inf_{h_u \in H} \|f - h_u\| = \inf_{u \in C_b(X)} \sup_{x,y \in X} |f(x,y) - u(x) + u(y)|. \tag{8}$$

Here, given a topological space Y, $C_b(Y)$ denotes the Banach space of bounded continuous real-valued functions on it with the standard uniform norm $\|u\| = \sup_{y \in Y} |u(y)|$.

Definition 3.1 We say a function $u \in C_b(X)$ is an *exact solution* to the best approximation problem (8), if infimum in (8) is attained at it:

$$m(f;H) = \sup_{x,y \in X} |f(x,y) - u(x) + u(y)|.$$

Let us consider the cost function

$$\varphi(x,y) = c(x,y) + m(f;H), \tag{9}$$

where

$$c(x,y) = \min(f(x,y), -f(y,x)). \tag{10}$$

The following two lemmas are particular cases of Lemmas 1 and 2 in [36], where more general spaces X are considered.

Lemma 3.1 *$m(f;H) = \inf\{\alpha \in \mathbb{R}_+ : Q(c+\alpha) \neq \varnothing)\}$, and a function u is an exact solution to (8) if and only if $u \in Q(\varphi)$.*

Lemma 3.2 *$Q(\varphi; \mathcal{L}^\infty(X)) \neq \varnothing$.*

In the proof of Lemma 3.2 the existence of strong lifting of $\mathcal{L}^\infty(X)$ is substantially used; see [36] for details.

The next theorem generalizes Theorem 4 in [36] and Theorem 16 in [35][10] where some sufficient conditions were given for f to be as in the theorem.

Theorem 3.1 *1. Suppose X satisfies $X = X + \mathbb{R}^n_+$ and the function c, as given by (10), satisfies (4); then there exists a function $v \in C^\infty(X)$ which is an exact solution to problem (8).*

2. Suppose that either f is non-positive and satisfies (4) or f is non-negative and $(-f)$ satisfies (4); then c satisfies (4).

Proof. 1. By Lemma 3.2, $Q(\varphi; \mathcal{L}^\infty(X)) \neq \varnothing$. Since c satisfies inequality (4), the function φ, as given by (9), satisfies it as well. Now, from Theorem 2.1 it follows that there is a function $v \in Q(\varphi) \cap C^\infty(X)$, and applying Lemma 3.1 completes the proof.

2. If f is non-positive then $c(x,y) = f(x,y)$, and if f is non-negative then $c(x,y) = -f(y,x)$, and in both cases the result follows. □

4 Applications to utility theory

4.1 Preferences admitting Lipschitz continuous utility functions

Let X be a metric space with a distance d. Its elements may be considered as states of some system. By a (original) *preference* on X we mean an arbitrary

[10] See also the first statement of Theorem 3 in [36].

binary relation $\preceq$ (reflexivity and transitivity of $\preceq$ are not supposed): given two states $x, y \in X$, we say that y is preferred to x if $x \preceq y$. We say that y is strictly preferred to x and write it as $x \prec y$ if $x \preceq y$, $\neg(y \preceq x)$.[11]

Besides the original preference $\preceq$ on X, we will consider a revealed (extended) one, $\preceq_*$. It is defined as follows. Given $x, y \in X$, we denote by $T(x, y)$ the set of all chains (finite trajectories) $\tau = (x^1 \to x^2 \to \ldots \to x^m)$ in X leading from x to y ($m = m(\tau)$, $x^1 = x$, $x^m = y$). Every $\tau \in T(x, y)$ is appraised by

$$e(\tau) = \sum_{k=1}^{m} c(x^k, x^{k+1}) = \sum_{k \in J(\tau)} d(x^k, x^{k+1}), \tag{11}$$

where, for any $z, z' \in X$,

$$c(z, z') := \begin{cases} 0 & \text{if} \quad z \preceq z', \\ d(z, z') & \text{otherwise}, \end{cases} \tag{12}$$

$J(\tau) = \{k < m = m(\tau) : \neg(x^k \preceq x^{k+1})\}$. A chain $\tau \in T(x, y)$ is considered thus as a trajectory $x = x^1 \to x^2 \to \ldots \to x^m = y$ that starts at x and finishes at y. It comprises $m - 1$ elementary fragments $x^k \to x^{k+1}$, $k = 1, \ldots, m - 1$, and each fragment either improves the state ($x^k \preceq x^{k+1}$) or is a jump aside ($\neg(x^k \preceq x^{k+1})$). The value $e(\tau)$ may be treated as a total payment for moving along the trajectory provided that a displacement $x^k \to x^{k+1}$ improving the state is free while a jump fee equals the distance between the corresponding states, x^k and x^{k+1}. We define

$$x \preceq_* y \Leftrightarrow \inf_{\tau \in T(x,y)} e(\tau) = 0. \tag{13}$$

Remark 4.1 (cf. [28, Lemma 1]). Say $\tau \in T(x, y)$ is *regular* if it is of the form $y^0 \to x^1 \to y^1 \to x^2 \to \ldots \to x^m \to y^m \to x^{m+1}$ where $x^k \preceq y^k$, $k = 1, \ldots, m$, $y^0 = x$, $x^{m+1} = y$. Since d satisfies the triangle inequality, it is profitable to replace two neighboring jumps $x^k \to x^{k+1}$ and $x^{k+1} \to x^{k+2}$ with the single one $x^k \to x^{k+2}$, and we derive from (11) that it suffices to take only regular τ in (13). Thus,

$$x \preceq_* y \Leftrightarrow \inf_{\tau \in T_{reg}(x,y)} e(\tau) = 0,$$

where $T_{reg}(x, y)$ is the subset in $T(x, y)$ consisting of regular τ.

It follows from (11),(12),(13) that $\preceq_*$ is a preorder (a reflexive and transitive binary relation) on X. Clearly $x \preceq y \Rightarrow x \preceq_* y$.

[11] Writing $\neg(y \preceq x)$ means that $y \preceq x$ fails.

Remark 4.2 Notice that

$$\inf_{\tau \in T(x,y)} e(\tau) = \inf_{\tau \in T_{reg}(x,y)} e(\tau) = c_*(x,y), \tag{14}$$

where c_* is the reduced cost function (see (2)) for the cost function c given by (12). The definition of $\preceq_*$, as given by (13), may be then rewritten as

$$x \preceq_* y \Leftrightarrow c_*(x,y) = 0. \tag{15}$$

The preference $\preceq_*$ generates the corresponding strict preference $\prec_*$,

$$x \prec_* y \Leftrightarrow (x \preceq_* y,\ \neg(y \preceq_* x)).$$

The next result is an immediate consequence of (11),(13),(15).

Lemma 4.1 *The following conditions are equivalent: (a)* $x \prec y \Rightarrow x \prec_* y$*, (b) if* $x \prec y$ *then there is a number* $\delta = \delta(x,y) > 0$ *such that* $e(\tau) > \delta$ *for every (for every regular)* $\tau \in T(y,x)$*, (c)* $x \prec y \Rightarrow c_*(y,x) > 0$.

Definition 4.1 Given a preference $\preceq$ on X, a function $u : X \to \mathbb{R}$ is called a *utility function* for it if for any $x, y \in X$ two implications hold as follows:

$$x \preceq y \Rightarrow u(x) \leq u(y),$$

$$x \prec y \Rightarrow u(x) < u(y).$$

Remark 4.3 Condition (b) of Lemma 4.1 strengthens the acyclicity assumption: if $x \prec y$ then no $\tau = (y = x^1 \to x^2 \to \ldots \to x^m = x) \in T(y,x)$ can exist with $x^k \preceq x^{k+1}$, $k = 1, \ldots, m-1$. Obviously, this acyclicity assumption is necessary for the existence of an utility function for $\preceq$. But it is not sufficient: a simplest counter-example is the standard lexicographical order on $\mathbb{R}^2$ as $\preceq$.

Remark 4.4 An equivalent formulation of the acyclicity assumption is as follows: if $x \prec y$ then $e(\tau) > 0$ for every $\tau \in T(y,x)$.

Definition 4.2 Say a preference $\preceq$ satisfies the *strengthened acyclicity assumption* if statement (b) of Lemma 4.1 holds true, that is for every $x \prec y$ there is a number $\delta = \delta(x,y) > 0$ such that $e(\tau) > \delta$ for all $\tau \in T(y,x)$.

The next lemma says that replacing a given distance by an equivalent one can affect neither the revealed preference $\preceq_*$ nor the strengthened acyclicity assumption.

