Abstract convexity and the Monge—Kantorovich duality

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Abstract

In the present survey, we reveal links between abstract convex analysis and two variants of the Monge—Kantorovich problem (MKP), with given marginals and with a given marginal difference. It includes: (1) the equivalence of the validity of duality theorems for MKP and appropriate abstract convexity of the corresponding cost functions; (2) a characterization of a (maximal) abstract cyclic monotone map $F : X \to L \subset \mathbb{R}^X$ in terms connected with the constraint set

$$Q_0(\varphi) := \{u \in \mathbb{R}^Z : u(z_1) - u(z_2) \le \varphi(z_1, z_2) \quad \forall z_1, z_2 \in Z = \operatorname{dom} F\}$$

of a particular dual MKP with a given marginal difference and in terms of L-subdifferentials of L-convex functions; (3) optimality criteria for MKP (and Monge problems) in terms of abstract cyclic monotonicity and non-emptiness of the constraint set $Q_0(\varphi)$, where φ is a special cost function on $X \times X$ determined by the original cost function c on $X \times Y$. The Monge—Kantorovich duality is applied then to several problems of mathematical economics relating to utility theory, demand analysis, generalized dynamics optimization models, and economics of corruption, as well as to a best approximation problem.

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1 Introduction

Abstract convexity or convexity without linearity may be defined as a theory which deals with applying methods of convex analysis to non-convex objects. Today this theory becomes an important fragment of non-linear functional analysis, and it has numerous applications in such different fields as non-convex global optimization, various non-traditional duality schemes for particular classes of sets and functions, non-smooth analysis, mass transportation problems, mathematical economics, approximation theory, and measure theory; for history and references, see, e.g., [15], [30], [41], [43], [53], [54] [59], [60], [62]...[1]

In this survey, we'll dwell on connections between abstract convexity and the Monge—Kantorovich mass transportation problems; some applications to mathematical economics and approximation theory will be considered as well.

Let us recall some basic notions relating to abstract convexity. Given a nonempty set Ω and a class H of real-valued functions on it, the H-*convex envelope* of a function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ is defined to be the function $co_H(f)(\omega) := \sup\{h(\omega) : h \in H(f)\}$, $\omega \in \Omega$, where $H(f)$ comprises functions in H majorized by f, $H(f) := \{h \in H : h \leq f\}$. Clearly, $H(f) = H(co_H(f))$. A function f is called H-*convex* if $f = co_H(f)$.

In what follows, we take $\Omega = X \times Y$ or $\Omega = X \times X$, where X and Y are compact topological spaces, and we deal with H being a convex cone or a linear subspace in $C(\Omega)$. The basic examples are $H = \{h_{uv} : h_{uv}(x, y) = u(x) - v(y), (u, v) \in C(X) \times C(Y)\}$ for $\Omega = X \times Y$ and $H = \{h_u : h_u(x, y) = u(x) - u(y), u \in C(X)\}$ for $\Omega = X \times X$. These examples are closely connected with two variants of the Monge—Kantorovich problem (MKP): with given marginals and with a given marginal difference.

Given a cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ and finite positive regular Borel measures, σ_1 on X and σ_2 on Y, $\sigma_1 X = \sigma_2 Y$, the MKP with marginals σ_1 and σ_2 is to minimize the integral

$$\int_{X \times Y} c(x, y)\, \mu(d(x, y))$$

subject to constraints: $\mu \in C(X \times Y)^*_+$, $\pi_1\mu = \sigma_1$, $\pi_2\mu = \sigma_2$, where $\pi_1\mu$ and

[1] Abstract convexity is, in turn, a part of a broader field known as generalized convexity and generalized monotonicity; see [14] and references therein.

$\pi_2\mu$ stand for the marginal measures of μ.[2]

A different variant of MKP, the MKP with a given marginal difference, relates to the case $X = Y$ and consists in minimizing the integral

$$\int_{X\times X} c(x,y)\,\mu(d(x,y))$$

subject to constraints: $\mu \in C(X\times X)^*_+$, $\pi_1\mu - \pi_2\mu = \sigma_1 - \sigma_2$.

Both variants of MKP were first posed and studied by Kantorovich [17, 18] (see also [19, 20, 21]) in the case where $X = Y$ is a metric compact space with its metric as the cost function c. In that case, both variants of MKP are equivalent but, in general, the equivalence fails to be true.

The MKP with given marginals is a relaxation of the Monge 'excavation and embankments' problem [49], a non-linear extremal problem, which is to minimize the integral

$$\int_X c(x, f(x))\,\sigma_1(dx)$$

over the set $\Phi(\sigma_1, \sigma_2)$ of measure-preserving Borel maps $f : (X, \sigma_1) \to (Y, \sigma_2)$. Of course, it can occur that $\Phi(\sigma_1, \sigma_2)$ is empty, but in many cases it is nonempty and the measure μ_f on $X \times Y$,

$$\mu_f B = \sigma_1\{x : (x, f(x)) \in B\}, \quad B \subset X \times Y,$$

is positive and has the marginals $\pi_1\mu_f = \sigma_1$, $\pi_2\mu_f = \sigma_2$. Moreover, if μ_f is an optimal solution to the MKP then f proves to be an optimal solution to the Monge problem.

Both variants of MKP may be treated as problems of infinite linear programming. The dual MKP problem with given marginals is to maximize

$$\int_X u(x)\sigma_1(dx) - \int_Y v(y)\sigma_2(dy)$$

over the set

$$Q'(c) := \{(u,v) \in C(X)\times C(Y) :\ u(x) - v(y) \le c(x,y) \quad \forall (x,y) \in X\times Y\},$$

and the dual MKP problem with a given marginal difference is to maximize

$$\int_X u(x)\,(\sigma_1 - \sigma_2)(dx)$$

[2] For any Borel sets $B_1 \subseteq X$, $B_2 \subseteq Y$, $(\pi_1\mu)(B_1) = \mu(B_1\times Y)$, $(\pi_2\mu)(B_2) = \mu(X\times B_2)$.

over the set

$$Q(c) := \{u \in C(X) : \ u(x) - u(y) \le c(x, y) \quad \forall x, y \in X\}.$$

As is mentioned above, in the classical version of MKP studied by Kantorovich, X is a metric compact space and c is its metric. In that case, $Q(c)$ proves to be the set of Lipschitz continuous functions with the Lipschitz constant 1, and the Kantorovich optimality criterion says that a feasible measure μ is optimal if and only if there exists a function $u \in Q(c)$ such that $u(x) - u(y) = c(x, y)$ whenever the point (x, y) belongs to the support of μ. This criterion implies the duality theorem asserting the equality of optimal values of the original and the dual problems.

Duality for MKP with general continuous cost functions on (not necessarily metrizable) compact spaces is studied since 1974; see papers by Levin [24, 25, 26] and references therein. A general duality theory for arbitrary compact spaces and continuous or discontinuous cost functions was developed by Levin and Milyutin [47]. In that paper, the MKP with a given marginal difference is studied, and, among other results, a complete description of all cost functions, for which the duality relation holds true, is given. Further generalizations (non-compact and non-topological spaces) see [29, 32, 37, 38, 42].

An important role in study and applications of the Monge—Kantorovich duality is played by the set $Q(c)$ and its generalizations such as

$$Q(c; E(X)) := \{u \in E(X) : \ u(x) - u(y) \le c(x, y) \quad \forall x, y \in X\},$$

where $E(X)$ is some class of real-valued functions on X. Typical examples are the classes: $\mathbb{R}^X$ of all real-valued functions on X, $l^\infty(X)$ of bounded real-valued functions on X, $U(X)$ of bounded universally measurable real-valued functions on X, and $\mathcal{L}^\infty(\mathbb{R}^n)$ of bounded Lebesgue measurable real-valued functions on $\mathbb{R}^n$ (Lebesgue equivalent functions are not identified).

Notice that the duality theorems and their applications can be restated in terms of abstract convexity of the corresponding cost functions. In that connection, mention an obvious equality $Q(c; E(X)) = H(c)$ where $H = \{h_u : u \in E(X)\}$. Conditions for $Q(c)$ or $Q_0(c) = Q(c; \mathbb{R}^Z)$ to be nonempty are some kinds of abstract cyclic monotonicity, and for specific cost functions c, they prove to be crucial in various applications of the Monge—Kantorovich duality. Also, optimality criteria for solutions to the MKP with given marginals and to the corresponding Monge problems can be given in terms of non-emptiness of $Q(\varphi)$ where φ is a particular function on $X \times X$ connected with the original cost function c on $X \times Y$.

The paper is organized as follows. Section 2 is devoted to connections between abstract convexity and infinite linear programming problems more general than MKP. In Section 3, both variants of MKP are regarded from the viewpoint of abstract convex analysis (duality theory; abstract cyclic monotonicity and optimality conditions for MKP with given marginals and for a Monge problem; further generalizations). In Section 4, applications to mathematical economics are presented, including utility theory, demand analysis, dynamics optimization, and economics of corruption. Finally, in Section 5 an application to approximation theory is given.

Our goal here is to clarify connections between the Monge - Kantorovich duality and abstract convex analysis rather than to present the corresponding duality results (and their applications) in maximally general form.

2 Abstract convexity and infinite linear programs

Suppose Ω is a compact Hausdorff topological space, and $c : \Omega \to \mathbb{R} \cup \{+\infty\}$ is a bounded from below universally measurable function on it. Given a convex cone $H \subset C(\Omega)$ such that $H(c) = \{h \in H : \ h \le c\}$ is nonempty, and a measure $\mu_0 \in C(\Omega)^*_+$, we consider two infinite linear programs, the original one, I, and the dual one, II, as follows.

The original program is to maximize the linear functional $\langle h, \mu_0 \rangle := \int_\Omega h(\omega)\, \mu_0(d\omega)$ subject to constraints: $h \in H$, $h(\omega) \le c(\omega)$ for all $\omega \in \Omega$. The optimal value of this program will be denoted as $v_I(c; \mu_0)$.

The dual program is to minimize the integral functional

$$c(\mu) := \int_\Omega c(\omega)\, \mu(d\omega)$$

subject to constraints: $\mu \ge 0$ (i.e., $\mu \in C(\Omega)^*_+$) and $\mu \in \mu_0 - H^0$, where H^0 stands for the conjugate (polar) cone in $C(\Omega)^*_+$,

$$H^0 := \{\mu \in C(\Omega)^* : \ \langle h, \mu \rangle \le 0 \ \text{ for all } \ h \in H\}.$$

The optimal value of this program will be denoted as $v_{II}(c; \mu_0)$.

Thus, for any $\mu_0 \in C(\Omega)^*_+$, one has

$$v_I(c;\mu_0) = \sup\{\langle h, \mu_0\rangle :\ h \in H(c)\}, \tag{1}$$

$$v_{II}(c;\mu_0) = \inf\{c(\mu) :\ \mu \geq 0, \mu \in \mu_0 - H^0\}. \tag{2}$$

In what follows, we endow $C(\Omega)^*$ with the weak* topology and consider $v_I(c;\cdot)$ and $v_{II}(c;\cdot)$ as functionals on the whole of $C(\Omega)^*$ by letting $v_I(c;\mu_0) = v_{II}(c;\mu_0) = +\infty$ for $\mu_0 \in C(\Omega)^* \setminus C(\Omega)^*_+$.

Clearly, both functionals are sublinear that is semi-additive and positive homogeneous. Furthermore, it is easily seen that the subdifferential of v_I at 0 is exactly the closure of $H(c)$,

$$\partial v_I(c;0) = \mathrm{cl}\, H(c). \tag{3}$$

Note that

$$v_I(c;\mu_0) \leq v_{II}(c;\mu_0). \tag{4}$$

Also, an easy calculation shows that the conjugate functional $v^*_{II}(c;u) := \sup\{\langle u, \mu_0\rangle - v_{II}(c;\mu_0) :\ \mu_0 \in C(\Omega)^*\}$, $u \in C(\Omega)$, is the indicator function of $\mathrm{cl}\, H(c)$,

$$v^*_{II}(c;u) = \begin{cases} 0, & u \in \mathrm{cl}\, H(c); \\ +\infty, & u \notin \mathrm{cl}\, H(c); \end{cases} \tag{5}$$

therefore, the second conjugate functional $v^{**}_{II}(c;\mu_0) := \sup\{\langle u, \mu_0\rangle - v^*_{II}(c;u) :\ u \in C(\Omega)\}$ is exactly $v_I(c;\mu_0)$,

$$v^{**}_{II}(c;\mu_0) = v_I(c;\mu_0),\ \mu_0 \in C(\Omega)^*. \tag{6}$$

As is known from convex analysis (e.g., see [47] where a more general duality scheme was used), the next result is a direct consequence of (6).

Proposition 2.1 *Given $\mu_0 \in \mathrm{dom}\, v_I(c;\cdot) := \{\mu \in C(\Omega)^*_+ :\ v_I(c;\mu) < +\infty\}$, the following assertions are equivalent:*

(a) $v_I(c;\mu_0) = v_{II}(c;\mu_0)$;

(b) the functional $v_{II}(c;\cdot)$ is weakly lower semi-continuous (lsc) at μ_0.*

Say c is *regular* if it is lsc on Ω and, for every $\mu_0 \in \mathrm{dom}\, v_I(c;\cdot)$,

$$v_{II}(c;\mu_0) = \inf\{c(\mu) :\ \mu \geq 0, \mu \in \mu_0 - H^0,\ \|\mu\| \leq M\|\mu_0\|\}, \tag{7}$$

where $M = M(c;H) > 0$. Note that if $\mu_0 \notin \mathrm{dom}\, v_I(c;\cdot)$ then, by (4), $v_{II}(c;\mu_0) = +\infty$; therefore, for such μ_0, (7) is trivial. Thus, for a regular c, (7) holds true for all $\mu_0 \in C(\Omega)^*$.

Proposition 2.2 *(i) If c is regular, then $v_{II}(c;\cdot)$ is weakly* lsc on $C(\Omega)^*_+$ hence both statements of Proposition 2.1 hold true whenever $\mu_0 \in C(\Omega)^*_+$.*

(ii) If, in addition, $\mu_0 \in \operatorname{dom} v_I(c;\cdot)$ then there exists an optimal solution to program II.

Proof. (i) It suffices to show that for every real number C the Lebesgue sublevel set $L(v_{II}(c;\cdot);C) := \{\mu_0 \in C(\Omega)^*_+ : v_{II}(c;\mu_0) \le C\}$ is weakly* closed. According to the Krein–Shmulian theorem (see [11, Theorem V.5.7]), this is equivalent to that the intersections of $L(v_{II}(c;\cdot);C)$ with the balls $B_{C_1}(C(\Omega)^*) := \{\mu_0 \in C(\Omega)^* : \|\mu_0\| \le C_1\}$, $C_1 > 0$, are weakly* closed. Since c is regular, one has

$$L(v_{II}(c;\cdot);C) \cap B_{C_1}(C(\Omega)^*) = \{\mu_0 : (\mu_0,\mu) \in L'(C,C_1)\}, \tag{8}$$

where

$$\begin{aligned} L'(C,C_1) := \{(\mu_0,\mu) \in C(\Omega)^*_+ \times C(\Omega)^*_+ :\ & \|\mu_0\| \le C_1, \|\mu\| \le M\|\mu_0\|, \\ & c(\mu) \le C, \mu \in \mu_0 - H^0\}. \end{aligned} \tag{9}$$

Note that the functional $\mu \mapsto c(\mu)$ is weakly* lcs on $C(\Omega)^*_+$ because of lower semi-continuity of c as a function on Ω, and it follows from here that $L'(C,C_1)$ is weakly* closed hence weakly* compact in $C(\Omega)^* \times C(\Omega)^*$. Being a projection of $L'(C,C_1)$ onto the first coordinate, the set $L(v_{II}(c;\cdot);C) \cap B_{C_1}(C(\Omega)^*)$ is weakly* compact as well, and the result follows.

(ii) This follows from the weak* compactness of the constraint set of (7) along with the weak* lower semi-continuity of the functional $\mu \mapsto c(\mu)$. $\square$

We say that the *regularity assumption* is satisfied if every H-convex function is regular.

The next result is a direct consequence of Proposition 2.2.