Lemma 4.2 *Let d^1 and d^2 be equivalent distances on X, i.e. there are positive constants A, B such that $Ad^2(x,y) \le d^1(x,y) \le Bd^2(x,y)$ for all $x, y \in X$. Consider two cost functions, c^1 and c^2, that answer d^1 and d^2 according to (12); then for any $x, y \in X$ the equivalence holds true*

$$c^1_*(x,y) = 0 \Leftrightarrow c^2_*(x,y) = 0,$$

and both the preorder $\preceq_*$ (see (13) or (15)) and the strengthened acyclicity assumption don't depend on what of two distances is used.

Proof. For any $x, y \in X$ we consider two functions on $T(y,x)$, e^1 and e^2, given by (11) and responding to c^1 and c^2 respectively. Clearly, one has $Ac^2(x,y) \le c^1(x,y) \le Bc^2(x,y)$; therefore, $Ac^2_*(x,y) \le c^1_*(x,y) \le Bc^2_*(x,y)$ and, for every $\tau \in T(y,x)$, $Ae^2(\tau) \le e^1(\tau) \le Be^2(\tau)$, and the result follows. □

Definition 4.3 Given a metric space (X,d), a function $u : X \to \mathbb{R}$ is called *d-Lipschitz continuous* if $|u(x) - u(y)| \le Ld(x,y)$ for all $x, y \in X$ where $L > 0$. The class of functions satisfying this inequality will be denoted as $Lip_L(X,d)$.

Clearly, $Lip_L(X,d) = Lip_1(X, Ld)$, that is one can assume $L = 1$ by passing to an equivalent distance.

We shall need the following simple lemma.

Lemma 4.3 *For any $a, b, c \in \mathbb{R}_+$, one has*

$$a - b \le c \Rightarrow \frac{a}{1+a} - \frac{b}{1+b} \le \frac{c}{1+c}, \tag{16}$$

and for any $a, b \in \mathbb{R}$,

$$a \le b \Leftrightarrow \frac{a}{1+|a|} \le \frac{b}{1+|b|}, \quad a < b \Leftrightarrow \frac{a}{1+|a|} < \frac{b}{1+|b|}. \tag{17}$$

The next result is a direct consequence of Lemma 4.3.

Corollary 4.1 *If $u : X \to \mathbb{R}$ is a non-negative utility function for $\preceq$ and $u \in Lip_1(X,d)$, then the function $v = \frac{u}{1+u}$ has the same properties as u, and $v(X) \subset [0,1]$.*

Theorem 4.1 (cf. [28, Theorem 1]) *Suppose that $\preceq$ is a preference on a separable metric space (X,d). The following statements are then equivalent: (a) there exists a (non-negative) d-Lipschitz continuous utility function for $\preceq$; (b) $\preceq$ satisfies the strengthened acyclicity assumption.*

Proof. (a)⇒(b) Suppose u is a d-Lipschitz continuous utility function for $\preceq$. We will assume by passing, if needed, to $L^{-1}u$ that $u \in Lip_1(X,d)$. Then, by (12), $u \in Q_0(c)$. If now $x \prec y$, then $\delta(x,y) := u(y) - u(x) > 0$, and we get, for every $\tau = (y = x^1, x^2, \ldots, x^m = x) \in T(y,x)$,

$$e(\tau) = \sum_{k=1}^{m-1} c(x^k, x^{k+1}) \geq \sum_{k=1}^{m-1} (u(x^k) - u(x^{k+1})) = u(y) - u(x) = \delta(x,y) > 0.$$

(b)⇒(a) Since, by implication (b)⇒(a) of Lemma 4.1, $x \prec y \Rightarrow x \prec_* y$, every utility function for $\preceq_*$ will be a utility function for $\preceq$. Therefore, it suffices to present a (non-negative) d-Lipschitz continuous utility function for $\preceq_*$. Let us consider the function

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{c_*(x,x^k)}{1 + c_*(x,x^k)}$$

where the sequence (x^k) is dense in X. Since c_* satisfies the triangle inequality, one has $c_*(x,x^k) - c_*(y,x^k) \leq c_*(x,y)$, $k = 1,2,\ldots$. Suppose $x \preceq_* y$, then, by (15), $c_*(x,y) = 0$, and we get $c_*(x,x^k) \leq c_*(y,x^k)$, $k = 1,2,\ldots$. Now, taking into account equivalence (17) from Lemma 4.3, one gets $u(x) \leq u(y)$. Furthermore, if $x \prec y$, then $c_*(y,x) > 0$ by Lemma 4.1, and applying the triangle inequality yields

$$-d(x,x^k) \leq -c_*(x,x^k) \leq c_*(y,x) - c_*(y,x^k) \leq c_*(x^k,x) \leq d(x^k,x). \tag{18}$$

Let us chose a subsequence (x^{k_n}) that tends to x; then

$$\lim_{n\to\infty} d(x,x^{k_n}) = \lim_{n\to\infty} d(x^{k_n},x) = 0,$$

and we derive from (18) that

$$\lim_{n\to\infty} c_*(x,x^{k_n})) = 0, \tag{19}$$

$$\lim_{n\to\infty} c_*(y,x^{k_n}) = c_*(y,x)) > 0. \tag{20}$$

We see from (19), (20) that $c_*(x,x^{k_n}) < c_*(y,x^{k_n})$ for large n, and since $c_*(x,x^k) \leq c_*(y,x^k)$ for all k, we derive from (17) that $u(x) < u(y)$.

It remains to show that u is d-Lipschitz continuous. We have $c_*(x,x^k) - c_*(y,x^k) \leq c_*(x,y) \leq c(x,y) \leq d(x,y)$ for any $x,y \in X$, and applying implication (16) from Lemma 4.3 with $a = c_*(x,x^k)$, $b = c_*(y,x^k)$, and $c = d(x,y)$ yields

$$\frac{c_*(x,x^k)}{1 + c_*(x,x^k)} - \frac{c_*(y,x^k)}{1 + c_*(y,x^k)} \leq \frac{d(x,y)}{1 + d(x,y)}.$$

This implies $u(x) - u(y) \leq \frac{d(x,y)}{1+d(x,y)} \leq d(x,y)$, that is $u \in Lip_1(X,d)$. □

Corollary 4.2 *Suppose $\preceq$ is a preference on a separable metrizable space X, the following statements are then equivalent: (a) there exists a (non-negative bounded) continuous utility function for $\preceq$; (b) there exists a bounded distance*[12] *d on X such that d determines topology of X and $\preceq$ satisfies the strengthened acyclicity assumption with respect to d.*

Proof. (a)⇒(b) Suppose u is a continuous utility function for $\preceq$. We will assume by passing, if needed, to $\frac{1}{2}+\frac{1}{2}\frac{u}{1+|u|}$ that u is non-negative and bounded. Let d_1 be an arbitrary distance that determines topology of X, and

$$d(x,y) := |u(x)-u(y)| + \frac{d_1(x,y)}{1+d_1(x,y)}.$$

Clearly, $u \in Lip_1(X,d)$, and d is a bounded distance determining topology of X. Statement (b) follows then from Theorem 4.1.

(b)⇒(a) This follows directly from Theorem 4.1. □

Theorem 4.2 *Suppose that Ω is a metrizable topological space, (X,d_X) is a separable metric space, and for every $\omega \in \Omega$ a preference $\preceq_\omega$ is given on X. Let d be a distance on $\Omega \times X$ consistent with the product topology and satisfying $d((\omega,x),(\omega,y)) = d_X(x,y)$ for all $\omega \in \Omega$, $x,y \in X$. The following statements are then equivalent: (a) there exists a function $u : \Omega \times X \to [0,1]$ such that $u \in Lip_1(\Omega \times X, d)$ and, for every $\omega \in \Omega$, $u(\omega,\cdot)$ belongs to $Lip_1(X,d_X)$ and is a utility function for $\preceq_\omega$; (b) the preference $\preceq$ on $\Omega \times X$, determined by*

$$(\omega,x) \preceq (\omega',y) \Leftrightarrow \omega=\omega',\ x \preceq_\omega y, \tag{21}$$

satisfies the strengthened acyclicity assumption.

Remark 4.5 Obviously, if $\preceq$ satisfies the strengthened acyclicity assumption then, for every $\omega \in \Omega$, $\preceq_\omega$ satisfies it, too.

Proof. (a)⇒(b) It follows from the definition of the preference $\preceq$ on $\Omega \times X$ (see (21)) that u is a utility function for $\preceq$. Then $u(\omega,x)-u(\omega',y) \le 0$ if $(\omega,x) \preceq (\omega',y)$, and since $u(\omega,x)-u(\omega',y) \le d((\omega,x),(\omega',y))$ for all $(\omega,x),(\omega',y) \in \Omega \times X$, one gets $u(\omega,x)-u(\omega',y) \le c((\omega,x),(\omega',y))$ for all $(\omega,x),(\omega',y) \in \Omega \times X$, where

$$c((\omega,x),(\omega',y)) := \begin{cases} 0 & \text{if } \omega=\omega',\ x \preceq_\omega y, \\ d((\omega,x),(\omega',y)) & \text{otherwise.} \end{cases} \tag{22}$$

If now $(\omega,x) \prec (\omega',y)$ then $\omega=\omega'$ and $x \prec_\omega y$, hence $u(\omega,x) < u(\omega,y)$. Then, for $\delta(x,y) := u(\omega,y)-u(\omega,x) > 0$ and any chain (trajectory) in $\Omega \times X$ leading

[12] A distance d is called bounded if there is $C>0$ such that $d(x,y) \le C$ for all $x,y \in X$.

from $(\omega, y) = (\omega^1, x^1)$ to $(\omega, x) = (\omega^m, x^m)$, $\tau = ((\omega^1, x^1) \to (\omega^2, x^2) \to \ldots \to (\omega^m, x^m))$, one has

$$e(\tau) = \sum_{k=1}^{m-1} c((\omega^k, x^k), (\omega^{k+1}, x^{k+1})) \geq \sum_{k=1}^{m-1} \Big(u(\omega^k, x^k) - u(\omega^{k+1}, x^{k+1}) \Big)$$
$$= u(\omega, y) - u(\omega, x) = \delta(x, y),$$

and the strengthened acyclicity assumption is thus established.[13]

(b)⇒(a) Consider on $(\Omega \times X) \times (\Omega \times X)$ the cost function c as given by (22) and the corresponding reduced cost function c_*, take in X a dense countable set $\{x^1, x^2, \ldots\}$, and let

$$u(\omega, x) := \sum_{k=1}^{\infty} \frac{c_*((\omega, x), (\omega, x^k))}{2^k \left(1 + c_*((\omega, x), (\omega, x^k))\right)}. \tag{23}$$

Obviously, $u(X) \subset [0, 1]$. Since c_* satisfies the triangle inequality, one has

$$c_*((\omega, x), (\omega, x^k)) - c_*((\omega', y), (\omega', x^k)) \leq c_*((\omega, x), (\omega', y)).$$

Now, taking into account (16) and arguing as in the proof of Theorem 4.1, (b)⇒(a), one gets

$$u(\omega, x) - u(\omega', y) \leq \sum_{k=1}^{\infty} \frac{c_*((\omega, x), (\omega', y))}{2^k \left(1 + c_*((\omega, x), (\omega', y))\right)} = \frac{c_*((\omega, x), (\omega', y))}{1 + c_*((\omega, x), (\omega', y))}, \tag{24}$$

and as $c_* \leq c \leq d$, the inequality holds

$$u(\omega, x) - u(\omega', y) \leq \frac{d((\omega, x), (\omega', y))}{1 + d((\omega, x), (\omega', y))} \leq d((\omega, x), (\omega', y)) = d_X(x, y);$$

that is $u \in Lip_1(\Omega \times X, d)$ and $u(\omega, \cdot) \in Lip_1(X, d_X)$. It remains to show that $u(\omega, \cdot)$ is a utility function for $\preceq_\omega$. Let $x \preceq_\omega y$, then $c((\omega, x), (\omega, y)) = 0$; therefore,

$$c_*((\omega, x), (\omega, x^k)) - c_*((\omega, y), (\omega, x^k)) \leq c_*((\omega, x), (\omega, y)) = 0,$$

and we get from (23), along with (17), that $u(\omega, x) \leq u(\omega, y)$. Suppose now that $x \prec_\omega y$, then, by Lemma 4.1, $c_*((\omega, y), (\omega, x)) > 0$. Let us chose a subsequence (x^{k_n}) that tends to x, then $\lim_{n\to\infty} d_X(x, x^{k_n}) = \lim_{n\to\infty} d_X(x^{k_n}, x) = 0$, and applying the triangle inequality gives

$$-d_X(x, x^k) \leq -c_*((\omega, x), (\omega, x^k)) \leq c_*((\omega, y), (\omega, x)) - c_*((\omega, y), (\omega, x^k))$$
$$\leq c_*((\omega, x^k), (\omega, x)) \leq d_X(x^k, x).$$

[13] When Ω is separable, the validity of this assumption follows immediately from Theorem 4.1.

It follows that

$$\lim_{n\to\infty} c_*((\omega,x),(\omega,x^{k_n})) = 0, \tag{25}$$

$$\lim_{n\to\infty} c_*((\omega,y),(\omega,x^{k_n})) = c_*((\omega,y),(\omega,x)) > 0, \tag{26}$$

and since $c_*((\omega,x),(\omega,x^k)) \le c_*((\omega,y),(\omega,x^k))$ for all k, we derive from (25),(26), along with (17), that $u(\omega,x) < u(\omega,y)$. □

Remark 4.6 A similar result on a jointly continuous representation for closed preorders $\preceq_\omega$ was obtained earlier in [21, 26]. For the case of closed total preorders, see also [40].

4.2 Preferences admitting smooth utility functions

In this subsection, we deal with preferences on a subset X of $\mathbb{R}^n$. In what follows, $d_X(x,y) = \|x-y\|$ where $\|\cdot\|$ stands for the Euclidean norm.

Theorem 4.3 *Suppose X is closed or open and satisfies $X + \mathbb{R}^n_+ = X$, $\preceq$ is a preference on X satisfying both the strengthened acyclicity assumption and the implication*

$$x \preceq y \Rightarrow (x+z) \preceq (y+z) \quad \forall z \in \mathbb{R}^n_+; \tag{27}$$

then there is a utility function v for $\preceq$ such that $v \in C^\infty(X)$ and $\|\nabla v(x)\| \le 1$ for all $x \in X$. If, in addition, $\preceq$ is monotone, i.e. $x \le y$ implies $x \preceq y$, then v is non-decreasing.

Proof. By Theorem 4.1 and Corollary 4.1, there is a d_X-Lipschitz continuous utility function u for $\preceq$ such that $u(X) \subset (0,1)$ and $u \in Lip_1(X,d_X)$. Taking into account (27), along with the equality $d_X(x,y) = \|x-y\|$, we see that $c(x+z,y+z) \le c(x,y)$ whenever $x,y \in X$, $z \in \mathbb{R}^n_+$. It follows from Remark 2.2 that assumption (4) of Theorem 2.1 is satisfied, and applying the theorem gives us a function $v \in Q(c) \cap C^\infty(X)$ such that $v(X) \subset (0,1)$. Moreover, as follows from the proof of Theorem 2.1, one can take $v = \Phi(u)|_X$. Since by (12), $c(x,y) \le d_X(x,y)$, one has $Q(c) \subset Lip_1(X,d_X)$; therefore, $v \in Lip_1(X,d_X)$, and as v is smooth, this may be rewritten as $\|\nabla v(x)\| \le 1$.

Let us show that v is a utility function for $\preceq$. Suppose that $x \preceq y$, then $x+z \preceq y+z$ for all $z \in \mathbb{R}^n_+$, hence $u(x+z) \le u(y+z)$, and (6) implies $v(x) = \Phi(u)(x) \le \Phi(u)(y) = v(y)$. If now $x \prec y$ then $u(x) < u(y)$, and since u is continuous, there is $\delta > 0$ such that $u(x+z) < u(y+z)$ whenever $z \in \mathbb{R}^n_+$, $\|z\| < \delta$. Let $B = \{z = (z_1,\dots,z_n) \in \mathbb{R}^n_+ : \|z\| < \delta\}$. Taking into account that $u(x+z) \le u(y+z)$ for all $z \in \mathbb{R}^n_+$, it follows

$$\int_B u(x+z)\eta(-z)\,dz_1\dots dz_n < \int_B u(y+z)\eta(-z)\,dz_1\dots dz_n,$$

$$\int_{\mathbb{R}^n_+\setminus B} u(x+z)\eta(-z)\,dz_1\dots dz_n \le \int_{\mathbb{R}^n_+\setminus B} u(y+z)\eta(-z)\,dz_1\dots dz_n,$$

and we deduce from (6) that $v(x) = \Phi(u)(x) < \Phi(u)(y) = v(y)$, that is v is a utility function. Finally, notice that for a monotone preference, non-decrease of an utility function is obvious. □

Theorem 4.4 *Suppose that $X + \mathbb{R}^n_+ = X$, Ω is a metrizable topological space, d is a distance on $\Omega \times X$ such as in Theorem 4.2, for every $\omega \in \Omega$ a preference $\preceq_\omega$ satisfying (27) is given on X, and condition (b) of Theorem 4.2 is fulfilled. Then there exists a function $v : \Omega\times X \to \mathbb{R}$ such that: (i) $v \in Lip_1(\Omega\times X, d)$; (ii) for every $\omega \in \Omega$, $v(\omega,\cdot)$ is a utility function for $\preceq_\omega$, $v(\omega,\cdot) \in C^\infty(X)$, and $\|\nabla_x v(\omega,x)\| \le 1$, $x \in X$.*[14] *If all $\preceq_\omega$ are monotone then all $v(\omega,\cdot)$ are non-decreasing.*

Remark 4.7 A similar result is given in [37] for the case where all $\preceq_\omega$ are preorders.