Corollary 2.1 *Suppose the regularity assumption is satisfied, then the duality relation $v_I(c;\mu_0) = v_{II}(c;\mu_0)$ holds true whenever c is H-convex and $\mu_0 \in C(\Omega)^*_+$. If, in addition, $\mu_0 \in \operatorname{dom} v_I(c;\cdot)$, then these optimal values are finite, and there exists an optimal solution to program II.*

We now give three examples of convex cones H, for which the regularity assumption is satisfied. In all the examples, $\Omega = X \times Y$, where X, Y are compact Hausdorff spaces.

Example 2.1 Suppose $H = \{h = h_{uv} : h_{uv}(x,y) = u(x) - v(y),\ u \in C(X),\ v \in C(Y)\}$. Since H is a vector subspace and $\mathbf{1}_\Omega \in H$, one has $\|\mu\| = \langle \mathbf{1}_\Omega, \mu\rangle = \langle \mathbf{1}_\Omega, \mu_0\rangle = \|\mu_0\|$ whenever $\mu \in \mu_0 - H^0$, $\mu \geq 0$, $\mu_0 \geq 0$; therefore, (7) holds with $M = 1$, and the regularity assumption is thus satisfied.

Remark 2.1 . As follows from [42, Theorem 1.4, (b)⇔(c)] (see also [43, Theorem 10.3]), a function $c : \Omega = X \times Y \to \mathbb{R} \cup \{+\infty\}$ is H-convex relative to H from Example 2.1 if and only if it is bounded below and lsc. (Note that, since Ω is compact, every lsc function c is automatically bounded below.)

Example 2.2 Let $X = Y$ and $H = \{h = h_u : h_u(x,y) = u(x) - u(y),\ u \in C(X)\}$, then $H^0 = \{\nu \in C(\Omega)^* : \pi_1\nu - \pi_2\nu = 0\}$, where $\pi_1\nu$ and $\pi_2\nu$ are (signed) Borel measures on X as given by $\langle u, \pi_1\nu\rangle = \int_{X\times X} u(x)\,\nu(d(x,y))$, $\langle u, \pi_2\nu\rangle = \int_{X\times X} u(y)\,\nu(d(x,y))$ for all $u \in C(X)$. Observe that any H-convex function $c : \Omega = X \times X \to \mathbb{R} \cup \{+\infty\}$ is lsc (hence, bounded from below), vanishes on the diagonal ($c(x,x) = 0 \quad \forall x \in X$), and satisfies the triangle inequality $c(x,y) + c(y,z) \geq c(x,z)$ whenever $x, y, z \in X$. Moreover, it follows from [47, Theorem 6.3] that every function with such properties is H-convex. Let $\mu_0, \mu \in C(\Omega)^*_+$ and $\mu \in \mu_0 - H^0$. Then $\nu = \mu - \mu_0 \in -H^0 = H^0$, hence $\pi_1\mu - \pi_2\mu = \pi_1\mu_0 - \pi_2\mu_0$, and (2) is rewritten as

$$v_{II}(c;\mu_0) = \inf\{c(\mu) : \mu \geq 0,\ \pi_1\mu - \pi_2\mu = \pi_1\mu_0 - \pi_2\mu_0\}. \tag{10}$$

Furthermore, since c is lsc, vanishes on the diagonal, and satisfies the triangle inequality, it follows from [47, Theorem 3.1] that (10) is equivalent to

$$v_{II}(c;\mu_0) = \inf\{c(\mu) : \mu \geq 0,\ \pi_1\mu = \pi_1\mu_0,\ \pi_2\mu = \pi_2\mu_0\}. \tag{11}$$

Therefore,

$$\|\mu\| = \langle \mathbf{1}_\Omega, \mu\rangle = \langle \mathbf{1}_X, \pi_1\mu\rangle = \langle \mathbf{1}_X, \pi_1\mu_0\rangle = \langle \mathbf{1}_\Omega, \mu_0\rangle = \|\mu_0\| \tag{12}$$

whenever μ satisfies the constraints of (11); therefore, (7) holds with $M = 1$, and the regularity assumption is thus satisfied.

Example 2.3 Let $X = Y$ and $H = \{h = h_{u\alpha} : h_{u\alpha}(x,y) = u(x) - u(y) - \alpha,\ u \in C(X),\ \alpha \in \mathbb{R}_+\}$, then $(-\mathbf{1}_\Omega) \in H$, and for any $\mu \in \mu_0 - H^0$ one has $\|\mu\| - \|\mu_0\| = \langle \mathbf{1}_\Omega, \mu - \mu_0\rangle \leq 0$. Therefore, (7) holds with $M = 1$, and the regularity assumption is satisfied.

Remark 2.2 . Taking into account Example 2.2, it is easily seen that any function $c : \Omega = X \times X \to \mathbb{R} \cup \{+\infty\}$ of the form $c(x, y) = \varphi(x, y) - \alpha$, where $\alpha \in \mathbb{R}_+, \varphi$ is lsc, vanishes on the diagonal, and satisfies the triangle inequality, is H-convex relative to H from Example 2.3. On the other hand, it is clear that any H-convex function c satisfies the condition $c(x, x) = const \leq 0 \quad \forall x \in X$.

Now suppose that $\mu_0 = \delta_\omega$ is the Dirac measure (delta function) at some point $\omega \in \Omega$, $\langle u, \delta_\omega \rangle := u(\omega)$ whenever $u \in C(\Omega)$. We shall show that in this case some duality results can be established without the regularity assumption.

Observe that for all $\omega \in \Omega$ one has $v_I(c; \delta_\omega) = v_I(co_H(c); \delta_\omega) = co_H(c)(\omega)$.

Proposition 2.3 *Two statements hold as follows:*

(i) If c is H-convex, then the duality relation $v_I(c; \delta_\omega) = v_{II}(c; \delta_\omega)$ is valid whenever $\omega \in \Omega$;

(ii) If, for a given $\omega \in \Omega$, $v_I(c; \delta_\omega) = v_{II}(c; \delta_\omega)$, then $v_I(co_H(c); \delta_\omega) = v_{II}(c; \delta_\omega) = v_{II}(co_H(c); \delta_\omega)$.

Proof. (i) By using the definition of v_I and taking into account that c is H-convex, one gets $v_I(c; \delta_\omega) = co_H(c)(\omega) = c(\omega)$. Further, since $\mu = \delta_\omega$ satisfies constraints of the dual program, it follows that $v_{II}(c; \delta_\omega) \leq c(\omega)$; hence $v_I(c; \delta_\omega) \geq v_{II}(c; \delta_\omega)$, and applying (4) completes the proof.

(ii) Since $c \geq co_H(c)$, it follows that $v_{II}(c; \delta_\omega) \geq v_{II}(co_H(c); \delta_\omega)$; therefore, $v_I(co_H(c); \delta_\omega) = v_I(c; \delta_\omega) = v_{II}(c; \delta_\omega) \geq v_{II}(co_H(c); \delta_\omega)$, and taking into account (4), the result follows. □

Let us define a function

$$c_\#(\omega) := v_{II}(c; \delta_\omega). \tag{13}$$

Clearly, $c_\# \leq c$.

Lemma 2.1 $H(c) = H(c_\#)$.

Proof. If $h \in H(c)$, then, for every $\mu \geq 0, \mu \in \delta_\omega - H^0$, one has $c(\mu) \geq \langle h, \mu \rangle \geq h(\omega)$, hence $c_\#(\omega) = \inf\{c(\mu) : \mu \geq 0, \mu \in \delta_\omega - H^0\} \geq h(\omega)$, that is $h \in H(c_\#)$.

If now $h \in H(c_\#)$, then $h \in H(c)$ because $c_\# \leq c$. □

The next result is a direct consequence of Lemma 2.1.

Corollary 2.2 *For every $\omega \in \Omega$, $c(\omega) \geq c_{\#}(\omega) \geq co_H(c)(\omega)$.*

It follows from Corollary 2.2 that if c is H-convex, then $c_{\#} = c$.

Corollary 2.3 *$c_{\#}$ is H-convex if and only if $c_{\#} = co_H(c)$.*

Proof. If $c_{\#}$ is H-convex, then $c_{\#}(\omega) = \sup\{h(\omega) : h \in H(c_{\#})\}$, and applying Lemma 2.1 yields $c_{\#}(\omega) = \sup\{h(\omega) : h \in H(c)\} = co_H(c)(\omega)$. If $c_{\#}$ fails to be H-convex, then there is a point $\omega \in \Omega$ such that $c_{\#}(\omega) > \sup\{h(\omega) : h \in H(c_{\#})\}$, and applying Lemma 2.1 yields $c_{\#}(\omega) > \sup\{h(\omega) : h \in H(c)\} = co_H(c)(\omega)$. $\square$

Proposition 2.4 *The following statements are equivalent:*

(a) $c_{\#}$ is H-convex;

(b) the duality relation $v_I(c; \delta_\omega) = v_{II}(c; \delta_\omega)$ holds true whenever $\omega \in \Omega$;

(c) for all $\omega \in \operatorname{dom} co_H(c) := \{\omega \in \Omega : co_H(c)(\omega) < +\infty\}$, the functional $v_{II}(c; \cdot)$ is weakly lsc at δ_ω.*

Proof. Taking into account that $v_I(c; \delta_\omega) = co_H(c)(\omega)$, the equivalence $(a) \Leftrightarrow (b)$ is exactly the statement of Corollary 2.3. The equivalence $(b) \Leftrightarrow (c)$ is a particular case of Proposition 2.1. $\square$

We now consider two more general mutually dual linear programs, as follows. Suppose that E, E' is a pair of linear spaces in duality relative to a bilinear form $\langle e, e' \rangle_E$, $e \in E$, $e' \in E'$. We endow them with the corresponding weak topologies: $\sigma(E, E')$ and $\sigma(E', E)$. Given a convex cone K in E, a functional $e'_0 \in E'$, and a weakly continuous (i.e., continuous relative to the weak topology in the Banach space $C(\Omega)$ and the weak topology $\sigma(E, E')$ in E) linear map $A : E \to C(\Omega)$ such that the set $\{e \in K : Ae \leq c\}$ is nonempty, one has to find the optimal values

$$v'_I(c; e'_0) := \sup\{\langle e, e'_0 \rangle_E : e \in K, \, Ae \leq c\}, \tag{14}$$

$$v'_{II}(c; e'_0) := \inf\{c(\mu) : \mu \geq 0, \, A^*\mu \in e'_0 - K^0\}, \tag{15}$$

where K^0 is the convex cone in E' conjugate to K,

$$K^0 := \{e' \in E' : \langle e, e' \rangle_E \leq 0 \quad \text{for all } e \in K\}.$$

Clearly, both the functionals, (14) and (15), are sublinear, and $v'_I(c; e'_0) \leq v'_{II}(c; e'_0)$. Similarly to Proposition 2.1, the next result is a particular case of Lemma 5.1 (see also Remark 1 after it) in [47].

Proposition 2.5 *Given $e'_0 \in \operatorname{dom} v'_I(c;\cdot) := \{e' \in E' : v'_I(c;e') < +\infty\}$, the following assertions are equivalent:*

(a) $v'_I(c;e'_0) = v'_{II}(c;e'_0)$;

(b) the functional $v'_{II}(c;\cdot)$ is weakly lower semi-continuous at e'_0.

Let us define $H := AK$; then $H^0 = (A^*)^{-1}(K^0)$.

Remark 2.3 . Note that if

$$\operatorname{dom} v'_I(c;\cdot) \subseteq A^* C(\Omega)^*_+, \tag{16}$$

then, for every $e'_0 \in \operatorname{dom} v'_I(c;\cdot)$,

$$v'_I(c;e'_0) = v_I(c;\mu_0) \quad \text{and} \quad v'_{II}(c;e'_0) = v_{II}(c;\mu_0)$$

whenever $\mu_0 \in (A^*)^{-1}(e'_0)$. Also note that, for $e'_0 \notin \operatorname{dom} v'_I(c;\cdot)$, one has $v'_I(c;e'_0) = v'_{II}(c;e'_0) = +\infty$.

The next result follows then from Corollary 2.1.

Corollary 2.4 *Suppose the regularity assumption is satisfied. If (16) is valid, then the duality relation $v'_I(c;e'_0) = v'_{II}(c;e'_0)$ holds true whenever c is H-convex and $e'_0 \in E'$. If, in addition, $e'_0 \in \operatorname{dom} v'_I(c;\cdot)$ then there exists an optimal solution to program II.*

3 Abstract convexity and the Monge - Kantorovich problems (MKP)

In this section, we consider two variants of the Monge—Kantorovich problem (MKP), with given marginals and with a given marginal difference. Both the problems are infinite linear programs, and abstract convexity plays important role in their study. Abstract cyclic monotonicity along with optimality criteria for MKP will be studied as well.

3.1 MKP with given marginals

Let X and Y be compact Hausdorff topological spaces[3], σ_1 and σ_2 finite positive regular Borel measures on them, $\sigma_1 X = \sigma_2 Y$, and $c : X \times Y \to$

[3]For the sake of simplicity, we assume X and Y to be compact; however, the corresponding duality theorem (Theorem 3.1 below) holds true for any Hausdorff completely regular spaces; see [42, Theorem 1.4] and [43, Theorem 10.3]. See also [29, Theorem 1].

$\mathbb{R} \cup \{+\infty\}$ an universally measurable function bounded from below. The natural projecting maps of $X \times Y$ onto X and Y will be denoted as π_1 and π_2, respectively.

The MKP with given marginals is to find the optimal value

$$\mathcal{C}(c; \sigma_1, \sigma_2) := \inf\{c(\mu) : \mu \geq 0, \pi_1\mu = \sigma_1, \pi_2\mu = \sigma_2\} \tag{17}$$

where

$$c(\mu) := \int_{X \times Y} c(x, y)\, \mu(d(x, y)), \tag{18}$$

$$\begin{aligned}(\pi_1\mu)B_1 &= \mu\pi_1^{-1}(B_1) = \mu(B_1 \times Y) \text{ for every Borel set } B_1 \subset X,\\ (\pi_2\mu)B_2 &= \mu\pi_2^{-1}(B_2) = \mu(X \times B_2) \text{ for every Borel set } B_2 \subset Y.\end{aligned}$$

The dual problem is to find the optimal value

$$\mathcal{D}(c; \sigma_1, \sigma_2) := \sup\{\langle u, \sigma_1\rangle - \langle v, \sigma_2\rangle : (u, v) \in Q'(c)\}, \tag{19}$$

where

$$Q'(c) = \{(u, v) \in C(X) \times C(Y) : u(x) - v(y) \leq c(x, y),\ (x, y) \in X \times Y\}. \tag{20}$$

Clearly, always

$$\mathcal{D}(c; \sigma_1, \sigma_2) \leq \mathcal{D}'(c; \sigma_1, \sigma_2) \leq \mathcal{C}(c; \sigma_1, \sigma_2), \tag{21}$$

where $\mathcal{D}'(c; \sigma_1, \sigma_2)$ stands for supremum of $\int_X u(x)\, \sigma_1(dx) - \int_Y v(y)\, \sigma_2(dy)$ over all pairs of bounded Borel functions (u, v) satisfying $u(x) - v(y) \leq c(x, y)$ whenever $x \in X, y \in Y$.

Let H be as in Example 2.1, $E := C(X) \times C(Y)$, $E' := C(X)^* \times C(Y)^*$, $\langle e, e'\rangle_E := \langle u, \sigma_1'\rangle - \langle v, \sigma_2'\rangle$ for all $e = (u, v) \in E$, $e' = (\sigma_1', \sigma_2') \in E'$, $K := E$, and $A : E \to C(X \times Y)$ is given by $Ae(x, y) := u(x) - v(y)$, $e = (u, v)$. Clearly, $H = AK$, $Q'(c) = A^{-1}(H(c))$, and

$$\mathcal{C}(c; \sigma_1, \sigma_2) = v'_{II}(c; e'_0), \quad \mathcal{D}(c; \sigma_1, \sigma_2) = v'_I(c; e'_0),$$

where $e'_0 = (\sigma_1, \sigma_2)$, $v'_I(c; e'_0)$ and $v'_{II}(c; e'_0)$ are given by (14) and (15), respectively. Note that (16) is satisfied.