Proof. By Theorem 4.2, there is a function $u : \Omega \times X \to (0,1)$ such that $u \in Lip_1(\Omega\times X, d)$ and, for every $\omega \in \Omega$, $u(\omega,\cdot)$ belongs to $Lip_1(X, d_X)$ and is a utility function for $\preceq_\omega$. We define a function $v : \Omega \times X \to (0,1)$,

$$v(\omega,x) := \Phi(u(\omega,\cdot))(x) = \int_{\mathbb{R}^n_+} u(\omega, x+z)\eta(-z)\,dz_1\dots dz_n,$$

and arguing as in the proof of Theorem 4.3, convince ourselves that the statement of the theorem holds true. □

Remark 4.8 Both theorems, 4.3 and 4.4, remain true for $d_X(x,y) = \|x-y\|$ where $\|\cdot\|$ is an arbitrary (non-Euclidean) norm, if one replaces $\|\nabla v(x)\| \le 1$ (resp. $\|\nabla_x v(\omega,x)\| \le 1$) with $\|\nabla v(x)\|_* \le 1$ (resp. $\|\nabla_x v(\omega,x)\|_* \le 1$) where $\|\cdot\|_*$ is the dual norm.

4.3 Applications to rational choice problem

Let $\mathcal{M}$ be a family of non-empty subsets in a separable metric space (X,d), and suppose that for every $M \in \mathcal{M}$ a certain non-empty subset $\gamma(M) \subset M$ is chosen.

Definition 4.4 A choice function γ is called *utility-rational* if there is a real-valued function u on X such that, for every $M \in \mathcal{M}$,

$$\gamma(M) = \{x \in M : u(x) = \max_{y\in M} u(y)\}. \tag{28}$$

In such a case, one says that u *rationalizes* γ. If, in addition, u is d-Lipschitz continuous or X is a domain in $\mathbb{R}^n$ and u is smooth (C^r), we will say that γ is *d-Lipschitz-utility-rational* or *C^r-smooth-utility-rational.*

[14] $\nabla_x v(\omega,x) := (\partial v(\omega,x)/\partial x_1, \dots, \partial v(\omega,x)/\partial x_n)$.

Definition 4.5 Say a trajectory $z^0 \to z^1 \to \ldots \to z^m$ in X is *improving* if there are sets $M_i \in \mathcal{M}$, $i = 0, 1, \ldots, m+1$ such that $z^i \in \gamma(M_i) \cap M_{i+1}$, $i = 0, 1, \ldots, m$.

As is well-known, if γ is utility-rational then it satisfies the *strong axiom of revealed preference*: no improving trajectory $z^0 \to z^1 \to \ldots \to z^m$ can exist in X with $z^0 \in \gamma(M)$ and $z^m \in M \setminus \gamma(M)$, where $M_0 = M_{m+1} = M$. The same assumption for $m = 1$ only is known as the *weak axiom of revealed preference*: no $M, M_1 \in \mathcal{M}$, $z^0 \in \gamma(M) \cap M_1$, $z^1 \in (M \setminus \gamma(M)) \cap \gamma(M_1)$ can exist.

Definition 4.6 We say that γ satisfies the *strengthened axiom of revealed preference*, if for every $M \in \mathcal{M}$, every $x \in M \setminus \gamma(M)$, and every $y \in \gamma(M)$, there exists $\delta = \delta(M, x, y) > 0$ such that given a trajectory $y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x^{m+1} = x$ with $x^k \in M_k$, $y^k \in \gamma(M_k)$, $M_k \in \mathcal{M}$, $k = 1, \ldots, m$, one has

$$\sum_{k=1}^{m+1} d(y^{k-1}, x^k) \geq \delta. \tag{29}$$

If (29) holds for $m = 1$ only (that is, $d(y, x^1) + d(y^1, x) \geq \delta$ whenever $M, M_1 \in \mathcal{M}$, $x \in M \setminus \gamma(M)$, $x^1 \in M_1$, $y^1 \in \gamma(M_1)$, $y \in \gamma(M)$), we will say that γ satisfies the *strengthened weak axiom of revealed preference.*

Remark 4.9 Given a trajectory $z^0 \to z^1 \to \ldots \to z^m$ where $z^0 \in \gamma(M), z^m \in M \setminus \gamma(M)$, $z^k \in M_{k+1} \cap \gamma(M_k)$, $k = 0, 1, \ldots, m$, and $M_0 = M_{m+1} = M$, one can associate it with the trajectory $y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x^{m+1} = x$ where $y^0 = x^1 = z^0$, $y^1 = x^2 = z^1, \ldots, y^m = x^{m+1} = z^m = x$. Since $x^k \in M_k$, $y^k \in \gamma(M_k)$ $(k = 1, \ldots, m)$, and $d(y^{k-1}, x^k) = 0$ $(k = 1, \ldots, m+1)$, (29) fails. It follows that the strengthened axiom of revealed preference implies the strong axiom of revealed preference, and the strengthened weak axiom of revealed preference implies the weak axiom of revealed preference.

Lemma 4.4 *The strengthened weak axiom of revealed preference implies that each $\gamma(M)$ is closed in M.*

Proof. Indeed, otherwise there are $x \in M \setminus \gamma(M)$ and a sequence $(z^1, z^2, \ldots) \subset \gamma(M)$ such that $\lim_{k \to \infty} d(x, z^k) = 0$. But this contradicts (29) for $m = 1$, $M_1 = M$, $y = y^0 = x^1 = z^1$, and $y^1 = x^2 = z^k$, where k is large enough. □

Given a choice function γ, we associate it with a preference $\preceq_\gamma$ as follows:

$$x \preceq_\gamma y \Leftrightarrow [x = y \ \text{ or } \exists M \in \mathcal{M} : \ x \in M, \ y \in \gamma(M)].$$

Lemma 4.5 *The strengthened axiom of revealed preference for γ implies the strengthened acyclicity assumption for $\preceq_\gamma$.*

Proof. This follows easily from definitions if one takes into account that $x \prec_\gamma y \Rightarrow [\exists M \in \mathcal{M} : \ x \in M \setminus \gamma(M), \ y \in \gamma(M)]$. $\square$

Lemma 4.6 *Suppose (28) holds, so that γ is utility-rational with a rationalizing function u, then*

$$x \prec_\gamma y \Leftrightarrow [\exists M \in \mathcal{M} : \ x \in M \setminus \gamma(M), \ y \in \gamma(M)],$$

and u is a utility function for $\preceq_\gamma$.

Proof. Clearly $x \preceq_\gamma y \Leftrightarrow u(x) \leq u(y)$. Suppose $x \prec_\gamma y$; then, by the definition of $\preceq_\gamma$, there is a set $M \in \mathcal{M}$ such that $x \in M \setminus \gamma(M)$, $y \in \gamma(M)$. If now $x \in M \setminus \gamma(M)$, $y \in \gamma(M)$ for some $M \in \mathcal{M}$ then, by (28), $u(x) < u(y)$. It follows that no set $M' \in \mathcal{M}$ can exist with the property $y \in M'$, $x \in \gamma(M')$, hence $x \prec_\gamma y$, and u is a utility function for $\preceq_\gamma$. $\square$

The next result is an easy consequence of Theorem 4.1 for the preference $\preceq_\gamma$ if one takes into account Lemmas 4.5 and 4.6.

Theorem 4.5 ([28, Corollary 2]) *The following statements are equivalent: (a) γ is d-Lipschitz-utility-rational; (b) γ satisfies the strengthened axiom of revealed preference.*

Theorem 4.6 *Suppose that X is closed or open in $\mathbb{R}^n$, $X + \mathbb{R}^n_+ = X$, a family $\mathcal{M}$ satisfies*

$$M \in \mathcal{M}, \ z \in \mathbb{R}^n_+ \Rightarrow z + M \in \mathcal{M}, \tag{30}$$

a choice function γ satisfies the strengthened axiom of revealed preference, and

$$\gamma(z + M) \supseteq z + \gamma(M), \quad M \in \mathcal{M}, \ z \in \mathbb{R}^n_+; \tag{31}$$

then there is a function $u \in C^\infty(X)$ that rationalizes γ and satisfies the inequality $\|\nabla u(x)\| \leq 1$, $x \in X$.