Theorem 3.1 ([42, Theorem 1.4]). *The following statements are equivalent:*

(a) c is H-convex;

(b) c is bounded below and lsc;

(c) the duality relation $\mathcal{C}(c;\sigma_1,\sigma_2) = \mathcal{D}(c;\sigma_1,\sigma_2)$ *holds for all* $\sigma_1 \in C(X)^*_+$, $\sigma_2 \in C(Y)^*_+$.

Moreover, if these equivalent statements hold true then, for any positive measures σ_1, σ_2 *with* $\sigma_1 X = \sigma_2 Y$, *there exists an optimal solution to the MKP with marginals* σ_1, σ_2.

Proof. $(a) \Leftrightarrow (b)$ See Remark 2.1.

$(a) \Rightarrow (c)$ Taking into account Example 2.1, this follows from Corollary 2.4.

$(c) \Rightarrow (a)$ Since $\mu = \delta_{(x,y)}$ is the sole positive measure with marginals $\sigma_1 = \delta_x, \sigma_2 = \delta_y$, one gets $\mathcal{C}(c;\delta_x,\delta_y) = c(x,y)$. Now, taking into account Remark 2.3, we see that $v_I(c;\delta_{(x,y)}) = \mathcal{D}(c;\delta_x,\delta_y)$, $v_{II}(c;\delta_{(x,y)}) = \mathcal{C}(c;\delta_x,\delta_y)$; therefore, $c = c_\#$, and applying Proposition 2.4 completes the proof.

Finally, the latter statement of the theorem is a particular case of the latter assertion of Corollary 2.4. □

3.2 MKP with a given marginal difference

Let X be a compact Hausdorff topological space[4], $\rho \in C(X\times X)^*$ a signed measure satisfying $\rho X := \langle \mathbf{1}_X, \rho\rangle = 0$, and $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ an universally measurable function bounded from below. As before, π_1 and π_2 stand for the projecting maps of $X \times X$ onto the first and the second coordinates, respectively. The corresponding marginals of a measure $\mu \in C(X \times X)^*_+$ are designated as $\pi_1\mu$ and $\pi_2\mu$.

The MKP with a given marginal difference is to find the optimal value

$$\mathcal{A}(c;\rho) := \inf\{c(\mu) : \mu \geq 0,\ \pi_1\mu - \pi_2\mu = \rho\}, \tag{22}$$

where

$$c(\mu) := \int_{X\times X} c(x,y)\,\mu(d(x,y)). \tag{23}$$

The dual problem is to find the optimal value

$$\mathcal{B}(c;\rho) := \sup\{\langle u,\rho\rangle : u \in Q(c)\}, \tag{24}$$

[4]For the sake of simplicity, we assume X to be compact; however, the corresponding duality theorems (Theorems 3.2 and 3.3 below) hold true for more general spaces (in particular, for any Polish space); see [32, Theorems 9.2 and 9.4], [42, Theorem 1.2] and [43, Theorem 10.1 and 10.2].

where

$$Q(c) = \{u \in C(X) : u(x) - u(y) \le c(x,y) \quad \forall x, y \in X\}. \tag{25}$$

(Note that $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho) = +\infty$ when $\rho X \neq 0$.)

Suppose H is as in Example 2.2, $E := C(X)$, $E' := C(X)^*$, $\langle e, e' \rangle_E := \langle u, \sigma \rangle$ for all $e = u \in E$, $e' = \sigma \in E'$, $K := E$, and $A : E \to C(X \times X)$ is given by $Ae(x,y) := u(x) - u(y)$, $e = u$. Clearly, $H = AK$, $Q(c) = A^{-1}(H(c))$ (hence $H(c)$ is nonempty if and only if $Q(c)$ is such), and

$$\mathcal{A}(c;\rho) = v'_{II}(c;e'_0), \quad \mathcal{B}(c;\rho) = v'_I(c;e'_0), \tag{26}$$

where $e'_0 = \rho$, $v'_I(c;e'_0)$ and $v'_{II}(c;e'_0)$ are given by (14) and (15), respectively. Note that

$$\operatorname{dom} v'_I(c;\cdot) = \operatorname{dom} \mathcal{A}(c;\cdot) \subseteq A^* C(X \times X)^*_+ = \{\rho \in C(X)^* : \rho X = 0\}.$$

Let $U(X)$ stands for the class of all bounded universally measurable functions on X,

$$Q(c; U(X)) := \{v \in U(X) : v(x) - v(y) \le c(x,y) \quad \forall (x,y) \in X \times X\}.$$

Theorem 3.2 (cf. [47, Theorems 3.1, 3.2 and 4.4], [42, Theorem 1.2], [43, Theorem 10.1]). *Suppose that c is an universally measurable function vanishing on the diagonal $D = \{(x,x) : x \in X\}$ and satisfying the triangle inequality, the following statements are then equivalent:*

(a) c is H-convex relative to H from Example 2.2, that is $Q(c) \neq \varnothing$ and

$$c(x,y) = \sup\{u(x) - u(y) : \ u \in Q(c)\} \quad \text{for all} \ \ x, y \in X; \tag{27}$$

(b) c is bounded below and lsc;

(c) $Q(c) \neq \varnothing$, and the duality relation $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho)$ holds for all $\rho \in C(X)^*$;

(d) $Q(c;U(X)) \neq \varnothing$, and the duality relation $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho)$ holds for all $\rho \in C(X)^*$, $\rho X = 0$.

Moreover, if these equivalent statements hold, then, for any ρ, $\rho X = 0$, and for any positive measures σ_1, σ_2 with $\sigma_1 - \sigma_2 = \rho$, there is a measure $\mu \in C(X \times X)^*_+$ such that $\pi_1\mu = \sigma_1$, $\pi_2\mu = \sigma_2$ and $\mathcal{A}(c;\rho) = \mathcal{C}(c;\sigma_1,\sigma_2) = c(\mu)$.

Proof. $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious; as for $(b) \Rightarrow (a)$, see Example 2.2. The implication $(a) \Rightarrow (c)$ and the latter statement of the theorem follow from Corollary 2.4 if one takes into account Example 2.2 along with identities (26). The proof will be complete if we show that (d) implies (a). Suppose (27) fails; then

$$c(x_0, y_0) > \sup\{u(x_0) - u(y_0) : u \in Q(c)\} = \mathcal{B}(c; \delta_{x_0} - \delta_{y_0}) \tag{28}$$

for some $x_0, y_0 \in X$ with $x_0 \neq y_0$; hence, $\mathcal{B}(c; \delta_{x_0} - \delta_{y_0}) < +\infty$. We define the function

$$c'(x, y) := \min\{c(x, y) - v(x) + v(y), N\} + v(x) - v(y), \tag{29}$$

where $v \in Q(c; U(X))$,

$$N > \max\{0, \mathcal{B}(c; \delta_{x_0} - \delta_{y_0}) - v(x_0) + v(y_0)\}. \tag{30}$$

Clearly, it is bounded and universally measurable, and $c' \leq c$. Furthermore, c' satisfies the triangle inequality (this is easily derived from non-negativeness of $c(x, y) - v(x) + v(y)$); therefore, $w(x) := c'(x, y_0)$ belongs to $Q(c; U(X))$. Consider

$$\mathcal{B}(c; \rho; U(X)) := \sup\{\langle v', \rho\rangle : v' \in Q(c; U(X))\}$$

and note an obvious inequality

$$\mathcal{A}(c; \rho) \geq \mathcal{B}(c; \rho; U(X)) \quad \forall \rho,\ \rho X = 0. \tag{31}$$

Now, taking into account (29) - (31), one gets

$$\begin{aligned} \mathcal{A}(c; \delta_{x_0} - \delta_{y_0}) &\geq \mathcal{B}(c; \delta_{x_0} - \delta_{y_0}; U(X)) \geq w(x_0) - w(y_0) \\ &= c'(x_0, y_0) > \mathcal{B}(c; \delta_{x_0} - \delta_{y_0}), \end{aligned}$$

which contradicts the duality relation. □

The next Proposition supplements Theorem 3.2.

Proposition 3.1 . *Suppose $c : X \times X \to \mathbb{R}$ is bounded universally measurable, vanishes on the diagonal, and satisfies the triangle inequality, then $Q(c; U(X))$ is nonempty.*

Proof. Let us fix an arbitrary point $y_0 \in X$ and consider the function $v(x) = c(x, y_0)$. Clearly, it is universally measurable, real-valued and bounded, and as c satisfies the triangle inequality, one has $v \in Q(c; U(X))$. □

Remark 3.1 . Suppose that $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ satisfies the triangle inequality and vanishes on the diagonal. As follows from [47, Theorem 3.3[5]], $Q(c; U(X))$ is nonempty if c is Baire measurable or if its Lebesgue sublevel sets $L(c;\alpha) = \{(x, y) \in X \times X : c(x, y) \le \alpha\}$, $\alpha \in \mathbb{R}$, are the results of applying the A-operation to Baire subsets of $X \times X$. (If X is metrizable, the latter means that all $L(c; \alpha)$ are analytic (Souslin).)

Now consider the case where the cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ vanishes on the diagonal but fails to satisfy the triangle inequality, and define the *reduced cost function* c_* associated with it as follows:

$$\begin{aligned} c_*(x,y) : &= \inf_n \inf \left\{ \sum_{i=1}^{n+1} c(x_{i-1}, x_i) : x_i \in X, x_0 = x, x_{n+1} = y \right\} \\ &= \lim_{n\to\infty} \inf \left\{ \sum_{i=1}^{n+1} c(x_{i-1}, x_i) : x_i \in X, x_0 = x, x_{n+1} = y \right\}. \end{aligned} \tag{32}$$

Clearly, $c_* \le c$, c_* satisfies the triangle inequality (we assume, by definition, that $+\infty + (-\infty) = +\infty$), and $H(c_*) = H(c)$; therefore, $Q(c_*) = Q(c)$, and if $Q(c)$ is nonempty, then c_* does not take the value $-\infty$ and is bounded from below. We get

$$\mathcal{B}(c;\rho) = \mathcal{B}(c_*;\rho) \le \mathcal{A}(c_*;\rho) \le \mathcal{A}(c;\rho) \quad \forall \rho \in C(X)^*. \tag{33}$$

Proposition 3.2 *Suppose c_* is universally measurable. If $Q(c)$ is nonempty and $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho)$ for all $\rho \in C(X)^*$, then c_* is H-convex and $c_* = co_H(c) = c_\#$ where $c_\#$ is given by (13). In such a case,*

$$c_*(x, y) = \sup_{u \in Q(c)} (u(x) - u(y)) \quad \text{for all } x, y \in X. \tag{34}$$

Proof. It follows from (33) that $\mathcal{A}(c_*;\rho) = \mathcal{B}(c_*;\rho)$ for all $\rho \in C(X)^*$. Note that c_* vanishes on the diagonal because c vanishes on the diagonal and $Q(c_*) = Q(c) \ne \varnothing$. Now, applying Theorem 3.2 yields H-convexity of c_*, and as $H(c) = H(c_*)$, one gets $c_* = co_H(c)$. Finally, the duality relation $\mathcal{A}(c; \delta_x - \delta_y) = \mathcal{B}(c; \delta_x - \delta_y)$ may be rewritten as $v_I(c; \delta_{(x,y)}) = v_{II}(c; \delta_{(x,y)})$ (see Remark 2.3), and applying Proposition 2.4 and Corollary 2.3 yields $c_\# = co_H(c)$. □

[5] See also [32, Theorem 9.2 (III)], where a more general result is proved.

Remark 3.2 . As is proved in [47, Lemma 4.2], if the Lebesgue sublevel sets of c, $L(c;\alpha) = \{(x,y) \in X \times X : c(x,y) \le \alpha\}$, $\alpha \in \mathbb{R}$, are the results of applying the A-operation to Baire subsets of $X \times X$, then Lebesgue sublevel sets of c_*, $L(c_*;\alpha)$, $\alpha \in \mathbb{R}$, have the same property hence c_* proves to be universally measurable.

The next result is a direct consequence of (33) and Theorem 3.2.

Proposition 3.3 *If c_* is H-convex and $\mathcal{A}(c;\rho) = \mathcal{A}(c_*;\rho)$, then $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho)$.*

Remark 3.3 . As is established in [32, Theorem 9.6] and (for a metrizable case) in [47, Theorem 2.1], a *reduction theorem* is true: if the Lebesgue sublevel sets of c are the results of applying the A-operation to Baire subsets of $X \times X$, then $\mathcal{A}(c;\rho) = \mathcal{A}(c_*;\rho)$ provided that the equality holds

$$\mathcal{A}(c;\rho) = \lim_{N\to\infty} \mathcal{A}(c \wedge N;\rho), \tag{35}$$

where $(c \wedge N)(x,y) = \min\{c(x,y), N\}$. (Note that, for a bounded c, (35) is trivial.)

Taking into account Remarks 3.2 and 3.3, the next result is derived from Propositions 3.2, 3.3 and the reduction theorem.

Theorem 3.3 (cf. [47, Theorems 3.1 and 3.2], [32, Theorem 9.4], [43, Theorem 10.2]). *Suppose that c is bounded from below and vanishes on the diagonal, and that its sublevel sets are the results of applying the A-operation to Baire subsets of $X \times X$. The following statements are then equivalent:*

(a) the reduced cost function c_* is H-convex, and condition (35) is satisfied whenever $\rho X = 0$;

(b) $Q(c)$ is nonempty, and the duality relation $\mathcal{A}(c;\rho) = \mathcal{B}(c;\rho)$ holds for all $\rho \in C(X)^*$.

Proof. $(a) \Rightarrow (b)$ Taking into account the reduction theorem (see Remark 3.3), this follows from Proposition 3.3.

$(b) \Rightarrow (a)$ In accordance with Remark 3.2, c_* is universally measurable; then, by Proposition 3.2, it is H-convex. It remains to show that (35) is satisfied. First, note that, being a bounded function, every $u \in Q(c)$ belongs to $Q(c \wedge N)$, where $N = N(u) > 0$ is large enough; therefore,

$$\mathcal{B}(c;\rho) = \lim_{N\uparrow\infty} \mathcal{B}(c \wedge N;\rho).$$

Now, by using the monotonicity of $\mathcal{A}(c;\rho)$ in c, one gets

$$\begin{aligned}\mathcal{A}(c;\rho) &\geq \limsup_{N\uparrow\infty}\mathcal{A}(c\wedge N;\rho)\geq \liminf_{N\uparrow\infty}\mathcal{A}(c\wedge N;\rho)\\ &\geq \lim_{N\uparrow\infty}\mathcal{B}(c\wedge N;\rho)=\mathcal{B}(c;\rho),\end{aligned}$$

which clearly implies (35). $\square$

Corollary 3.1 *Suppose c is Baire measurable, bounded from below, and vanishes on the diagonal. Then c_* is H-convex if and only if $Q(c)$ is nonempty and $\mathcal{A}(c;\rho)=\mathcal{B}(c;\rho)$ for all $\rho\in C(X)^*$.*

3.3 A connection between two variants of MKP

Given compact Hausdorff topological spaces X and Y, we define $X\oplus Y$ to be the formal union $X\cup Y$ of disjoint copies of X and Y with the topology of direct sum: by definition, a set G is open in $X\oplus Y$ if $G\cap X$ is open in X and $G\cap Y$ is open in Y. Clearly, $X\oplus Y$ is compact, both X and Y are open-closed in it, and $C(X\oplus Y)=C(X)\times C(Y)$. Furthermore, $C(X\oplus Y)^*=C(X)^*\times C(Y)^*$, that is a pair $(\sigma_1,\sigma_2)\in C(X)^*\times C(Y)^*$ is identified with a measure $\hat{\sigma}\in C(X\oplus Y)^*$,

$$\hat{\sigma}B=\sigma_1(B\cap X)+\sigma_2(B\cap Y) \text{ for any Borel } B\subseteq X\oplus Y,$$

and every $\hat{\sigma}\in C(X\oplus Y)^*$ is obtained in such a way. We shall write this as $\hat{\sigma}=(\sigma_1,\sigma_2)$.

Given $\sigma_1\in C(X)^*_+$ and $\sigma_2\in C(Y)^*_+$, we associate them with the measures $\hat{\sigma}_1=(\sigma_1,0),\hat{\sigma}_2=(0,\sigma_2)\in C(X\oplus Y)^*_+$. Similarly, every $\mu\in C(X\times Y)^*_+$ is associated with the measure $\hat{\mu}\in C((X\oplus Y)\times(X\oplus Y))^*_+$,

$$\hat{\mu}B:=\mu(B\cap(X\times Y)) \text{ for any Borel } B\subseteq(X\oplus Y)\times(X\oplus Y). \tag{36}$$

Given a cost function $c:X\times Y\to\mathbb{R}\cup\{+\infty\}$, every pair $(u,v)\in Q'(c)$ is identified with a function $w\in C(X\oplus Y)$, $w|X=u$, $w|Y=v$, which belongs to $Q(\hat{c})$ for

$$\hat{c}(z,z'):=\sup\{w(z)-w(z'):\ w=(u,v)\in Q'(c)\},\ z,z'\in X\oplus Y, \tag{37}$$

where $Q'(c)\subset C(X\oplus Y)$ is defined as in (20). Clearly, $\hat{c}$ is lsc, vanishes on the diagonal, and satisfies the triangle inequality, c majorizes the restriction of $\hat{c}$ onto $X\times Y$, and $Q(\hat{c})=Q'(c)$. Note that if c coincides with the restriction of $\hat{c}$ onto $X\times Y$ then $\mathcal{C}(c;\sigma_1,\sigma_2)=\mathcal{C}(\hat{c};\hat{\sigma}_1,\hat{\sigma}_2)$.

Proposition 3.4 (cf. [42, Theorem 1.5] and [26, Lemma 7]). *I. Given a cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent:*

(a) c is H-convex relative to H from Example 2.1;

(b) c is the restriction to $X \times Y$ of a function $\hat{c}$ on $(X \oplus Y) \times (X \oplus Y)$, which is H-convex relative to $H \subset C((X \oplus Y) \times (X \oplus Y))$ from Example 2.2.

If these equivalent statements hold, then $Q(\hat{c}) = Q'(c)$ and

$$\mathcal{A}(\hat{c}; \hat{\sigma}_1 - \hat{\sigma}_2) = \mathcal{B}(\hat{c}; \hat{\sigma}_1 - \hat{\sigma}_2) = \mathcal{C}(c; \sigma_1, \sigma_2) = \mathcal{D}(c; \sigma_1, \sigma_2) > -\infty$$

whenever $\sigma_1 \in C(X)^*_+, \sigma_2 \in C(Y)^*_+$, $\sigma_1 X = \sigma_2 Y$.

II. If $c \in C(X \times Y)$, then there is a continuous function $\hat{c}$ satisfying (b).

Proof. *I.* This follows easily from Theorems 3.1 and 3.2 if one takes $\hat{c}$ as given by (37).

II. Define $\hat{c}$ as follows:

$$\hat{c}(z_1, z_2) = \begin{cases} c(x, y), & \text{if } z_1 = x \in X, \quad z_2 = y \in Y; \\ c_1(x_1, x_2), & \text{if } z_1 = x_1 \in X, \quad z_2 = x_2 \in X; \\ c_2(y_1, y_2), & \text{if } z_1 = y_1 \in Y, \quad z_2 = y_2 \in Y; \\ c_3(y, x), & \text{if } z_1 = y \in Y, \quad z_2 = x \in X, \end{cases} \tag{38}$$

where

$$\begin{aligned} c_1(x_1, x_2) &= \max_{y \in Y} \left(c(x_1, y) - c(x_2, y) \right), \\ c_2(y_1, y_2) &= \max_{x \in X} \left(c(x, y_2) - c(x, y_1) \right), \\ c_3(y, x) &= \max_{x_1 \in X, y_1 \in Y} \left(c(x_1, y_1) - c(x_1, y) - c(x, y_1) \right). \end{aligned}$$

Clearly, $\hat{c}$ is continuous, vanishes on the diagonal, and $\hat{c}|_{X \times Y} = c$. Moreover, a direct testing shows that it satisfies the triangle inequality. Then $\hat{c}$ is H-convex with respect to H from Example 2.2, and the result follows. □

The next result supplements Theorem 3.1.

Proposition 3.5 (cf. [26, Theorem 5]). *Suppose $c \in C(X \times Y)$, $\sigma_1 \in C(X)^*_+$, $\sigma_2 \in C(Y)^*_+$, and $\sigma_1 X = \sigma_2 Y$, then there is an optimal solution $(u, v) \in C(X) \times C(Y)$ to the dual MKP, that is, $(u, v) \in Q'(c)$ and*

$$\int_X u(x)\, \sigma_1(dx) - \int_Y v(y)\, \sigma_2(dy) = \mathcal{D}(c; \sigma_1, \sigma_2).$$

Proof. Take a function $\hat{c} \in C((X \oplus Y) \times (X \oplus Y))$ from Proposition 3.4, II (see (38)) and fix arbitrarily a point $z_0 \in X \oplus Y$. Since $\hat{c}$ is continuous and vanishes on the diagonal, the set

$$Q(\hat{c}; z_0) := \{w \in Q(\hat{c}) :\, w(z_0) = 0\}$$

is compact in $C(X \oplus Y)$ and there exists a function $w_0 = (u, v) \in Q(\hat{c}; z_0)$ such that $\langle w_0, \hat{\sigma}_1 - \hat{\sigma}_2 \rangle = \max\{\langle w, \hat{\sigma}_1 - \hat{\sigma}_2 \rangle :\, w \in Q(\hat{c}; z_0)\}$. Now, taking into account an obvious equality $Q(\hat{c}) = Q(\hat{c}; z_0) + \mathbb{R}$ and applying Proposition 3.4, one gets

$$\int_X u(x)\, \sigma_1(dx) - \int_Y v(y)\, \sigma_2(dy) = \langle w_0, \hat{\sigma}_1 - \hat{\sigma}_2 \rangle$$

$$= \max\{\langle w, \hat{\sigma}_1 - \hat{\sigma}_2 \rangle :\, w \in Q(\hat{c})\} = \mathcal{B}(\hat{c}; \hat{\sigma}_1 - \hat{\sigma}_2) = \mathcal{D}(c; \sigma_1, \sigma_2). \qquad \square$$

3.4 Abstract cyclic monotonicity and optimality conditions for MKP

Given a set X and a subset L in $\mathbb{R}^X$, a multifunction $F : X \to L$ is called *L-cyclic monotone* if, for every cycle $x_1, \ldots, x_m, x_{m+1} = x_1$ in $\operatorname{dom} F = \{x \in X :\, F(x) \neq \varnothing\}$, the inequality holds

$$\sum_{k=1}^{m} (l_k(x_k) - l_k(x_{k+1})) \geq 0 \tag{39}$$

whenever $l_k \in F(x_k)$, $k = 1, \ldots, m$. By changing the sign of this inequality, one obtains the definition of *L-cyclic antimonotone* multifunction. Clearly, F is L-cyclic monotone if and only if $-F$ is $(-L)$-cyclic antimonotone.

We say a function $U : X \to \mathbb{R} \cup \{+\infty\}$ is *L-convex* if it is H-convex relative to

$$H := \{h_{l\alpha} :\, h_{l\alpha}(x) = l(x) - \alpha,\, (l, \alpha) \in L \times \mathbb{R}\}. \tag{40}$$

A function $V : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *L-concave* if $U = -V$ is $(-L)$-convex.

Examples of L-cyclic monotone multifunctions are *L-subdifferentials* of L-convex functions, $\partial_L U : X \to L$, where

$$\partial_L U(x) := \{l \in L :\, l(z) - l(x) \leq U(z) - U(x) \quad \text{for all } z \in X\}. \tag{41}$$

Similarly, examples of L-cyclic antimonotone multifunctions are *L-superdifferentials* of L-concave functions, $\partial^L V : X \to L$, where

$$\partial^L V(x) := \{l \in L : \, l(z) - l(x) \geq V(z) - V(x) \quad \text{for all } z \in X\}. \tag{42}$$

It is obvious from the above definitions that, for every L-concave V,

$$\partial^L V = \partial_{(-L)}(-V). \tag{43}$$

Remark 3.4 . The classic monotone (antimonotone) multifunctions can be considered as examples of L-cyclic monotone (resp., antimonotone) ones, answering the case where X is a Hausdorff locally convex space, $L = X^*$ is the dual space, and $l(x) = \langle x, l \rangle$, $x \in X$, $l \in L$. Close notions of c-monotonicity (c-antimonotonicity) and of c-subdifferentials of c-convex functions (c-superdifferentials of c-concave functions) are widespread in literature; e.g., see [12, 55, 63]. A connection between the corresponding L-concepts and c-concepts is discussed in [40].

Given a multifunction $F : X \to L$, we denote $Z = \operatorname{dom} F := \{z \in X : F(z) \neq \varnothing\}$ and consider two functions $Z \times Z \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\varphi_F(z_1, z_2) = \varphi_{F,L}(z_1, z_2) := \inf\{l(z_1) - l(z_2) : l \in F(z_1)\}, \tag{44}$$

$$\psi_F(z_1, z_2) = \psi_{F,L}(z_1, z_2) := \inf\{l(z_1) - l(z_2) : l \in F(z_2)\}. \tag{45}$$

Clearly, $\psi_{F,L}(z_1, z_2) = \varphi_{(-F),(-L)}(z_2, z_1)$.

Remark 3.5 . Note that if $\sup\limits_{l \in L} |l(z)| < \infty$ for every $z \in Z$, then both the functions are real-valued.

Given a function $\zeta : Z \times Z \to \mathbb{R} \cup \{-\infty\}$ vanishing on the diagonal ($\zeta(z, z) = 0 \quad \forall z \in Z$), we consider the set

$$Q_0(\zeta) := \{u \in \mathbb{R}^Z : \, u(z_1) - u(z_2) \leq \zeta(z_1, z_2) \quad \forall z_1, z_2 \in Z\}. \tag{46}$$

It follows from (46) that if $Q_0(\zeta)$ is nonempty, then ζ is real-valued. Clearly, $Q_0(\zeta) = Q_0(\zeta_*)$ where ζ_* is the reduced cost function associated with ζ (for the definition of the reduced cost function, see (32)). Also, observe that if Z is a topological space and ζ is a bounded continuous function on $Z \times Z$ vanishing on the diagonal, then $Q_0(\zeta) = Q(\zeta)$. (Here, $Q(\zeta)$ is defined for a compact Z as in (27), and if Z is not compact, we define $Q(\zeta)$ to be the set of all *bounded* continuous functions u satisfying (46).)

Theorem 3.4 ([40, Theorem 2.1]). *A multifunction $F : X \to L$ is L-cyclic monotone if and only if $Q_0(\varphi_F)$ is nonempty.*

Theorem 3.5 ([40, Theorem 2.2]). *Suppose $F : X \to L$ is L-cyclic monotone. Given a function $u : Z = \operatorname{dom} F \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent:*

(a) $u \in Q_0(\varphi_F)$;

(b) u is a restriction to Z of some L-convex function $U : X \to \mathbb{R} \cup \{+\infty\}$, and $F(z) \subseteq \partial_L U(z)$ for all $z \in Z$.

The next result extending a classical convex analysis theorem due to Rockafellar is an immediate consequence of Theorems 3.4 and 3.5.

Corollary 3.2 ([39], [40], [53][6]). *A multifunction $F : X \to L$ is L-cyclic monotone if and only if there is a L-convex function $U : X \to \mathbb{R} \cup \{+\infty\}$ such that $F(x) \subseteq \partial_L U(x)$ for all $x \in X$.*

Suppose $F : X \to L$ is a L-cyclic monotone multifunction. We say F is *maximal L-cyclic monotone* if $F = T$ for any L-cyclic monotone multifunction T such that $F(x) \subseteq T(x)$ whenever $x \in X$.

Theorem 3.6 ([40, Theorem 2.3]). *A multifunction $F : X \to L$ is maximal L-cyclic monotone if and only if $F = \partial_L U$ for all L-convex functions $U : X \to \mathbb{R} \cup \{+\infty\}$ satisfying $U|\operatorname{dom} F \in Q_0(\varphi_F)$.*

Remark 3.6 . Theorem 3.6 is an abstract version of the corresponding classical result due to Rockafellar [58]. In classical setting, X is a Hausdorff locally convex space, $L = X^*$ is the conjugate space, and $l(x) = \langle x, l \rangle$. In this case, L-convex functions, their L-subdifferentials, and L-cyclic monotone multifunctions are, respectively, convex lsc functions, their subdifferentials, and classical cyclic monotone multifunction $X \to X^*$. Rockafellar's theorem says that maximal cyclic monotone multifunctions are exactly the subdifferentials of lsc convex functions, and if U_1 and U_2 are two such functions with $\partial U_1 = \partial U_2$, then $U_1 - U_2$ is a constant function. However, in general case both these assertions fail: there is a L-convex function, for which $\partial_L U$ is not maximal, and there are two L-convex functions, U_1 and U_2, such that

[6] See also [4, 12, 56, 61, 63], where close abstract results related to c-cyclic monotonicity and c-subdifferentials of c-convex functions (c-cyclic antimonotonicity and c-superdifferentials of c-concave functions) may be found.

the multifunction $F = \partial_L U_1 = \partial_L U_2$ is maximal L-cyclic monotone but the difference $U_1 - U_2$ is not constant. The corresponding counter-example can be seen in [40, Example 2.1].

Let X, Y be compact Hausdorff topological spaces. Given a cost function $c \in C(X \times Y)$, we consider the MKP with marginals σ_1 and σ_2, $\sigma_1 X = \sigma_2 Y$. Recall (see subsection 3.1), that it is to find the optimal value

$$\mathcal{C}(c; \sigma_1, \sigma_2) = \inf\{c(\mu) : \mu \in \Gamma(\sigma_1, \sigma_2)\},$$

where $c(\mu)$ is given by (18),

$$\Gamma(\sigma_1, \sigma_2) = \{\mu \in C(X \times Y)^*_+ : \pi_1 \mu = \sigma_1, \pi_2 \mu = \sigma_2\}.$$

We consider the set of real-valued functions on X,

$$L := \{-c(\cdot, y) : y \in \operatorname{spt} \sigma_2\}. \tag{47}$$

where the symbol spt stands for the support of the corresponding measure. Every $\mu \in \Gamma(\sigma_1, \sigma_2)$ can be associated with the multifunction $F_\mu : X \to L$,

$$F_\mu(x) := \{-c(\cdot, y) : (x, y) \in \operatorname{spt} \mu\}. \tag{48}$$

(F_μ is well-defined because the projection of (a compact set) $\operatorname{spt} \mu$ onto Y is exactly $\operatorname{spt} \sigma_2$.)

Note that for $F = F_\mu$ function (44) takes the form

$$\varphi_{F_\mu}(z_1, z_2) = \inf_{y:(z_1,y)\in \operatorname{spt}\mu} (c(z_2, y) - c(z_1, y)). \tag{49}$$

Furthermore, since c is continuous and $\operatorname{spt} \mu$ is compact, infimum in (49) is attained whenever $z_1 \in Z = \operatorname{dom} F_\mu = \pi_1(\operatorname{spt} \mu) = \operatorname{spt} \sigma_1$, and the function φ_{F_μ} is continuous and vanishes on the diagonal in $Z \times Z$; therefore, $Q_0(\varphi_{F_\mu}) = Q(\varphi_{F_\mu})$.

Theorem 3.7 (cf. [40, Theorem 5.1] and [44, Theorem 2.1]). *Given a measure $\mu \in \Gamma(\sigma_1, \sigma_2)$, the following statements are equivalent:*

(a) μ is an optimal solution to the MKP, that is $c(\mu) = \mathcal{C}(c; \sigma_1, \sigma_2)$;

(b) the set $Q_0(\varphi_{F_\mu}) = Q(\varphi_{F_\mu})$ is nonempty;

(c) F_μ is L-cyclic monotone.