Proof. Taking into account that (31) implies (27) for $\preceq_\gamma$, this is a direct consequence of Theorem 4.3 along with Lemma 4.5. $\square$

In the sequel, we will strengthen conditions (30) and (31) as follows:

$$M \in \mathcal{M}, \ z \in \mathbb{R}^n, \ z + M \subset X \Rightarrow z + M \in \mathcal{M}, \tag{32}$$

$$M \in \mathcal{M}, \ z \in \mathbb{R}^n, \ z + M \in \mathcal{M} \Rightarrow \gamma(z + M) = z + \gamma(M). \tag{33}$$

Notice that (33) may be rewritten equivalently as

$$M \in \mathcal{M}, \ z \in \mathbb{R}^n, \ z + M \in \mathcal{M} \Rightarrow \gamma(z + M) \supseteq z + \gamma(M). \tag{34}$$

Indeed, if $M \in \mathcal{M}$ and $z + M \in \mathcal{M}$ then $M = -z + (z + M)$, and (34) implies $\gamma(M) \supseteq -z + \gamma(z + M) \supseteq -z + z + \gamma(M) = \gamma(M)$; therefore, $\gamma(z + M) = z + \gamma(M)$.

Remark 4.10 Suppose $X = \mathbb{R}^n$, $\mathcal{M}$ satisfies (32), and (31) holds with the equality sign, then (33) is satisfied. Indeed, given $M \in \mathcal{M}$ and $z \in \mathbb{R}^n$, we find $z^0 \in \mathbb{R}^n_+$, such that $(z^0 + z) \in \mathbb{R}^n_+$, and take $M_0 = -z^0 + M$. Then $M_0 \in \mathcal{M}$, $M = z^0 + M_0$, $z + M = (z + z^0) + M_0$, and (31) implies $\gamma(M) = z^0 + \gamma(M_0)$, $\gamma(z + M) = z + z^0 + \gamma(M_0) = z + \gamma(M)$.

We now show that if $X = \mathbb{R}^n$ and conditions (32), (33) are satisfied then in the strengthened axiom of revealed preference (see Definition 4.6) it suffices to consider a particular class of trajectories as follows.

Definition 4.7 Given $M \in \mathcal{M}$, $x \in M \setminus \gamma(M)$, and $y \in \gamma(M)$, we say a trajectory $y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x^{m+1} = x$ is *strongly regular* if

$$x^{k+1} = y^k \in M_{k+1} \cap \gamma(M_k), \quad k = 0, 1, \ldots, m-1$$

and $y^m \in \gamma(M_m)$, where $M_1, \ldots, M_m \in \mathcal{M}$ and $M_0 = M$. Clearly, for such a trajectory one has $\sum_1^{m+1} d(y^{k-1}, x^k) = d(y^m, x)$.

Lemma 4.7 *Suppose $X = \mathbb{R}^n$ and conditions (32), (33) are satisfied, the following statements are then equivalent: (a) the strengthened axiom of revealed preference holds; (b) for every $M \in \mathcal{M}$, every $x \in M \setminus \gamma(M)$, and every $y \in \gamma(M)$ there is a number $\delta = \delta(M, x, y) > 0$ such that $d(y^m, x) \geq \delta$ whenever $y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x^{m+1} = x$ is a strongly regular trajectory.*

Proof. It suffices to show that $(b) \Rightarrow (a)$. To this end, take $M \in \mathcal{M}$, $x \in M \setminus \gamma(M)$, $y \in \gamma(M)$, and consider an arbitrary trajectory

$$y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x^{m+1} = x$$

with $x^k \in M_k$, $y^k \in \gamma(M_k)$, $k = 1, \ldots, m$. The result will follow if we find a strongly regular trajectory

$$y = \overline{y}^0 \to \overline{x}^1 \to \overline{y}^1 \to \ldots \to \overline{x}^m \to \overline{y}^m \to x \tag{35}$$

satisfying $\sum\limits_{k=1}^{m+1} d(y^{k-1}, x^k) \geq d(\overline{y}^m, x)$. We define $\overline{y}^k = \overline{x}^{k+1} = z^k$ for $k = 1, \ldots, m-1$ and $\overline{y}^m = z^m$, where $z^0 = y^0 = y$ and $z^k = \sum\limits_{j=0}^{k} y^j - \sum\limits_{j=1}^{k} x^j$, $k = 1, \ldots, m$. Since $x^1 \in M_1$ and $y = y^0 \in \gamma(M)$, we have $z^0 = y^0 = x^1 + (y^0 - x^1) \in \gamma(M) \cap (M_1 + y^0 - x^1)$. Similarly, since $y^k \in \gamma(M_k)$, $z^k = y^k + \sum_{j=1}^{k}(y^{j-1} - x^j) \in$

$\gamma(M_k)+\sum_{j=1}^{k}(y^{j-1}-x^j)$, and since $x^{k+1}\in M_{k+1}$, $z^k=x^{k+1}+\sum_{j=1}^{k+1}(y^{j-1}-x^j)\in M_{k+1}+\sum_{j=1}^{k+1}(y^{j-1}-x^j)$. We derive from (33)

$$z^k\in\gamma(\overline{M}_k)\cap\overline{M}_{k+1},\quad k=0,1,\dots,m-1,$$

where $\overline{M}_0=M$ and

$$\overline{M}_k=M_k+\sum_{j=1}^{k}(y^{j-1}-x^j)\in\mathcal{M},\quad k=1,\dots,m.$$

It follows that trajectory (35) is strongly regular, and since $d(\overline{y}^m,x)=d(z^m,x)=\|\sum\limits_{j=1}^{m+1}(y^{j-1}-x^j)\|\le\sum\limits_{j=1}^{m+1}\|y^{j-1}-x^j\|=\sum\limits_{j=1}^{m+1}d(y^{j-1},x^j)$, the proof is complete. □

Theorem 4.7 *Suppose $X=X+\mathbb{R}^n_+$, conditions (32), (33) are satisfied, and there exist $M_0\in\mathcal{M}$ and $n+1$ points in $\gamma(M_0)$, $y,y^1,y^2,\dots,y^n$, such that the interior of $M_0\setminus\gamma(M_0)$ is non-empty and the vectors $y^1-y,y^2-y,\dots,y^n-y$ are linearly independent; then the strong axiom of revealed preference fails to be true.*

Proof. First suppose that $X=\mathbb{R}^n$. Given $x\in M_0\setminus\gamma(M_0)$, one has a representation $x-y=\sum_{i=1}^{n}\alpha_i(y^i-y)$, where the coefficients α_i are uniquely determined. Moreover, since the interior of $M_0\setminus\gamma(M_0)$ is non-empty, we can find x in $M_0\setminus\gamma(M_0)$ for which all α_i are rational, so that $\alpha_i=\frac{p_i}{q}$ where $q\ge1$ and p_i, $i=1,\dots,n$, are integers. Suppose for the sake of definiteness, that there are both positive and negative p_i[15], and by passing, if needed, to a new enumeration, assume $p_i>0$ for $1\le i\le r$, $p_i<0$ for $r+1\le i\le m$, and $p_i=0$ for $m+1\le i\le n$ where $1\le r<m\le n$. We get

$$x=y+\sum_{i=1}^{m}p_i(y^i-y)+(q-1)(y-x). \tag{36}$$

By letting $p_0:=0$, $M_0(0):=M_0$, we define the sets

$$M_i(1)=\begin{cases}M_{i-1}(p_{i-1})+y^i-y & \text{for } 1\le i\le r,\\ M_{i-1}(|p_{i-1}|)+y-y^i & \text{for } r+1\le i\le m,\\ M_{i-1}(|p_{i-1}|)+y-x & \text{for } i=m+1,\end{cases}$$

and for $j>1$,

$$M_i(j)=\begin{cases}M_i(j-1)+y^i-y & \text{for } 1\le i\le r,\\ M_i(j-1)+y-y^i & \text{for } r<i\le m,\\ M_i(j-1)+y-x & \text{for } i=m+1.\end{cases}$$

[15]The case that all p_i are non-negative or all of them are non-positive is examined similarly.