Proof. $(a) \Rightarrow (b)$ By Proposition 3.5, there is an optimal solution (u, v) to the dual MKP; therefore,

$$\mathcal{D}(c; \sigma_1, \sigma_2) = \int_X u(x)\,\sigma_1(dx) - \int_Y v(y)\,\sigma_2(dy), \tag{50}$$

and taking into account the duality relation $\mathcal{C}(c; \sigma_1, \sigma_2) = \mathcal{D}(c; \sigma_1, \sigma_2)$ (see Theorem 3.1), (50) can be rewritten as

$$c(\mu) = \int_X u(x)\,\sigma_1(dx) - \int_Y v(y)\,\sigma_2(dy). \tag{51}$$

Furthermore, since $\pi_1\mu = \sigma_1$, $\pi_2\mu = \sigma_2$, and $(u, v) \in Q'(c)$, (51) implies

$$u(x) - v(y) = c(x, y) \text{ whenever } (x, y) \in \operatorname{spt}\mu. \tag{52}$$

Note that $\pi_1(\operatorname{spt}\mu)$ is closed as the projection of a compact set; therefore, $Z = \operatorname{dom} F_\mu = \pi_1(\operatorname{spt}\mu) = \operatorname{spt}\sigma_1$, and (52) means

$$u(z) - v(y) = c(z, y) \text{ whenever } z \in Z,\ l = -c(\cdot, y) \in F_\mu(z). \tag{53}$$

Now, given any $z, z' \in Z$, and $l \in F_\mu(z) = \{-c(\cdot, y) : (z, y) \in \operatorname{spt}\mu\}$, we derive from (53) $u(z') - u(z) = u(z') - c(z, y) - v(y) \le c(z', y) - c(z, y)$, and taking infimum over all y with $(z, y) \in \operatorname{spt}\mu$, yields $u(z') - u(z) \le \varphi_{F_\mu}(z, z')$ hence $(-u) \in Q(\varphi_{F_\mu})$.

$(b) \Rightarrow (a)$ Since every measure from $\Gamma(\sigma_1, \sigma_2)$ vanishes outside the set $\operatorname{spt}\sigma_1 \times \operatorname{spt}\sigma_2$, one can consider μ as a measure on $X_\mu \times Y_\mu$ (instead of $X \times Y$), where $X_\mu = \operatorname{spt}\sigma_1$, $Y_\mu = \operatorname{spt}\sigma_2$. It suffices to show that μ is an optimal solution to the MKP on $X_\mu \times Y_\mu$.

Note that $u \in Q(\varphi_{F_\mu})$ means

$$u(z_1) - u(z_2) \le c(z_2, y) - c(z_1, y) \tag{54}$$

whenever $(z_1, y) \in \operatorname{spt}\mu$. Let us define

$$v(y) := -\inf_{z:(z,y)\in\operatorname{spt}\mu} (u(z) + c(z, y)),\ y \in Y_\mu. \tag{55}$$

Since $\operatorname{spt}\mu$ is compact and u, c are continuous, the infimum in the right-hand side of (55) is attained and v proves to be a bounded lsc function on Y_μ. Moreover, it follows from (54) that

$$-u(z) - v(y) \le c(z, y) \quad \forall (z, y) \in X_\mu \times Y_\mu \tag{56}$$

and

$$-u(z) - v(y) = c(z,y) \quad \forall (z,y) \in \operatorname{spt}\mu. \tag{57}$$

Note now that (57) implies

$$\int_{X_\mu} (-u)(x)\,\sigma_1(dx) - \int_{Y_\mu} v(y)\,\sigma_2(dy) = c(\mu). \tag{58}$$

We derive from (58) that $\mathcal{D}'(c;\sigma_1,\sigma_2) \geq c(\mu) \geq \mathcal{C}(c;\sigma_1,\sigma_2)$, and as always $\mathcal{D}'(c;\sigma_1,\sigma_2) \leq \mathcal{C}(c;\sigma_1,\sigma_2)$ (see (21)), μ is optimal.

$(b) \Leftrightarrow (c)$ This is a particular case of Theorem 3.4. $\square$

Remark 3.7 . A different proof of a similar theorem is given in [40, Theorem 5.1] and [44, Theorem 2.1], where non-compact spaces are considered. A close result saying that optimality of μ and c-cyclic antimonotonicity of $\operatorname{spt}\mu$ are equivalent may be found in [12].

We now turn to the Monge problem. Recall (see Introduction) that it is to minimize the functional

$$\mathcal{F}(f) := \int_X c(x, f(x))\,\sigma_1(dx)$$

over the set $\Phi(\sigma_1,\sigma_2)$ of measure-preserving Borel maps $f : (X,\sigma_1) \to (Y,\sigma_2)$. (A map f is called measure-preserving if $f(\sigma_1) = \sigma_2$, that is $\sigma_1 f^{-1}(B_Y) = \sigma_2 B_Y$ for every Borel set $B_Y \subseteq Y$.) Any $f \in \Phi(\sigma_1,\sigma_2)$ is associated with a measure $\mu_f = (\mathrm{id}_X \times f)(\sigma_1) \in C(X \times Y)^*_+$, as given by

$$\int_{X\times Y} w(x,y)\,\mu_f(d(x,y)) := \int_X w(x, f(x))\,\sigma_1(dx) \quad \forall w \in C(X\times Y),$$

or, equivalently,

$$\mu_f B := \sigma_1\{x \in X : (x, f(x)) \in B\}$$

whenever $B \subseteq (X \times Y)$ is Borel. It is easily seen that $\mu_f \in \Gamma(\sigma_1,\sigma_2)$ and

$$\mathcal{F}(f) = c(\mu_f). \tag{59}$$

The measure μ_f is called a (feasible) Monge solution to MKP. It follows from (59) that if there is an optimal solution to MKP which is the Monge solution

μ_f, then f is an optimal solution to the Monge problem and optimal values of both problems coincide,

$$\mathcal{C}(c;\sigma_1,\sigma_2) = \mathcal{F}(f) = \mathcal{V}(c;\sigma_1,\sigma_2), \tag{60}$$

where $\mathcal{V}(c;\sigma_1,\sigma_2) = \inf\{\mathcal{F}(f) : f \in \Phi(\sigma_1,\sigma_2)\}$. In general case, $\mathcal{C}(c;\sigma_1,\sigma_2) \le \mathcal{V}(c;\sigma_1,\sigma_2)$, and $\Phi(\sigma_1,\sigma_2)$ can be empty; however, in some particular cases (60) holds true.

Remark 3.8 . When X and Y are subsets in $\mathbb{R}^n$, some existence (and uniqueness) results for optimal Monge solutions based on conditions of c-cyclic monotonicity (antimonotonicity) may be found in [3], [5], [6], [12], [39], [40], [55], [65]. In most of these publications, cost functions of the form $c(x,y) = \varphi(x-y)$ are considered. (Note that, since a pioneer paper by Sudakov [64],[7] much attention is paid to cost functions $c(x,y) = \|x-y\|$ for various norms $\|\cdot\|$ in $\mathbb{R}^n$; for such cost functions the optimal solution is not unique.) Several existence and uniqueness theorems for general cost functions are established in [39, 40].

Notice that for a continuous $f \in \Phi(\sigma_1,\sigma_2)$ and $\mu = \mu_f$ one has $\operatorname{spt}\mu = \{(z, f(z)) : z \in \operatorname{spt}\sigma_1\}$; therefore, F_μ as given by (48) is single-valued, $F_\mu(x) = -c(\cdot, f(x))$, and

$$\varphi_{F_\mu}(z_1,z_2) = \varphi_f(z_1,z_2) := c(z_2, f(z_1)) - c(z_1, f(z_1)).$$

The next optimality criterion is then a direct consequence of Theorem 3.7.

Corollary 3.3 (cf. [44, Corollary 2.2]). *Suppose $f \in \Phi(\sigma_1,\sigma_2)$ is continuous, then μ_f is an optimal solution to MKP if and only if $Q(\varphi_f)$ is nonempty.*

Remark 3.9 . If $f \in \Phi(\sigma_1,\sigma_2)$ is discontinuous, then the support of μ_f is the closure of the set $\{(z, f(z)) : z \in \operatorname{spt}\sigma_1\}$. In some cases, Corollary 3.3 and its generalizations following from Theorem 3.7 enable to find exact optimal solutions to concrete Monge problems; see [44, 45].

[7] In spite of a gap in Sudakov's proof (see [3]), its main idea proves to be fruitful.

3.5 Some generalizations

In this subsection we consider briefly some examples of H-convex functions similar to Example 2.2 and some sets of type $Q(c)$ and $Q_0(c)$ for cost functions c that can fail to vanish on the diagonal.

Given an arbitrary infinite set X, $l^\infty(X)$ and $l^\infty(X \times X)$ will denote the linear spaces of bounded real-valued functions on X and $X \times X$, respectively. They are dual Banach spaces relative to the uniform norms $\|u\|_\infty = \sup_{x\in X} |u(x)|$, $u \in l^\infty(X)$ and $\|w\|_\infty = \sup_{(x,y)\in X\times X} |w(x,y)|$, $w \in l^\infty(X \times X)$:

$$l^\infty(X) = l^1(X)^*, \quad l^\infty(X \times X) = l^1(X \times X)^*.$$

Here, $l^1(Z)$ stands for the space of real-valued functions v on Z with at most countable set $\operatorname{spt} v := \{z \in Z : v(z) \neq 0\}$ and $\|v\|_1 := \sum_{z\in \operatorname{spt} v} |v(z)| < \infty$, and the duality between $l^1(Z)$ and $l^\infty(Z)$ is given by the bilinear form

$$\langle v, u\rangle := \sum_{z\in \operatorname{spt} v} v(z)u(z),\ u \in l^\infty(Z),\ v \in l^1(Z).$$

Given a cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$, the reduced cost function c_* is defined as follows:

$$c_*(x,y) := \min\left(c(x,y),\ \inf_n \inf_{x_1,\dots,x_n} \sum_{i=1}^{n+1} c(x_{i-1},x_i)\right), \tag{61}$$

where $x_0 = x, x_{n+1} = y$. Clearly, it turns into (32) when c vanishes on the diagonal. Also, c_* satisfies the triangle inequality $c_*(x,y)+c_*(y,z) \geq c_*(x,z)$ for all $x,y,z \in X$ if one takes, by definition, that $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$.

Let us define a set

$$Q(c; l^\infty(X)) := \{u \in l^\infty(X) :\ u(x) - u(y) \leq c(x,y) \quad \forall x, y \in X\}. \tag{62}$$

Note that if $u \in Q(c; l^\infty(X))$, then

$$u(x_{i-1}) - u(x_i) \leq c(x_{i-1}, x_i), \quad i = 1, \dots, n+1,$$

and summing up these inequalities with $x_0 = x$, $x_{n+1} = y$ yields

$$u(x) - u(y) \leq \sum_{i=1}^{n+1}(u(x_{i-1}) - u(x_i)) \leq \sum_{i=1}^{n+1} c(x_{i-1}, x_i).$$

This implies $u \in Q(c_*; l^\infty(X))$, and as $c \geq c_*$, it follows that $Q(c; l^\infty(X)) = Q(c_*; l^\infty(X))$.

Proposition 3.6 (cf. [33, Lemma 2] and [37, Theorem 4.1].) *Suppose c_* is bounded from above, the following statements are then equivalent:*

(a) $Q(c; l^\infty(X)) \neq \varnothing$*;*

(b) $c_* \in l^\infty(X \times X)$*;*

(c) $c_*(x, y) > -\infty$ *for all* $x, y \in X$*;*

(d) $c_*(x, x) \geq 0$ *for all* $x \in X$*;*

(e) for all integers l and all cycles $x_0, \dots, x_{l-1}, x_l = x_0$ in X, the inequality holds $\sum_{i=1}^{l} c(x_{i-1}, x_i) \geq 0$*;*

(f) the function

$$\bar{c}(x, y) = \begin{cases} c_*(x, y), & \text{if } x \neq y; \\ 0, & \text{if } x = y; \end{cases} \tag{63}$$

is H-convex relative to $H := \{h_u(x, y) = u(x) - u(y) : u \in l^\infty(X)\}$.

Proof. $(a) \Rightarrow (b)$ Suppose that $u \in Q(c; l^\infty(X))$. Since $Q(c; l^\infty(X)) = Q(c_*; l^\infty(X))$, one has $u(x) - u(y) \leq c_*(x, y)$; therefore, c_* is bounded from below, and as, by hypothesis, c_* is bounded from above, $c_* \in l^\infty(X \times X)$.

$(b) \Rightarrow (a)$ Fix arbitrarily a point $x_0 \in X$ and set $u(x) := c_*(x, x_0)$. Clearly, $u \in l^\infty(X)$, and by the triangle inequality, $u(x) - u(y) = c_*(x, x_0) - c_*(y, x_0) \leq c_*(x, y)$ whenever $x, y \in X$, i.e., $u \in Q(c_*; l^\infty(X) = Q(c; l^\infty(X))$.

$(b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (b)$ Since c_* is bounded from above, one has $c_*(x, y) < M < +\infty$ for all $(x, y) \in X \times X$. Suppose $c_* \notin l^\infty(X)$, then there are points $(x_n, y_n) \in X \times X$ such that $c_*(x_n, y_n) < -n$, and applying the triangle inequality yields $c_*(x, y) \leq c_*(x, x_n) + c_*(x_n, y_n) + c_*(y_n, y) \leq 2M - n$; therefore $c_*(x, y) = -\infty$.

$(c) \Rightarrow (d)$ It follows from the triangle inequality that $c_*(x, x) \leq 2c_*(x, x)$ whenever $x \in X$. Therefore, if $c_*(x_0, x_0) < 0$ for some $x_0 \in X$, then $c_*(x_0, x_0) = -\infty$. (Moreover, in such a case, applying again the triangle inequality yields $c_*(x, y) \leq c_*(x, x_0) + c_*(x_0, x_0) + c_*(x_0, y) = -\infty$.)

$(d) \Rightarrow (c)$ Suppose $c_*(x, y) = -\infty$ for some $(x, y) \in X \times X$, then applying the triangle inequality yields $c_*(x, x) \leq c_*(x, y) + c_*(y, x) = -\infty$.

$(d) \Leftrightarrow (e)$ Obvious.

$(b) \Rightarrow (f)$ Take a point $x_0 \in X$ and define $u_{x_0}(x) := c_*(x, x_0)$. One has $h_{u_{x_0}}(x, y) = c_*(x, x_0) - c_*(y, x_0) \leq c_*(x, y) = \bar{c}(x, y)$ for any $x \neq y$, and $h_{u_{x_0}}(x, x) = 0 = \bar{c}(x, x)$ for all $x \in X$. Thus, $h_{u_{x_0}} \in H(\bar{c})$ whenever $x_0 \in X$.

Moreover, for $x_0 = y$ one gets $h_{u_y}(x,y) = \bar{c}(x,y)$, and H-convexity of $\bar{c}$ is thus established.

$(f) \Rightarrow (a)$ Obvious. $\square$

Remark 3.10 . It is easily seen that $Q(c; l^\infty(X)) = Q(\bar{c}; l^\infty(X))$ and $H(\bar{c}) = \{h_u : u \in Q(c; l^\infty(X))\}$.

Remark 3.11 . It follows easily from the proof of Proposition 3.6 that if c_* is bounded from above then either all the statements $(a) - (f)$ hold true or $c_*(x,y) = -\infty$ whenever $(x,y) \in X \times X$.

The following proposition is established by similar arguments, and so we omit its proof.

Proposition 3.7 (cf. [32, 35]). *Given a function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ such that $c_*(x,y) < +\infty$ whenever $x, y \in X$, the following statements are equivalent:*

(a) $Q_0(c) := \{u \in \mathbb{R}^X : u(x) - u(y) \leq c(x,y) \quad \forall x, y \in X\} \neq \varnothing$;

(b) $c_*(x,y) > -\infty$ *for all* $x, y \in X$;

(c) $c_*(x,x) \geq 0$ *for every* $x \in X$;

(d) for all integers l *and all cycles* $x_0, \ldots, x_{l-1}, x_l = x_0$ *in* X*, the inequality holds* $\sum_{i=1}^{l} c(x_{i-1}, x_i) \geq 0$.

(e) the function $\bar{c}$*, as given by (63), is* H*-convex with respect to* $H = \{h_u(x,y) = u(x) - u(y) : u \in \mathbb{R}^X\}$.

Remark 3.12 . It is easily seen that $H(\bar{c}) = \{h_u : u \in Q_0(c)\}$.

Remark 3.13 . If X is a domain in $\mathbb{R}^n$ and c is a smooth function vanishing on the diagonal, then either $Q_0(c)$ is empty or $Q_0(c) = \{u + const\}$ where $\nabla u(x) = -\nabla_y c(x,y)|_{y=x}$; see [33, 35, 36]. Second-order conditions (necessary ones and sufficient ones) for $Q_0(c)$ to be nonempty are given in [33, 35, 36, 46].

Let $E(X)$ be a closed linear subspace in $l^\infty(X)$ containing constant functions, separating points of X (that is, for any $x, y \in X$ there is a function $u \in E(X)$, $u(x) \neq u(y)$), and such that $u, v \in E(X)$ implies $uv \in E(X)$. Then $E(X)$ is a (commutative) Banach algebra with respect to the uniform norm $\|u\| = \sup_{x \in X} |u(x)|$ and the natural (pointwise) multiplication. (Also, $E(X)$ is a Banach lattice; see [43].) As is known from theory of Banach algebras [13, 51], the set $\varkappa X$ of all non-zero multiplicative linear functionals on

$E(X)$ is a weak* compact subset in $E(X)^*$, X is dense in $\varkappa X$ [8], and an isometry of Banach algebras (Gelfand's representation), $A : E(X) \to C(\varkappa X)$, $AE(X) = C(\varkappa X)$, holds as follows:

$$Au(\delta) := \langle u, \delta \rangle, \ u \in E(X), \delta \in \varkappa X.$$

Let us give three examples of Banach algebras $E(X)$. They are as follows:

1. $C^b(X)$ - the Banach algebra of bounded continuous real-valued functions on a completely regular Hausdorff topological space X. (In this case, $\varkappa X = \beta X$ is the Stone-Čech compactification of X.)

2. $U(X)$ - the Banach algebra of bounded universally measurable real-valued functions on a compact Hausdorff topological space X (we have yet met it in subsection 3.2).

3. $\mathcal{L}^\infty(\mathbb{R}^n)$ - the Banach algebra of bounded Lebesgue measurable real-valued functions on $\mathbb{R}^n$ (Lebesgue equivalent functions are not identified). This algebra will be of use in section 5.

Given a set X, a cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$, and an algebra $E(X)$, one can define the set

$$Q(c; E(X)) := \{u \in E(X) : \ u(x) - u(y) \leq c(x, y) \quad \forall x, y \in X\}$$

and a class of functions on $X \times X$,

$$H := \{h_u : \ h_u(x, y) = u(x) - u(y), u \in E(X)\}.$$

Clearly, $H(c) = Q(c; E(X))$ and c is H-convex if and only if

$$c(x, y) = \sup\{u(x) - u(y) : \ u \in Q(c; E(X))\}$$

whenever $x, y \in X$.

Moreover, $Q(c; E(X))$ proves to be the constraint set for an abstract (non-topological) variant of the dual MKP with a given marginal difference, and H-convexity arguments play important role in the corresponding duality results; see [37, 38] for details.

Similarly, given a cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$, one can take two algebras, $E_1(X)$ and $E_2(Y)$, and consider the set in their product,

$$Q'(c; E_1(X), E_2(Y)) := \{(u, v) : \ u(x) - v(y) \leq c(x, y) \quad \forall x \in X, y \in Y\},$$

[8] A point $x \in X$ is identified with the functional $\delta_x \in \varkappa X$, $\langle u, \delta_x \rangle = u(x)$, $u \in E(X)$.

and a class of functions on $X \times Y$,

$$H := \{h_{uv} : h_{uv}(x, y) = u(x) - v(y), (u, v) \in E_1(X) \times E_2(Y)\}.$$

Clearly, $H(c) = Q'(c; E_1(X), E_2(Y))$ and c is H-convex if and only if

$$c(x, y) = \sup\{u(x) - v(y) : (u, v) \in Q'(c; E_1(X), E_2(Y))\}$$

whenever $(x, y) \in X \times Y$.

Moreover, $Q'(c; E_1(X), E_2(Y))$ is the constraint set for an abstract variant of the dual MKP with given marginals, and H-convexity arguments play important role in the corresponding duality results; see [38].

4 Applications to mathematical economics

In this section, we present briefly several applications to mathematical economics. In all the applications, properties of the sets $Q(c)$ and $Q_0(c)$ for various particular cost functions c are considered. The corresponding results are based on conditions for these sets to be nonempty.

4.1 Utility theory

A *preorder* on a set X is a binary relation $\preceq$ which is reflexive ($x \preceq x$ for all $x \in X$) and transitive (for any $x, y, z \in X$, $x \preceq y$, $y \preceq z$ imply $x \preceq z$). A preorder $\preceq$ is called *total* if any two elements of X, x and y, are compatible, that is $x \preceq y$ or $y \preceq x$. A preorder $\preceq$ on a topological space X is called *closed* if its graph, $gr(\preceq) := \{(x, y) : x \preceq y\}$, is a closed subset in $X \times X$.

Any preorder $\preceq$ can be treated as a preference relation, and it determines two binary relations on X: the strict preference relation $\prec$,

$$x \prec y \iff x \preceq y \text{ but not } y \preceq x,$$

and the equivalence relation $\sim$,

$$x \sim y \iff x \preceq y \text{ and } y \preceq x.$$

A real-valued function u on X is said to be an *utility function* for a preorder $\preceq$ if for any $x, y \in X$ two conditions are satisfied as follows:

$$x \preceq y \Rightarrow u(x) \leq u(y), \tag{64}$$

$$x \prec y \Rightarrow u(x) < u(y). \tag{65}$$

Clearly, it follows from (64) that $x \sim y \Rightarrow u(x) = u(y)$.

The pair of conditions (64),(65) is equivalent to the single condition

$$x \preceq y \Leftrightarrow u(x) \leq u(y)$$

if and only if the preorder $\preceq$ is total. (Moreover, if $\preceq$ is total, then $x \prec y \Leftrightarrow u(x) < u(y)$ and $x \sim y \Leftrightarrow u(x) = u(y)$, that is, the preference relation is completely determined by its utility function.)

One of fundamental results in the mathematical utility theory is the famous theorem due to Debreu [9, 10], which asserts the existence of a continuous utility function for every total closed preorder on a separable metrizable space. We'll give here (see also [27, 28, 32]) some extensions of that theorem to the case where the preorder is not assumed to be total. The idea of our approach is to use a specific cost function c that vanishes on the graph of the preorder and has appropriate semicontinuity properties. With help of the duality theorem (Theorem 3.2) we'll obtain a representation

$$gr(\preceq) = \{(x, y) : u(x) \leq u(y) \quad \forall u \in H\} \tag{66}$$

with $H \subseteq Q(c)$. Moreover, sometimes it is possible to choose a countable $H = \{u_k : k = 1, 2, \ldots\}$, and in such a case

$$u_0(x) = \sum_{k=1}^{\infty} 2^{-k} \frac{u_k(x)}{1 + |u_k(x)|}$$

proves to be a continuous utility function for $\preceq$.

Theorem 4.1 ([22, 27]). *Let $\preceq$ be a closed preorder on a compact metrizable space X. Then $gr(\preceq)$ has a representation (66) with a countable H; hence there is a continuous utility function for $\preceq$.*

Proof.[9] Consider on $X \times X$ the cost function

$$c(x, y) = \begin{cases} 0, & \text{if } x \preceq y; \\ +\infty, & \text{otherwise.} \end{cases}$$

[9]This proof follows [27]; a proof in [22] is different.

It satisfies the triangle inequality and vanishes on the diagonal because $\preceq$ is transitive and reflexive. Also, it is lsc because $\preceq$ is closed. It follows from Theorem 3.2 that $Q(c)$ is nonempty and

$$c(x, y) = \sup_{u \in Q(c)} (u(x) - u(y));$$

therefore,

$$gr(\preceq) = \{(x, y) : u(x) \leq u(y) \quad \forall u \in Q(c)\}.$$

Since $C(X)$ is separable, one can choose a dense countable subset H in $Q(c)$. Then (66) holds with that H, and the result follows. $\square$

The next result is derived from Theorem 4.1.

Corollary 4.1 . ([28, 32]). *Theorem 4.1 is extended to X being a separable metrizable locally compact space.*

Theorem 4.2 ([31]). *Let $\preceq$ be a preorder on a separable metrizable space X, the following statements are then equivalent:*

(a) a representation (66) holds with a countable family $H \subset C^b(X)$;

(b) $\preceq$ is a restriction to X of a closed preorder $\preceq_1$ on X_1, where X_1 is a metrizable compactification of X.

If these equivalent statements hold true, then there is a continuous utility function for $\preceq$.

We consider now the following question. Given a closed preorder $\preceq_\omega$ depending on a parameter ω, when is there a *continuous utility*, i.e. a jointly continuous real-valued function $u(\omega, x)$ such that, for every ω, $u(\omega, \cdot)$ is a utility function for $\preceq_\omega$? This question arises in various parts of mathematical economics. In case of total preorders $\preceq_\omega$, some *sufficient* conditions for the existence of a continuous utility were obtained in [8, 48, 50, 52]. The corresponding existence results are rather special consequences of the following general theorem.

Theorem 4.3 ([28, 32]). *Suppose that Ω and X are metrizable topological spaces, and X, in addition, is separable locally compact. Suppose also that for every $\omega \in \Omega$ a preorder $\preceq_\omega$ is given on X, and that the set $\{(\omega, x, y) : x \preceq_\omega y\}$ is closed in $\Omega \times X \times X$. Then there exists a continuous utility $u : \Omega \times X \to [0, 1]$.*

Proof (the case where Ω is separable locally compact).[10] Let us define a preorder $\preceq$ on $\Omega \times X$,

$$(\omega_1, x_1) \preceq (\omega_2, x_2) \Longleftrightarrow \omega_1 = \omega_2, x_1 \preceq_{\omega_1} x_2.$$

It is obviously closed, and as $\Omega \times X$ is separable locally compact, the result follows from Corollary 4.1. □

Remark 4.1 . Observe that if all $\preceq_\omega$ are total then the condition that the set $\{(\omega, x, y) : x \preceq_\omega y\}$ is closed in $\Omega \times X \times X$ is *necessary* (as well as sufficient) for the existence of a continuous utility $u : \Omega \times X \to [0, 1]$.

Let $\mathcal{P}$ denote the set of all closed preorders on X. By identifying a preorder $\preceq \in \mathcal{P}$ with its graph in $X \times X$, we consider in $\mathcal{P}$ the topology t which is induced by the exponential topology on the space of closed subsets in the one-point compactification of $X \times X$ (for the definition and properties of the exponential topology, see [23]). Obviously, $(\mathcal{P}, t)$ is a metrizable space. The next result is obtained by applying Theorem 4.3 to $\Omega = (\mathcal{P}, t)$.

Corollary 4.2 (Universal Utility Theorem [28, 32]). *There exists a continuous function $u : (\mathcal{P}, t) \times X \to [0, 1]$ such that $u(\preceq, \cdot)$ is a utility function for $\preceq$ whenever $\preceq \in \mathcal{P}$.*

4.2 Demand analysis

Given a *price set* $P \subseteq \operatorname{int} \mathbb{R}^n_+$, we mean by a *demand function* any map $f : P \to \operatorname{int} \mathbb{R}^n_+$. We will say that an utility function $U : \mathbb{R}^n_+ \to \mathbb{R}_+$ *rationalizes* f if, for every $p \in P$,

$$q \in \mathbb{R}^n_+,\ p \cdot q \le p \cdot f(p) \Longrightarrow U(f(p)) \ge U(q). \tag{67}$$

Theorem 4.4 (cf. [46, Corollary 3]). *Given a function $f : P \to \operatorname{int} \mathbb{R}^n_+$, the following statements are equivalent:*

(a) there is a positive homogeneous utility function $U : \mathbb{R}^n_+ \to \mathbb{R}_+$, which is strictly positive on $f(P)$ and rationalizes f;

(b) there is a positive homogeneous continuous concave utility function $U : \mathbb{R}^n_+ \to \mathbb{R}_+$, which is strictly positive on $f(P)$ and rationalizes f;

[10] For the sake of simplicity, we restrict ourselves to the case where Ω is separable and locally compact. In the general case, the proof makes substantial use of a version of Michael's continuous selection theorem in a locally convex Fréchet space.

(c) for a cost function ξ on $P \times P$, as given by

$$\xi(p, p') := \ln(p' \cdot f(p)) - \ln(p' \cdot f(p')),$$

the set $Q_0(\xi)$ is nonempty;

(d) for every cycle $p^1, \ldots, p^l, p^{l+1} = p^1$ in P, the inequality holds true

$$\prod_{k=1}^{l} p^{k+1} \cdot f(p^k) \geq \prod_{k=1}^{l} p^k \cdot f(p^k);$$

(e) there is a strictly positive solution to the system

$$u(p) \geq \frac{p \cdot f(p)}{p \cdot f(p')} u(p') \quad \textit{for all} \quad p, p' \in P. \tag{68}$$

Proof. $(b) \Rightarrow (a)$ Obvious.

$(a) \Rightarrow (e)$ Define $u(p) := U(f(p))$, $p \in P$. Since for every $q \in \mathbb{R}^n_+$,

$$p \cdot \frac{p \cdot f(p)}{p \cdot q} q = p \cdot f(p),$$

it follows from (67) that

$$\frac{p \cdot f(p)}{p \cdot q} U(q) = U\left(\frac{p \cdot f(p)}{p \cdot q} q\right) \leq U(f(p)),$$

which implies (68) for $q = f(p')$.

$(e) \Rightarrow (c)$ Since a solution $u(p)$ to (68) is strictly positive, it follows that $v(p) := \ln u(p)$ makes sense and belongs to $Q_0(\xi)$.

$(c) \Leftrightarrow (d)$ This is an easy consequence of Proposition 3.7.

$(c) \Rightarrow (e)$ Suppose $v \in Q_0(\xi)$, then $u(p) = e^{v(p)}$ is strictly positive and satisfies (68).

$(e) \Rightarrow (b)$ Let us define

$$U(q) := \inf_{p' \in P} \frac{u(p')}{p' \cdot f(p')} p' \cdot q.$$

It follows easily from (68) that $U(f(p)) = u(p)$ whenever $p \in P$, hence U is strictly positive on $f(P)$. If now $p \cdot q \leq p \cdot f(p)$, then

$$U(q) \leq \frac{u(p)}{p \cdot f(p)} p \cdot q \leq u(p) = U(f(p)),$$

that is U rationalizes f. Since U is clearly upper semi-continuous concave (hence continuous; see [57, Theorem 10.2]) and positive homogeneous, the implication is completely established. $\square$

Remark 4.2 . Statement (d) can be considered as a particular (strengthened) version of the strong revealed preference axiom, and (e) generalizes the corresponding variant of the Afriat—Varian theory (see [1, 2, 66, 67]) to the case of infinite set of 'observed data'. Further results on conditions for rationalizing demand functions by concave utility functions with nice additional properties in terms of non-emptiness of sets $Q_0(\varphi)$ for various price sets P and some specific cost functions φ on $P \times P$ may be found in [46].

4.3 Dynamics models

In this subsection (see also [35, 36, 37]), we consider an abstract dynamic optimization problem resembling, in some respects, models of economic system development.

Suppose X is an arbitrary set and $a : X \to X$ is a multifunction with nonempty values. Its graph, $gr(a) = \{(x, y) : y \in a(x)\}$, may be considered as a continual net with vertices $x \in X$ and arcs $(x, y) \in gr(a)$, respectively. A finite sequence of elements of X, $\chi = (\chi(t))_{t=0}^{T}$ (where $T = T(\chi) < +\infty$ depends on χ), satisfying

$$\chi(t) \in a(\chi(t-1)), \quad t = 1, \dots, T,$$

is called a (finite) *trajectory*. We assume that the *connectivity hypothesis* is satisfied: for any $x, y \in X$, there is a trajectory χ that starts at x ($\chi(0) = x$) and finishes at y ($\chi(T) = y$).

Given a terminal function $l : X \times X \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} l \neq \varnothing$ and a cost function $c : X \times X \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} c = gr(a)$, the payment for moving along the trajectory χ equals

$$g(\chi) := l(\chi(0), \chi(T)) + \sum_{t=1}^{T} c(\chi(t-1), \chi(t)).$$

The problem is to minimize $g(\chi)$ over the set τ of all trajectories.