All these sets are shifts of M_0:

$$M_i(j) = M_0 + z^i(j), \tag{37}$$

where

$$z^1(j) = j(y^1 - y), \quad 1 \le j \le p_1,$$

$$z^i(j) = \begin{cases} \sum_{k<i} p_k(y^k - y) + j(y^i - y) & \text{for } 1 \le j \le p_i,\ 1 < i \le r, \\ \sum_{k<i} p_k(y^k - y) + j(y - y^i) & \text{for } 1 \le j \le |p_i|,\ r+1 \le i \le m, \\ \sum_{k<i} p_k(y^k - y) + j(y - x) & \text{for } 1 \le j \le q-1,\ i = m+1. \end{cases}$$

Taking into account (32), (33), it follows from (37) that

$$\gamma(M_i(j)) = \gamma(M_0) + z^i(j) \quad \forall (i,j). \tag{38}$$

Let us set

$$y^i(j) = y + z^i(j)$$

for $1 \le i \le m$, $1 \le j \le |p_i|$ and for $i = m+1$, $1 \le j \le q-1$. It follows from (38) that $y^i(j) \in \gamma(M_i(j))$ for all i, j, and if $i \le r$ then $y^i(1) = y + z^i(1) = y + z^{i-1}(p_{i-1}) + y^i - y = y^i + z^{i-1}(p_{i-1}) \in \gamma(M_{i-1}(p_{i-1}))$ (for $i = 1$ one has $y^1(1) = y + z^1(1) = y^1 \in \gamma(M_0)$) and, for $1 < j \le p_i$, $y^i(j) = y + z^i(j) = y^i + z^i(j-1) \in \gamma(M_i(j-1))$. Further, since $z^r(p_r) + y - y^{r+1} = z^{r+1}(1)$, we have $y^r(p_r) = y + z^r(p_r) = y^{r+1} + z^{r+1}(1) \in \gamma(M_{r+1}(1))$ hence $y^r(p_r) \in \gamma(M_r(p_r)) \cap \gamma(M_{r+1}(1))$. Similarly, for each i, $r+1 \le i \le m$, $y^i(|p_i|) = y + z^i(|p_i|) = y^{i+1} + z^{i+1}(1) \in \gamma(M_i(|p_i|)) \cap \gamma(M_{i+1}(1))$, and for $1 \le j < |p_i|$, $y^i(j) = y + z^i(j) = y^i + z^i(j) + y - y^i = y^i + z^i(j+1) \in \gamma(M_i(j+1))$ hence $y^i(j) \in \gamma(M_i(j)) \cap \gamma(M_i(j+1))$. Also, taking into account the equalities $z^{m+1}(1) = z^m(|p_m|) + y - x$ and $z^{m+1}(j+1) = z^{m+1}(j) + y - x$ for $1 \le j < q-1$, we get $y^m(|p_m|) = y + z^m(|p_m|) = x + z^{m+1}(1) \in M_0 + z^{m+1}(1) = M_{m+1}(1)$ hence $y^m(|p_m|) \in \gamma(M_m(|p_m|)) \cap M_{m+1}(1)$, and $y^{m+1}(j) = y + z^{m+1}(j) = x + z^{m+1}(j+1) \in \gamma(M_{m+1}(j)) \cap M_{m+1}(j+1)$, $1 \le j < q-1$. Finally, by (38), $y^{m+1}(q-1) \in \gamma(M_{m+1}(q-1))$. We obtain thus a trajectory

$$y \to y^1(1) \to y^1(2) \to \ldots \to y^1(p_1) \to y^2(1) \to y^2(2) \to \ldots$$
$$\to y^2(p_2) \to y^3(1) \to \ldots \to y^r(p_r) \to y^{r+1}(1) \to y^{r+1}(2) \to \ldots$$
$$\to y^{r+1}(|p_{r+1}|) \to y^{r+2}(1) \to \ldots \to y^{r+2}(|p_{r+2}|) \to \ldots$$
$$\to y^m(|p_m|) \to y^{m+1}(1) \to \ldots \to y^{m+1}(q-1),$$

which is improving and leads from y to $y^{m+1}(q-1)$, and since, by (36), $y^{m+1}(q-1) = y + z^{m+1}(q-1) = x$, the strong axiom of revealed preference fails to be true.

Thus, for $X = \mathbb{R}^n$, the proof is complete. If now X is a proper subset in $\mathbb{R}^n$, the result is derived from here by replacing M_0, $M_i(j)$ with their shifts $M_0 + z^*$, $M_i(j) + z^*$ where $z^* \in \mathbb{R}^n_+$ is taken from the condition that all these shifts are contained in X. $\square$

Theorem 4.8 *Suppose that $X = X + \mathbb{R}^n_+$, $\mathcal{M}$ satisfies (32) and consists of closed convex sets, $\gamma(M) \neq M$ for every $M \in \mathcal{M}$, and a choice function γ satisfies both the strengthened weak axiom of revealed preference and condition (33); then, for every $M \in \mathcal{M}$, $\gamma(M)$ is a closed convex subset in the boundary of M, and for each $t \in \mathbb{R}$ an implication holds*

$$[x, y \in \gamma(M), (1-t)x + ty \in M] \Rightarrow (1-t)x + ty \in \gamma(M). \tag{39}$$

Proof. Let us fix a point $x \in M \setminus \gamma(M)$. If $y \in \gamma(M)$ is an interior point of M then there exists $\varepsilon > 0$ such that the ε-ball with center at y is contained in M. Let us fix $0 < \varepsilon < \|x - y\|$ and take $0 < \delta < \frac{\varepsilon}{\|x-y\|}$, $1 - \delta < t < 1$, $x(t) = (1-t)x + ty$, and $M_1 = (x(t) - y) + M$. We have $x(t) - y = (1-t)(x-y)$, hence $\|x(t) - y\| < \varepsilon$ and $x(t) = y + (x(t) - y) \in \gamma(M_1)$. Further, since $\|x(t) - y\| < \varepsilon$, one has $2y - x(t) = y - (x(t) - y) \in M$ and $y = (x(t) - y) + (2y - x(t)) \in M_1$. We get a trajectory $y \to x(t)$ where $y \in \gamma(M)$, $x(t) \in M \setminus \gamma(M)$, and $x(t) \in \gamma(M_1)$, what is a contradiction with the weak axiom of revealed preference. Thus, we have established that each $\gamma(M)$ consists of boundary points. Also, $\gamma(M)$ is closed by Lemma 4.4.

Let us show that $\gamma(M)$ is convex. Suppose that $x, y \in \gamma(M)$ but $x(t) = (1-t)x + ty \notin \gamma(M)$ for some $0 < t < 1$. We consider the straight line passing through x and y, $L = \{(1-t)x + ty : t \in \mathbb{R}\}$, and taking into account that $\gamma(M)$ is closed, we will assume that $x, y \in \gamma(M) \cap L$ are chosen in such a way that $x(t) \notin \gamma(M)$ for all $0 < t < 1$. Fix such a t, which is close to 1, and consider $M_1 = x(t) - y + M$, $x^1 = x(t)$, and $y^1 = x - y + x(t)$. Here, we suppose, without loss of generality, that $M_1 \subset X$, otherwise we find a vector $z \in \mathbb{R}^n_+$ such that $z + x(t) - y \in \mathbb{R}^n_+$, hence $z + x(t) - y + M \subset X$, and consider $z + M$ in place of M. Then $x^1 \in M_1$, $y^1 \in \gamma(M_1)$, and since $d(y, x^1) + d(y^1, x) = \|y - x(t)\| + \|x(t) - y\| = 2(1-t)\|y - x\|$ becomes arbitrarily small when t is close enough to 1, we obtain a contradiction with the strengthened weak axiom of revealed preference.