Observe that the connectivity hypothesis can be rewritten as

$$c_*(x, y) < +\infty \quad \text{for all} \quad x, y \in X,$$

and the optimality of a trajectory $\bar{\chi}$ means exactly

$$g(\bar{\chi}) = \min\{l(x, y) + c_*(x, y) : x, y \in X\}. \tag{69}$$

Theorem 4.5 ([35, Theorem 7.1]). *Suppose X is a compact topological space, both functions on $X \times X$, l and c, are lsc, and $c(x, y) > 0$ for all $x, y \in X$. Then there exists an optimal trajectory.*

An important particular case of the above problem is to minimize the functional

$$g_1(\chi) := \sum_{t=1}^{T(\chi)} c(\chi(t-1), \chi(t))$$

over the set $\tau(X_1, X_2)$ of trajectories that start in X_1 and finish in X_2 (i.e., $\chi(0) \in X_1$, $\chi(T) \in X_2$), where X_1 and X_2 are given subsets of X. This problem is reduced to minimizing $g(\chi)$ over τ if one takes l to be the indicator function of $X_1 \times X_2$ (i.e., $l(x, y) = 0$ for $(x, y) \in X_1 \times X_2$ and $l(x, y) = +\infty$ otherwise).

The next result is a direct consequence of Theorem 4.5.

Corollary 4.3 ([35, Corollary 7.1]). *Let X and c be as in Theorem 4.5, and suppose that X_1 and X_2 are closed in X. Then there exists a trajectory $\bar{\chi} \in \tau(X_1, X_2)$ minimizing g_1 over $\tau(X_1, X_2)$.*

We now return to the general (non-topological) version of the problem.

Theorem 4.6 ([35, Theorem 7.2]). *A trajectory $\bar{\chi} = (\bar{\chi}(t))_{t=0}^{T}$ is optimal in τ if and only if: (a) the equality holds*

$$l(\bar{\chi}(0), \bar{\chi}(T)) + c_*(\bar{\chi}(0), \bar{\chi}(T)) = \min\{l(x, y) + c_*(x, y) : x, y \in X\}, \tag{70}$$

and (b) there is a function $u \in Q_0(c)$ satisfying

$$u(\bar{\chi}(t-1)) - u(\bar{\chi}(t)) = c(\bar{\chi}(t-1), \bar{\chi}(t)),\ t = 1, \dots, T. \tag{71}$$

An infinite sequence of elements of X, $\chi = (\chi(t))_{t=0}^{\infty}$, satisfying

$$\chi(t) \in a(\chi(t-1)), \quad t = 1, 2, \dots,$$

is called an *infinite trajectory*. Say an infinite trajectory $\chi = (\chi(t))_{t=0}^{\infty}$ is *efficient* if there exists $T_1 = T_1(\chi) < +\infty$ such that, for every $T \geq T_1$, the finite trajectory $\chi^T := (\chi(t))_{t=0}^{T}$ is optimal in τ.

The next result is derived from Theorem 4.6 with help of the Banach limit technique; see [35, Theorem 7.4] for details.

Theorem 4.7 *An infinite trajectory $\chi = (\chi(t))_{t=0}^{\infty}$ is efficient if and only if: (a) (70) holds for all $T \geq T_1$ and (b) there is a function $u \in Q_0(c)$ satisfying (71) for all t.*

4.4 Economics of corruption

Following [7] (see also [36, 56]), we briefly outline here some kind of principal-agents models relating to economics of corruption and dealing with distorting substantial economic information. Suppose there is a population of agents, each of them is characterized by his state (a variable of economic information), which is an element of some set X, and there is yet one agent called the principal (State, monopoly, social planner, insurance company and so on). The principal pays to an agent some amount of money $u(x)$ which depends on information x about agent's state. It is assumed that the actual state of the agent, y, cannot be observed directly by the principal; therefore, agents have a possibility to misrepresent at some cost[11] the relevant information to the principal. Thus, we assume that an agent can at the cost $c(x, y)$ to misrepresent his real state y into the state x without being detected. In such a case, his income equals $u(x) - c(x, y)$. The cost function c may take the value $+\infty$, which occurs when x is too far from y for falsifying y into x be possible without being detected. Also, it is assumed that $c(y, y) = 0$; therefore, if an agent gives true information to the principal, then his income equals the payoff $u(y)$. If now there is an $x \in X$ such that $u(x) - c(x, y) > u(y)$, then, for an agent with the actual state y, it proves to be profitable to falsify his state information. Say, in a model of collusion with a third party, an agent with the actual state y and a supervisor may agree to report the state x maximizing their total income $u(x) - c(x, y)$ and then to share between them the surplus $u(x) - c(x, y) - u(y) > 0$. Similar situations arise in other models (insurance fraud, corruption in taxation); see [7] for details.

Thus, given a cost function $c : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ vanishing on the diagonal, a question arises, whether the payoff function $u : X \to \mathbb{R}$ is *non-manipulable* or *collusion-proof* in the sense that it is in the interest of each agent to be honest. The answer is affirmative if and only if $u \in Q_0(c)$.

[11] For instance, by colluding with a third party (expert, supervisor, tax officer).

5 An application to approximation theory

In this section, we deal with some best approximation problems.

Let us consider a linear subspace in $l^\infty(X \times X)$,

$$H_0 := \{u(x) - u(y) : u \in l^\infty(X)\}. \tag{72}$$

Given a function $f \in l^\infty(X \times X)$, the problem is to find the value

$$m(f; H_0) := \min_{h \in H_0} \|f - h\|_\infty = \min_{u \in l^\infty(X)} \sup_{x,y \in X} |f(x, y) - u(x) + u(y)|. \tag{73}$$

Note that the minimum in (73) is attained at some $h = h_u \in H_0$, $h_u(x, y) = u(x) - u(y)$, because closed balls in the dual Banach space $l^\infty(X)$ are weak* compact and the functional on $l^\infty(X)$, $u \mapsto \sup_{x,y \in X} |f(x, y) - u(x) + u(y)|$ is weak* lsc. Moreover, (73) can be rewritten as

$$m(f; H_0) = \min\{\alpha > 0 : Q(c + \alpha; l^\infty) \neq \varnothing\}, \tag{74}$$

where

$$c(x, y) := \min(f(x, y), -f(y, x)), \quad x, y \in X, \tag{75}$$

and there exists a function u in $Q(c + m(f; H_0); l^\infty)$.

A topological analog of this problem is as follows. Given a completely regular Hausdorff topological space X, a subspace $\mathcal{H}_0$ in $C^b(X \times X)$,

$$\mathcal{H}_0 := \{u(x) - u(y) : u \in C^b(X)\}, \tag{76}$$

and a function $f \in C^b(X \times X)$, one has to find the value

$$m(f; \mathcal{H}_0) := \inf_{h \in \mathcal{H}_0} \|f - h\| = \inf_{u \in C^b(X)} \sup_{x,y \in X} |f(x, y) - u(x) + u(y)|. \tag{77}$$

Recall (see subsection 3.5) that, for every topological space X, $C^b(X)$ denotes the space of bounded continuous real-valued functions on it with the uniform norm $\|u\| = \sup_{x \in X} |u(x)|$. Clearly, (77) is equivalent to

$$m(f; \mathcal{H}_0) = \inf\{\alpha > 0 : Q(c + \alpha; C^b(X)) \neq \varnothing\}, \tag{78}$$

where c is given by (75).

Theorem 5.1 ([37, Theorem 5.1]). *For every $f \in l^\infty(X \times X)$,*

$$m(f; H_0) = -\inf \frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i), \tag{79}$$

and if X is a compact space, then for every $f \in C(X \times X)$,

$$m(f; \mathcal{H}_0) = -\inf \frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i). \tag{80}$$

Here, both infima, in (79) and (80), are taken over all integers n and all cycles $x_0, \dots, x_{n-1}, x_n = x_0$ in X.

Proof. As follows from (74), for every $\alpha > m(f; H_0)$ there is a function $u \in Q(c + \alpha; l^\infty)$. Then $u(x_{i-1}) - u(x_i) \le c(x_{i-1}, x_i)$, $i = 1, \dots, n$, and summing up these inequalities yields

$$0 = \sum_{i=1}^{n}(u(x_{i-1}) - u(x_i)) \le \sum_{i=1}^{n}(c(x_{i-1}, x_i) + \alpha) = \sum_{i=1}^{n} c(x_{i-1}, x_i) + n\alpha$$

(this follows also from implication $(a) \Rightarrow (e)$ of Proposition 3.6); therefore, $\alpha \ge -\inf \frac{1}{n} \sum\limits_{i=1}^{n} c(x_{i-1}, x_i)$, and

$$m(f; H_0) = \inf\{\alpha :\ Q(c + \alpha; l^\infty) \ne \varnothing\} \ge -\inf \frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i). \tag{81}$$

Suppose now that $\alpha < m(f; H_0)$. Then $Q(c + \alpha; l^\infty) = \varnothing$, and taking into account Remark 3.11, one has $(c+\alpha)_* \equiv -\infty$; therefore, there is a cycle $x_0, \dots, x_{n-1}, x_n = x_0$ such that $\sum_{i=1}^{n} c(x_{i-1}, x_i) + n\alpha < 0$. One obtains

$$\alpha < -\frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i) \le -\inf \frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i),$$

and as this holds true whenever $\alpha < m(f; H_0)$, one gets

$$m(f; H_0) \le -\inf \frac{1}{n} \sum_{i=1}^{n} c(x_{i-1}, x_i), \tag{82}$$

and (79) follows from (81),(82).

The proof of (80) is similar if one replaces $Q(c+\alpha; l^\infty)$ with $Q(c+\alpha)$ and takes into account that for every $c \in C(X\times X)$ and every $\alpha \in \mathbb{R}$ either $(c+\alpha)_* \in C(X \times X)$ or $(c+\alpha)_* \equiv -\infty$ (see [47, Lemma 2.4], where a more general result is established). $\square$

Corollary 5.1 *If X is a compact topological space and $f \in C(X\times X)$, then $m(f; H_0) = m(f; \mathcal{H}_0)$.*

Remark 5.1 . If X is a *non-compact* completely regular Hausdorff topological space, one can pass to its Stone-Čech compactification $X' = \beta X$. Taking into account the natural linear isometry $C^b(X) = C(X')$, the next result is an easy consequence of Theorem 5.1.

Corollary 5.2 *Theorem 5.1 is extended to X being any completely regular Hausdorff topological space provided that $f \in C(X'\times X')$, $C(X)$ in (76) is replaced with $C^b(X)$, and* max *in (77) is replaced with* sup.

Note that $C(\beta X \times \beta X)$ can be considered as the closure in $C^b(X\times X)$ of the subspace of finite sums $f(x,y) = \sum_1^n a_k(x)b_k(y)$, $a_k, b_k \in C^b(X), k = 1,\ldots,n$.

Say $u \in C^b(X)$ is an *exact solution* to the approximation problem if the infimum in the right-hand side of (77) is attained at it, that is $m(f;\mathcal{H}_0) = \sup_{x,y\in X} |f(x,y) - u(x) + u(y)|$. It follows from (78) that $u \in C^b(X)$ is an exact solution if and only if it belongs to $Q(c + m(f;\mathcal{H}_0); C^b(X))$; therefore, exact solutions exist if and only if $Q(c+m(f;\mathcal{H}_0); C^b(X))$ is nonempty.

Let $C^{n,\infty}$ be the linear space of bounded infinitely differentiable real-valued functions on $\mathbb{R}^n$, $\mathcal{H}_0^\infty$ a subspace in $C^{2n,\infty}$,

$$\mathcal{H}_0^\infty := \{u(x) - u(y) : \ u \in C^{n,\infty}\}.$$

Theorem 5.2 *Suppose $f(x,y) = g(x-y)$, where $g \in C^b(\mathbb{R}^n)$. Then*

$$m(f;\mathcal{H}_0) = m(f;\mathcal{H}_0^\infty) := \inf_{h\in\mathcal{H}_0^\infty} \|f-h\|,$$

and there is a function $u \in C^{n,\infty}$, which is an exact solution to the approximation problem:

$$m(f;\mathcal{H}_0) = m(f;\mathcal{H}_0^\infty) = \|f - h_u\|, \quad h_u(x,y) = u(x) - u(y). \tag{83}$$

To prove the theorem, some facts from the lifting theory [16][12] will be needed. The main of them is the existence of a strong lifting of $\mathcal{L}^\infty(\mathbb{R}^n)$. Recall (see subsection 3.5), that $\mathcal{L}^\infty(\mathbb{R}^n)$ is the space (Banach algebra and Banach lattice) of bounded Lebesgue measurable real-valued functions on $\mathbb{R}^n$ with the uniform norm on it, $\|u\| = \sup_{x\in\mathbb{R}^n}|u(x)|$, $u \in \mathcal{L}^\infty(\mathbb{R}^n)$. A homomorphism of Banach algebras (i.e., a multiplicative linear operator) $\rho : \mathcal{L}^\infty(\mathbb{R}^n) \to \mathcal{L}^\infty(\mathbb{R}^n)$ is said to be a *strong lifting* of $\mathcal{L}^\infty(\mathbb{R}^n)$ if four conditions are satisfied as follows:

1. ρ is a projector, that is $\rho^2 = \rho$;
2. for every $u \in \mathcal{L}^\infty(\mathbb{R}^n)$, $\rho(u) = u$ almost everywhere (a.e.), that is the set $\{x \in \mathbb{R}^n : \rho(u)(x) \neq u(x)\}$ is Lebesgue negligible;
3. for every $u \in \mathcal{L}^\infty(\mathbb{R}^n)$, $u = 0$ a.e. implies $\rho(u) \equiv 0$;
4. $\rho(u) \equiv u$ whenever $u \in C^b(\mathbb{R}^n)$.

It follows from these conditions along with linearity and multiplicativity of ρ that ρ is also a homomorphism of Banach lattices, i.e. $\rho(u \vee v) = \rho(u)\vee\rho(v)$ and $\rho(u\wedge v) = \rho(u)\wedge\rho(v)$ whenever $u, v \in \mathcal{L}^\infty(\mathbb{R}^n)$. Furthermore, $u \geq v$ a.e. implies $\rho(u)(x) \geq \rho(v)(x)$ for all $x \in \mathbb{R}^n$.

The Lebesgue space $L^\infty(\mathbb{R}^n)$ is a Banach algebra and a Banach lattice, and the operator $\pi : \mathcal{L}^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ mapping every function $u \in \mathcal{L}^\infty(\mathbb{R}^n)$ into its Lebesgue equivalence class is a homomorphism both of Banach algebras and of Banach lattices. Thus, π maps $\mathcal{L}^\infty(\mathbb{R}^n)$ onto the factor space $L^\infty(\mathbb{R}^n) = \mathcal{L}^\infty(\mathbb{R}^n)/\mathcal{N}_0$ where $\mathcal{N}_0$ is the subspace in $\mathcal{L}^\infty(\mathbb{R}^n)$ consisting of Lebesgue negligible functions, and the standard norm in $L^\infty(\mathbb{R}^n)$ is precisely the factor-norm with respect to π. Since $\rho(u) = \rho(v)$ whenever $u - v \in \mathcal{N}_0$, ρ generates a homomorphism of Banach algebras (and of Banach lattices) $\rho' : L^\infty(\mathbb{R}^n) \to \mathcal{L}^\infty(\mathbb{R}^n)$ (a strong lifting of $L^\infty(\mathbb{R}^n)$) such that $\pi \circ \rho' = \mathrm{id}_{L^\infty(\mathbb{R}^n)}$ and $\rho' \circ \pi = \rho$.

Proof of Theorem 5.2. It follows from (78) that for every k there is a function $u_k \in Q(c + m(f;\mathcal{H}_0) + \frac{1}{k}; C^b(\mathbb{R}^n))$. Fix an arbitrary point x_0 in $\mathbb{R}^n$ and assume without loss of generality that $u_k(x_0) = 0$. Then, for all $x \in \mathbb{R}^n$, one has

$$-c(x_0, x) - m(f;\mathcal{H}_0) - 1 \leq u_k(x) \leq c(x, x_0) + m(f;\mathcal{H}_0) + 1, \quad k = 1, 2, \dots;$$

therefore, the sequence (u_k) is bounded in $C^b(\mathbb{R}^n)$. Now, taking into account that $C^b(\mathbb{R}^n)$ is a closed linear subspace in $L^\infty(\mathbb{R}^n)$ and that $L^\infty(\mathbb{R}^n) =$

[12] See also [30, 43], where connections between the lifting theory and abstract convexity are given.