It remains to prove (39). Since $\gamma(M)$ is closed, the values $t_{\min} = \min\{t \in \mathbb{R} : (1-t)x + ty \in \gamma(M)\} \leq 0$ and $t_{\max} = \max\{t \in \mathbb{R} : (1-t)x + ty \in \gamma(M)\} \geq 1$ are well-defined. Clearly, (39) will be established if we show that $x(t) = (1-t)x(t_{\min}) + tx(t_{\max}) \notin M$ for $t \in \mathbb{R} \setminus [t_{\min}, t_{\max}]$. Suppose $x(t) \in M$ for some $t > t_{\max}$. We can assume that t is close to $t_{\max}$ hence $x(t)$ is close to $x(t_{\max})$. We consider $M_1 = x(t) - x(t_{\max}) + M$ and suppose that $M_1 \subset X$; otherwise we will pass from M to $M + z$ where $z \in \mathbb{R}_+$ is chosen from the condition that $z + x(t) - x(t_{\max}) \in \mathbb{R}_+$. Let us take a trajectory $y^* \to x^1 \to y^1 \to x^*$, where $x^* = x^1 = y^1 = x(t)$, $y^* = x(t_{\max})$, and notice that $x^* \in M \setminus \gamma(M)$, $y^* \in \gamma(M)$, $x^1 \in \gamma(M_1)$, $y^1 \in \gamma(M_1)$. Since $d(y^1, x^*) = 0$ and $d(y^*, x^1) = \|x(t_{\max}) - x(t)\|$ becomes arbitrarily small when t is close enough to $t_{\max}$, we obtain a contradiction with the strengthened weak axiom of revealed preference. Similarly, if $t < t_{\min}$

is close enough to $t_{\min}$ and $x(t) \in M$ then, arguing as above, we again get a contradiction with the strengthened weak axiom of revealed preference for $M_1 = x(t) - x(t_{\min}) + M$, $x^* = x^1 = y^1 = x(t)$, $y^* = x(t_{\min})$. $\square$

As follows from Theorems 4.7 and 4.8, conditions (32), (33) are strong enough, especially in combination with axioms of revealed preference. Let us consider three examples.

Example 4.1 Suppose $\mathcal{M}$ consists of all compact convex sets in X, and $\gamma(M)$ is given by (28), where $u(x) = a \cdot x$ is any non-zero linear function. Obviously (32), (33) are satisfied. Also, by Theorem 4.5, the strengthened axiom of revealed preference is satisfied, too.

Example 4.2 (For the sake of simplicity, we suppose $X = \mathbb{R}^n$.) Fix any $y^* \neq 0$ in X and consider $\mathcal{M} = \{M_* + z : z \in X\}$ and $\gamma(M_* + z) = \gamma(M_*) + z$ for every $z \in X$, where $M_* = \{0, y^*\}$ and $\gamma(M_*) = \{y^*\}$. Obviously, (32) and (33) are satisfied. Let us show that the strengthened axiom of revealed preference is satisfied, too. Suppose $M = M_0 \in \mathcal{M}$, $x \in M \setminus \gamma(M)$, $y \in \gamma(M)$, and let $y = y^0 \to x^1 \to y^1 \to \ldots \to x^m \to y^m \to x$ be a strongly regular trajectory leading from y to x. Then $x^{k+1} = y^k \in M_{k+1} \cap \gamma(M_k)$, $k = 0, 1, \ldots, m-1$, and $y^m \in \gamma(M_m)$ where $M_k = M_* + z^k \in \mathcal{M}$, $k = 0, 1, \ldots, m$. Since $x \in M_0 \setminus \gamma(M_0)$ and $M_* \setminus \gamma(M_*) = \{0\}$, we get $z^0 = x$. Similarly, $y^m \in \gamma(M_m)$ implies $y^m = y^* + z^m$, and $y^k \in M_{k+1} \cap \gamma(M_k)$ for $k = 0, 1, \ldots, m-1$ implies (a) $z^k = y^k - y^*$ and (b) either $y^k = z^{k+1}$ (this is the case $y^k \in M_{k+1} \setminus \gamma(M_{k+1})$) or $y^k = y^* + z^{k+1}$ (this is the case $y^k \in \gamma(M_{k+1})$) . It follows that, for each $k = 0, 1, \ldots, m-1$, $z^{k+1} - z^k$ is equal y^* or 0; therefore, $z^m = z^0 + \sum_{k=0}^{m-1} (z^{k+1} - z^k) = x + m_1 y^*$, and $y^m = y^* + z^m = x + (m_1 + 1)y^*$, where $m_1 \leq m$. Hence $d(y^m, x) \geq (m_1 + 1)\|y^*\| \geq \|y^*\|$ and, by Lemma 4.7, the strengthened axiom of revealed preference is satisfied. (This follows also from Theorem 4.5, if one takes into account that γ is rationalized by any linear function $u(x) = a \cdot x$ satisfying $a \cdot y^* > 0$.)

Example 4.3 Suppose $X = \mathbb{R}^n$, consider the vectors $e^1 = (1, 0, \ldots, 0)$, $e^2 = (0, 1, \ldots, 0), \ldots, e^n = (0, 0, \ldots, 1)$, $x = (x_1, x_2, \ldots, x_n)$, and define $M_0 = \{0, e^1, e^2, \ldots, e^n, x\}$, $\mathcal{M} = \{M_0 + z : z \in \mathbb{R}^n\}$, $\gamma(M_0) = \{0, e^1, e^2, \ldots, e^n\}$, $\gamma(M_0 + z) = \gamma(M_0) + z$. Obviously, (32) and (33) are satisfied. It follows easily from Lemma 4.7 that the strengthened axiom of revealed preference fails if, and only if, for every $\varepsilon > 0$, there exist integers $q \geq 1$, $p_1, \ldots, p_n$ such that $|qx_i + p_i| < \varepsilon$, $i = 1, \ldots, n$.[16]

[16]It seems plausible that this condition is satisfied for all $x \in \mathbb{R}^n$, otherwise one would have an example of a C^∞-smooth-utility-rational choice function γ that satisfies (33) but cannot be rationalized by a linear function.

Theorem 4.9 *Suppose $X = \mathbb{R}^n$, $\mathcal{M}$ satisfies (32), γ satisfies both (33) and the strengthened axiom of revealed preference, and there is a closed domain*[17] *$M \in \mathcal{M}$ with a smooth boundary containing $\gamma(M)$; then for every function $u \in C^\infty(\mathbb{R}^n)$ rationalizing γ and satisfying $\|\nabla u(x)\| \leq 1$*[18] *there exist a vector of unit length $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and a function $\lambda \in C^\infty(\mathbb{R}^n)$, $0 < \lambda \leq 1$, such that*

$$a_i \frac{\partial \lambda(x)}{\partial x_j} = a_j \frac{\partial \lambda(x)}{\partial x_i}, \quad x \in \mathbb{R}^n, \quad i, j \in \{1, \ldots, n\}, \tag{40}$$

and

$$\nabla u(x) = \lambda(x) a, \quad x \in \mathbb{R}^n. \tag{41}$$

Furthermore, a and λ are uniquely determined by u as follows:

$$\lambda(x) = \|\nabla u(x)\|, \quad a = \frac{\nabla u(x)}{\|\nabla u(x)\|} \quad \forall x \in \mathbb{R}^n, \tag{42}$$

and the linear function $u_1(x) = a \cdot x$ rationalizes γ as well.

Proof. Let y^* be any point in $\gamma(M)$, then y^* is a boundary point of M, and since the boundary of M is smooth, the normal to it at y^* is well defined. Moreover, it follows from (33) and (32) that each $x \in \mathbb{R}^n$ belongs to the set $\gamma(M + x - y^*)$ hence is a boundary point of $M + x - y^*$, and the normal to the boundary of $M + x - y^*$ at x does not depend on x. We take a to be the unit normal vector at y^*, which is directed outside M. Every $x \in \mathbb{R}^n$ lies on two smooth surfaces, the boundary of $M + x - y^*$ and the surface $\{z : u(z) = u(x)\}$, and, since u rationalizes γ, both the surfaces have the same tangent hyperplane at x. Then there is a coefficient $\lambda(x) > 0$ such that $\nabla u(x) = \lambda(x) a$. Moreover, since $u \in C^\infty(\mathbb{R}^n)$, $\|\nabla u(x)\| \leq 1$, it follows that $\lambda \in C^\infty(\mathbb{R}^n)$, $\lambda(x) \leq 1$, and (41) is thus established. Notice also that (40) and (42) are direct consequences of (41).

The proof will be completed if we show that an equivalence holds as follows:

$$u(x) \leq u(y) \Leftrightarrow a \cdot x \leq a \cdot y. \tag{43}$$

Notice that if there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\lambda(x) = \frac{dh(t)}{dt}|_{t = a \cdot x}, \quad x \in \mathbb{R}^n, \tag{44}$$

then h is increasing and $u(x) = h(a \cdot x) + const$, therefore, (43) holds. Let us prove the existence of such a function h. We assume, for the sake of definiteness, that

[17] A closed domain is a closed set that coincides with the closure of its interior.

[18] The existence of such functions follows from Theorem 4.6.