$L^1(\mathbb{R}^n)^*$ is a dual Banach space, (u_k) is bounded hence weak* precompact in $L^\infty(\mathbb{R}^n)$. We shall assume by passing, if needed, to a subsequence[13] that the sequence (u_k) converges weakly* to an element of $L^\infty(\mathbb{R}^n)$. In other words, there exists a function $v \in \mathcal{L}^\infty(\mathbb{R}^n)$ such that u_k converges weakly* to $\pi(v)$. It follows that the sequence $(u_k(x) - u_k(y)) \subset C(\mathbb{R}^{2n}) \subset L^\infty(\mathbb{R}^{2n})$ converges weakly* in $L^\infty(\mathbb{R}^{2n})$ to the element of $L^\infty(\mathbb{R}^{2n})$, which is the Lebesgue equivalence class of the function $v(x) - v(y)$. Now, as $u_k(x) - u_k(y) \leq c(x,y) + m(f;\mathcal{H}_0) + \frac{1}{k}$, and the positive cone $L^\infty_+(\mathbb{R}^{2n})$ is weakly* closed, one gets

$$v(x) - v(y) \leq c(x,y) + m(f;\mathcal{H}_0) \quad \text{a.e. in } \mathbb{R}^{2n}. \tag{84}$$

Let us define

$$N(y) := \{x \in \mathbb{R}^n : v(x) - v(y) > c(x,y) + m(f;\mathcal{H}_0)\}, \quad y \in \mathbb{R}^n.$$

It follows from (84) that the set

$$N := \{y \in \mathbb{R}^n : N(y) \text{ is not Lebesgue negligible}\}$$

is Lebesgue negligible. Consider y as a parameter and observe that, for every $y \notin N$, the inequality

$$v(x) - v(y) \leq c(x,y) + m(f;\mathcal{H}_0)$$

holds true for almost all $x \in \mathbb{R}^n$. Applying a strong lifting ρ to both sides of that inequality yields

$$\rho(v)(x) - v(y) \leq c(x,y) + m(f;\mathcal{H}_0) \tag{85}$$

for all $x \in \mathbb{R}^n$ and all $y \notin N$. Now, considering x as a parameter and applying ρ to both sides of (85) yields

$$\rho(v)(x) - \rho(v)(y) \leq c(x,y) + m(f;\mathcal{H}_0) \quad \forall x, y \in \mathbb{R}^n, \tag{86}$$

that is $\rho(v) \in Q(c + m(f;\mathcal{H}_0); \mathcal{L}^\infty(\mathbb{R}^n))$.

We define u to be the convolution of $\rho(v)$ and η,

$$u(x) = (\rho(v) * \eta)(x) := \int_{\mathbb{R}^n} \rho(v)(x-z)\eta(z)\,dz, \tag{87}$$

[13] Since $L^1(\mathbb{R}^n)$ is separable, the restriction of the weak* topology to any bounded subset of $L^\infty(\mathbb{R}^n)$ is metrizable.

where $\eta(z) := \pi^{-n/2} e^{-z \cdot z} = \pi^{-n/2} e^{-(z_1^2 + \ldots + z_n^2)}, \quad z = (z_1, \ldots, z_n) \in \mathbb{R}^n$. Since $\eta \in C^{n,\infty}$ and $\int_{\mathbb{R}^n} \rho(v)(x-z)\eta(z)\, dz = \int_{\mathbb{R}^n} \rho(v)(z)\eta(x-z)\, dz$, (87) implies $u \in C^{n,\infty}$.

Now, taking into account the form of the function f, one has $c(x,y) = \min(g(x-y), -g(y-x))$; hence, $c(x-z, y-z) = c(x,y)$, and (86) implies

$$\rho(v)(x-z) - \rho(v)(y-z) \le c(x,y) + m(f;\mathcal{H}_0) \quad \forall x, y \in \mathbb{R}^n. \tag{88}$$

Multiplying (88) by $\eta(z)$, integrating the obtained inequality by dz, and taking into account that $\int_{\mathbb{R}^n} \eta(z)\, dz = 1$, one gets $u(x) - u(y) \le c(x,y) + m(f;\mathcal{H}_0)$ for all $x, y \in \mathbb{R}^n$. Thus, $u \in Q_0(c + m(f;\mathcal{H}_0)) \cap C^{n,\infty}$, and the result follows. □

References

[1] Afriat S.N., The construction of utility functions from expenditure data. - *Intern. Econ. Rev.* 1967, **8**, 67–77.

[2] Afriat S.N., On a system of inequalities on demand analysis: an extension of the classical method. - *Intern. Econ. Rev.* 1973, **14**, 460–472.

[3] Ambrosio L., Lecture notes on optimal transport problems. In: Colli P. and Rodrigues J.F. (eds) *Mathematical Aspects of Evolving Interfaces*, Lecture Notes in Math. **1812**, 2003, pp.1–52.

[4] Balder E.J., An extension of duality-stability relations to non-convex optimization problems. - *SIAM J. Contr. Optim.* 1977, **15**, 329–343.

[5] Brenier Y., Polar factorization and monotone rearrangement of vector-valued functions. - Comm. Pure Appl. Math. 1991, **44**, 375–417.

[6] Cafarelli L., Feldman M., McCann R.J., Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs. - *J. Amer. Math. Soc.* 2002, **15**, 1–26.

[7] Carlier G., Ekeland I., Levin V.L., Shananin A.A., A system of inequalities arising in mathematical economics and connected with the Monge - Kantorovich problem. - *CAHIERS du CEREMADE* 2002, Ceremade - UMR 7534 - Univ. Paris Dauphine, No 0213, 3 Mai 2002, 13 pages; (http://www.ceremade.dauphine.fr/preprint/CMD/2002-13.pdf).

[8] Chichilnisky G., Continuous representation of preferences. - *Rev. Econ. Studies* 1980, **47**, 959–963.

[9] Debreu G., Representation of a preference ordering by a numerical function. In: R.Thrall et al. (eds) *Decision Processes*, Wiley, N.Y. 1954, pp.159–165.

[10] Debreu G., Continuity properties of Paretian utility. - *Intern. Econ. Rev.* 1964, **5**, 285–293.

[11] Dunford N., Schwartz J.T. *Linear Operators. Part I: General Theory*, Intersciense Publishers, N.Y., London 1958.

[12] Gangbo W., McCann R.J., The geometry of optimal transportation. - *Acta Math.* 1996, **177**, 113–161.

[13] Gelfand I.M., Raikov D.A., Shilov G.E. *Commutative Normed Rings*, Chelsea, N.Y. 1964.

[14] Hadjisavvas N., Komlosi S., Schaible S. (eds) *Handbook of Generalized Convexity and Generalized Monotonicity*, Springer, N.Y. 2005.

[15] Ioffe A.D., Abstract convexity and non-smooth analysis. - *Adv. Math. Econ.* 2001, **3**, 45–61.

[16] Ionescu Tulcéa A., Ionescu Tulcéa C. *Topics in the Theory of Lifting*, Springer, N.Y. 1969.

[17] Kantorovich L.V., On mass transfer. - *Dokl. Acad. Nauk SSSR* 1942, **37** (7-8), 199–201 (in Russian).

[18] Kantorovich L.V., On a problem of Monge. - *Uspekhi Matem. Nauk* 1948, **3**, no.2, 225–226 (in Russian).

[19] Kantorovich L.V., Akilov G.P. *Functional Analysis*, Nauka, Moscow 1984 (in Russian).

[20] Kantorovich L.V., Rubinshtein G.S., On a function space and some extremal problems. - *Dokl. Acad. Nauk SSSR* 1957, **115**, 1058–1061 (in Russian).

[21] Kantorovich L.V., Rubinshtein G.S., On a space of completely additive functions. - *Vestnik Leningrad Univ.* 1958, **13**(7), 52–59 (in Russian).

[22] Kiruta A.Ya., Rubinov A.M., Yanovskaya E.B. *Optimal Choice of Distributions in Complex Social and Economics Problems*, Nauka, Leningrad 1980 (in Russian).

[23] Kuratowski K. *Topology, vol.1*, Academic Press, N.Y.; PWN, Warsaw 1966.

[24] Levin V.L., Duality and approximation in the mass transfer problem. In: B.S.Mityagin (ed) *Mathematical Economics and Functional Analysis*, Nauka, Moscow 1974, pp.94–108 (in Russian).

[25] Levin V.L., On the problem of mass transfer. - *Soviet Math. Dokl.* 1975, **16**, 1349–1353 (English translation).

[26] Levin V.L., The Monge—Kantorovich mass transfer problem. In: B.A.Efimov (ed) *Methods of Functional Analysis in Mathematical Economics*, Nauka, Moscow 1978, pp.23–55 (in Russian).

[27] Levin V.L., Some applications of duality for the problem of translocation of masses with a lower semicontinuous cost function. Closed preferences and Choquet theory. - *Soviet Math. Dokl.* 1981, **24**, no. 2, 262–267 (English translation).

[28] Levin V.L., A continuous utility theorem for closed preorders on a σ-compact metrizable space. - *Soviet Math. Dokl.* 1983, **28**, no. 3, 715–718 (English translation).

[29] Levin V.L., The problem of mass transfer in a topological space, and probability measures having given marginal measures on product of two spaces. - *Soviet Math. Dokl.* 1984, **29**, 638–643 (English translation).

[30] Levin V.L. *Convex Analysis in Spaces of Measurable Functions and Its Applications in Mathematics and Economics*, Nauka, Moscow 1985 (in Russian).

[31] Levin V.L., Extremal problems with probability measures, functionally closed preorders and strong stochastic dominance. In: V.I.Arkin, A,Shiraev, R.Wets (eds) *Stochastic Optimization, LN in Control and Inform. Sci.* **81**, Springer, Berlin 1986, pp.435–447.

[32] Levin V.L., General Monge—Kantorovich problem and its applications in measure theory and mathematical economics. In: L.J. Leifman (ed) *Functional Analysis, Optimization, and Mathematical Economics (A collection of papers dedicated to memory of L.V. Kantorovich)*, Oxford University Press, N.Y., Oxford 1990, pp.141–176.

[33] Levin V.L., A formula for the optimal value of the Monge—Kantorovich problem with a smooth cost function and a characterization of cyclically monotone maps. - *Mat. Sbornik* 1990, **181**, no.12, 1694–1709 (in Russian); English translation in: *Math. USSR-Sb.* 1992, **71**, 533–548.

[34] Levin V.L., Some applications of set-valued mappings in mathematical economics. - *J. of Math. Economics* 1991, **20**, 69–87.

[35] Levin V.L., A superlinear multifunction arising in connection with mass transfer problems. - *Set-Valued Analysis* 1996, **4**, 41–65.

[36] Levin V.L., Reduced cost functions and their applications. - *J. of Math. Economics* 1997, **28**, 155–186.

[37] Levin V.L., Topics in the duality theory for mass transfer problem. In: V. Beneš, J. Štěpán (eds) *Distributions with Given Marginals and Moment Problems*, Kluwer, Dordrecht 1997, pp.243–252.

[38] Levin V.L., On duality theory for non-topological variants of the mass transfer problem. - *Sbornik: Mathematics* 1997, **188**, no.4, 571–602 (English translation).

[39] Levin V.L., Existence and uniqueness of a measure-preserving optimal mapping in a general Monge—Kantorovich problem. - *Funct. Anal. Appl.* 1998, **32**, 205–208 (English translation).

[40] Levin V.L., Abstract cyclical monotonicity and Monge solutions for the general Monge—Kantorovich problem. - *Set-Valued Analysis* 1999, **7**, 7–32.

[41] Levin V.L., Semiconic duality in convex analysis. - *Trans. Moscow Math. Soc.* 2000, **61**, 197–238 (English translation).

[42] Levin V.L., The Monge—Kantorovich problems and stochastic preference relations. - *Adv. Math. Econ.* 2001, **3**, 97–124.

[43] Levin V.L., Abstract convexity in measure theory and in convex analysis. - *J. Math. Sciences* 2003, **116**, issue 4, 3432–3467; (http://www.kluweronline.com/issn/1072-3374).

[44] Levin V.L., Optimal solutions of the Monge problem. - *Adv. Math. Econ.* 2004, **6**, 85–122.

[45] Levin V.L., Optimality conditions and exact solutions to the two-dimensional Monge—Kantorovich problem. In: A.M.Vershik (ed) *Representation Theory, Dynamical Systems. XI* (special issue) - *Zapiski nauchnyh seminarov POMI (Notes of scientific seminars of St.-Petersburg's branch of the Steklov Institute Math.)* 2004, **312**, 150–164 (in Russian);(http://www.pdmi.ras.ru/znsl/index.html); English translation to appear in *J. Math. Sciences.*

[46] Levin V.L., A method in demand analysis connected with the Monge—Kantorovich problem. - *Adv. Math. Econ.* 2005, **7**, 47–93.

[47] Levin V.L., Milyutin A.A., The problem of mass transfer with a discontinuous cost function and a mass statement of the duality problem for convex extremal problems. - *Russian Math. Surveys* 1979, **34**, 1–78 (English translation).

[48] Mas-Colell A., On the continuous representation of preorders. - *Intern. Econ. Rev.* 1977, **18**, 509–513.

[49] Monge G., Mémoire sur la théorie des déblais et de remblais. In: *Histoire de l'Académie Royale des Sciences de Paris, avec les Mémoires de Mathématique et de Physique pour la même année*, 1781, pp.666–704.

[50] Mount K.R., Reiter S., Construction of a continuous utility function for a class of preferences. - *J. of Math. Economics* 1976, **3**, 227–245.

[51] Naimark M.A. *Normed Algebras*, Wolters-Noordhoff, Groningen 1972.

[52] Neuefeind W., On continuous utility. - *J. of Economic Theory* 1972, **5**, 174–176.

[53] Pallaschke D., Rolewicz S. *Foundations of Mathematical Optimization (Convex analysis without linearity)*, Kluwer, Dordrecht 1997.

[54] Penot J.P., The bearing of duality on microeconomics. - *Adv. Math. Econ.* 2005, **7**, 113–139.

[55] Rachev S.T., Rüschendorf L. *Mass Transportation Problems*, Vol.1:Theory, Vol.2:Applications. Springer, Berlin 1998.

[56] Rochet J.-C., A necessary and sufficient condition for rationalizability in a quasi-linear context. - *J. of Math. Economics* 1987, **16**, 191–200.

[57] Rockafellar R.T. *Convex Analysis*, Princeton Univ. Press, Princeton 1970.

[58] Rockafellar R.T., On the maximal monotonicity of the subdifferential mappings. - *Pacific J. Math.* 1970, **33**, 209–216.

[59] Rubinov A.M. *Abstract Convexity and Global Optimization*, Kluwer, Dordrecht 2000.

[60] Rubinov A.M., Abstract convexity: examples and applications. - *Optimization* 2000, **47**, 1–33.

[61] Rüschendorf L., On c-optimal random variables. - *Statist. Probab. Letters* 1996, **27**, 267–270.

[62] Singer I. *Abstract Convex Analysis*, Wiley-Interscience Publication, N.Y. 1997.

[63] Smith C., Knott M., On Höffding-Fréchet bounds and cyclic monotone relations. - *J. Multivar. Anal.* 1992, **40**, 328–334.

[64] Sudakov V.N., Geometric problems in the theory of infinite-dimensional probability distributions. - *Proc. Steklov Inst. Math.* 1979, **141**, 1–178; translated from the Russian: *Trudy Mat. Inst. Steklov* 1976, **141**.

[65] Trudinger N.S., Wang X.J., On the Monge mass transfer problem. - *Calc. Var. PDE* 2001, **13**, 19–31.

[66] Varian H.R., The nonparametric approach to demand analysis. - *Econometrica* 1982, **50**, 945–973.

[67] Varian H.R., Non-parametric tests of consumer behaviour. - *Rev. Econ. Studies* 1983, **V(1)**, no.160, 99–110.