$a_1 \neq 0$, and pass from $x = (x_1, \ldots, x_n)$ to new variables $y = (y_1, \ldots, y_n) = Ax$, where $y_1 = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, $y_k = x_k$ $(k = 2, \ldots, n)$. Let $y = Ax$, and $\mu(y) := \lambda(A^{-1}(y)) = \lambda(x)$. One has

$$\frac{\partial \lambda}{\partial x_1} = \frac{\partial \mu}{\partial y_1} a_1, \quad \frac{\partial \lambda}{\partial x_k} = \frac{\partial \mu}{\partial y_1} a_k + \frac{\partial \mu}{\partial y_k} \quad (k = 2, \ldots, n);$$

and by taking into account (40), we derive from here

$$\frac{\partial \mu}{\partial y_k} = \frac{\partial \lambda}{\partial x_k} - \frac{a_k}{a_1} \frac{\partial \lambda}{\partial x_1} = 0 \quad \forall k \neq 1.$$

It follows that $\mu(y)$ depends on y_1 only, i.e. $\lambda(x)$ is a function of $a \cdot x$; then there is a smooth function $\lambda_0 : \mathbb{R} \to (0, 1]$ such that $\lambda(x) = \lambda_0(a \cdot x)$, and (44) is satisfied for h being a primitive function of λ_0. □

Corollary 4.3 *Suppose $X = \mathbb{R}^n$, $\mathcal{M}$ satisfies (32), γ satisfies both (33) and the strengthened axiom of revealed preference, and there is a closed convex set $M_* \in \mathcal{M}$ with non-empty interior and smooth boundary; then there is a linear function rationalizing γ.*

Proof. Taking into account that the interior of M_* is non-empty, the strengthened axiom of revealed preference implies $\gamma(M_*) \neq M_*$. Let $\mathcal{M}_1 := \{M_* + z : z \in \mathbb{R}^n\}$. Since $\mathcal{M}_1 \subset \mathcal{M}$, γ satisfies the strengthened axiom of revealed preference with respect to $\mathcal{M}_1$; then, by Theorem 4.8, the boundary of M_* contains $\gamma(M_*)$, and applying Theorem 4.9 concludes the proof. □

Remark 4.11 The proof of Theorem 4.9 is similar to that of Theorem 3.2 in [37] where the existence of a linear utility function is established for a class of closed total preorders on X. I would like to eliminate an annoying defect slipped in the proof of Theorem 3.2 in [37]. It relates to the boundary of the set $M_\varepsilon(y^*) = \{x \in X : x \preceq y^*\} \cap B_\varepsilon(y^*)$. It is said in [37, p.101] that the boundary of $M_\varepsilon(y^*)$ is $\{x : u(x) = u(y^*)\} \cap B_\varepsilon(y^*)$. Of course, this is nonsense. In fact, the boundary of $M_\varepsilon(y^*)$ consists of two pieces, viz. $\{x : u(x) = u(y^*), \|x - y^*\| < \varepsilon\}$ and $\{x \in M_\varepsilon(y^*) : \|x - y^*\| = \varepsilon\}$. Nevertheless, the theorem is correct and its proof, as given in [37], holds true taking into account that, for $y \in \mathit{int}\, X$, $M_\varepsilon(y)$ is the shift of $M_\varepsilon(y^*)$, $y \in \{x : u(x) = u(y), \|x - y\| < \varepsilon\}$ is a boundary point of $M_\varepsilon(y)$, and the boundary of $M_\varepsilon(y)$ is smooth near y.

4.4 On a class of interval orders admitting smooth representations

Definition 4.8 Recall (see e.g. [2, Chapter 6]) that a binary relation $\succ$ on a set X is called an *interval order* if it is asymmetric ($x \succ y \Rightarrow \neg(y \succ x)$) and satisfies

the following condition:

$$(x \succ y \text{ and } x' \succ y') \Rightarrow (x \succ y' \text{ or } x' \succ y).$$

Given an interval order $\succ$, one can associate it with a preference $\preceq$ as follows: $x \preceq y \Leftrightarrow \neg(x \succ y)$. Since $\succ$ is asymmetric, one has

$$x \succ y \Rightarrow y \preceq x.$$

It follows from Definition 4.8 that $\succ$ is transitive; however generally $\preceq$ is not such.

Definition 4.9 Given an interval order $\succ$, a pair (u, v) of real-valued functions on X is called its *representation* if, for any $x, y \in X$, an equivalence holds as follows:

$$x \succ y \Leftrightarrow u(x) > v(y). \tag{45}$$

Clearly, (45) implies

$$x \preceq y \Leftrightarrow u(x) \leq v(y). \tag{46}$$

In what follows, X is supposed to be a subset in $\mathbb{R}^n$ which is stable with respect to shifts in positive directions and either closed or open. A representation (u, v) of $\succ$ is called *Lebesgue measurable* (resp. *continuous, smooth*) if both the functions, u and v, are Lebesgue measurable (resp. continuous, smooth). In the next theorem we describe a class of interval orders on X for which the existence of Lebesgue measurable and of smooth representations are equivalent.

Theorem 4.10 *Suppose X is closed or open and satisfies $X = X + \mathbb{R}^n_+$, and $\succ$ is an interval order on X such that, for every $z \in \mathbb{R}^n_+$, an equivalence holds as follows:*

$$x \succ y \Leftrightarrow (x + z) \succ (y + z). \tag{47}$$

If $\succ$ has a Lebesgue measurable representation then there is a representation (u, v) such that $u, v \in C^\infty(X)$ and $u(X) \subset (0, 1)$, $v(X) \subset (0, 1)$.

Proof. Let (u_0, v_0) be a Lebesgue measurable representation of $\succ$. Taking into account Lemma 4.3 and passing, if needed, from u_0, v_0 to $\frac{1}{2} + \frac{1}{2}\frac{u_0}{1+|u_0|}$, $\frac{1}{2} + \frac{1}{2}\frac{v_0}{1+|v_0|}$, we will assume that $u_0, v_0 \in \mathcal{L}^\infty(X)$ and $u_0(X) \subset (0, 1)$, $v_0(X) \subset (0, 1)$. Let us define $u = \Phi(u_0)$, $v = \Phi(v_0)$ where Φ is given by (6); then $u, v \in C^\infty(X)$, and $u(X) \subset (0, 1)$, $v(X) \subset (0, 1)$. Furthermore, if $x \succ y$ then, by (47), $(x+z) \succ (y+z)$ for all $z \in \mathbb{R}^n_+$, hence $u_0(x+z) > v_0(y+z)$, and we derive from (6) that $u(x) > v(y)$. If now $x \succ y$ fails then $x \preceq y$, and taking into account (46) along with (47) and (6), it follows that $u(x) \leq v(y)$. It follows that $x \succ y \Leftrightarrow u(x) > v(y)$, and (u, v) is thus a C^∞-smooth representation of $\succ$. $\square$

Remark 4.12 Clearly, (47) is equivalent to

$$x \preceq y \Leftrightarrow (x+z) \preceq (y+z) \quad \forall z \in \mathbb{R}^n_+,$$

which is a strengthening of (27).

Remark 4.13 A sufficient condition for an interval order on a connected separable topological space to have a representation (u, v) such that u is lower semi-continuous and v is upper semi-continuous may be found in [1] (see also [2, Theorem 6.4.5]). A criterion for an interval order on a connected topological space to have a continuous representation is given in [3] (see also [2, Theorem 6.5.5]). By combining Theorem 4.10 with these theorems, one can obtain sufficient conditions for the existence of smooth representations. We omit precise formulations of the corresponding results.

5 Conclusion

In this paper, sufficient conditions were obtained for the existence of a smooth feasible solution to a dual Monge–Kantorovich problem with a fixed marginal difference on a closed or open subset X in $\mathbb{R}^n$ which is stable with respect to shifts in positive directions. Based on the existence theorem, a united method is developed for solving various problems in approximation theory and mathematical economics. Among other things, the following results were obtained.

We have considered a problem of best approximating a bounded continuous function on $X \times X$ by differences of the form $u(x) - u(y)$ and we have given a condition for the existence of a smooth exact solution to it.

We have proved the existence of a smooth utility function for a (generally non-reflexive and non-transitive) preference relation on X satisfying the strengthened acyclicity assumption and the condition that adding one and the same positive vector to each of two comparable alternatives cannot affect the preference relation between them. A similar result (the existence of a smooth joint utility) is obtained for a class of preferences depending on a parameter.

We have studied a general rational choice problem on a family of subsets of X, and we have given conditions for the existence of a linear (or smooth) utility function rationalizing the choice. These conditions are a strengthened axiom of revealed preference and a translation invariance requirement (or some its weakening).

We have proved that for an interval order $\succ$ on X satisfying $x \succ y \Leftrightarrow (x+z) \succ (y+z)$ whenever $z \in \mathbb{R}^n_+$ the existence of a smooth representation and of a Lebesgue measurable one are equivalent.

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