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A MODEL OF RESOURCE REDISTRIBUTION

In this paper we investigate various aspects of resource redistribution procedures that suggest processes of exchange. The definition we have adopted of the elementary act of interaction stipulates that any participant at each stage may use only information on the stages of a fixed number of other participants. Sufficient conditions are formulated for the existence of a procedure from the given class that obtains a distribution of resources that is optimal in a certain sense.

1. Statement of the Problem

We will consider a system including m participants, each of which is represented by a concave (convex from above) function $f_k(x_k)$, $x_k \in R^n$, $k = 1, 2, \ldots, m$. The matrix $x = (x_1, \ldots, x_k, \ldots, x_m)$ of size $n \times m$ with nonnegative components we will call states of the system. We will let $f(x) = \sum_{k=1}^{m} f_k(x_k)$. For each subset of participants $\alpha \subset M = \{1, 2, \ldots, m\}$ we define for $x \geqslant 0$ a point-set mapping $\xi_\alpha(x)$:

$$\xi_\alpha(x) = \{z \mid z \in G_\alpha(x), f(z) = \max_{y \in G_\alpha(x)} f(y)\}, \qquad (1)$$

where $G_\alpha(x) = \{y \mid y = (y_1, \ldots, y_k, \ldots, y_m), y_k \in R^n, y \geqslant 0,$

$$\sum_{k\in\alpha} y_k = \sum_{k\in\alpha} x_k,\ y_k = x_k \text{ for } k \notin \alpha\Big\}. \qquad (2)$$

It follows from relations (1) and (2) that $f(\xi_\alpha(x)) \geqslant f(x)$ for any α and $x \geqslant 0$.

We will define a system A of subsets of the set M, $A = \{\alpha\}$, $\alpha \subset M$, $\alpha \neq \Phi$. In the following, A will usually contain all the possible subsets of fixed size θ, $2 \leqslant \theta \leqslant m$.

Definition 1. We will call a point-set mapping $\xi_\alpha(x)$, given by (1) and (2), a feasible transformation or a feasible bargain if $\alpha \in A$.

Definition 2. The point $x = (x_1, \ldots, x_k, \ldots, x_m)$, $x \geqslant 0$, we will call optimal if it is a solution to the problem:

$$f(y) = \sum_{k=1}^{m} f_k(y_k) \to \max, \sum_{k=1}^{m} y_k = \sum_{k=1}^{m} x_k,\ y_k \geqslant 0,\ k = 1, \ldots m. \ (3)$$

We will denote the greatest value of $f(y)$ in (3) by $f^*(x)$.

Definition 3. We will call the sequence of states $x^s, s = 0, 1, \ldots$, an optimizing sequence, if $x^s \in \xi_{\alpha_s}(x^{s-1})$, $\alpha_s \in A$, $s = 1, 2, \ldots$, and $f(x^s) \to f^*(x^0)$ as $s \to \infty$.

We will explain the definitions. We will interpret the functions $f_k(x_k)$ as utilities of the vector of resources x_k for the k-th participant measured in the same units. Let $x^0 = (x_1^0, \ldots, x_k^0, \ldots, x_m^0)$ be an arbitrary initial distribution of resources. The transformation $\xi_\alpha(x^0)$ corresponds to the following interaction. The participants of the set α combine their resources and find an arbitrary solution of the problem:

$$\sum_{k\in\alpha} f_k(y_k) \to \max, \quad \sum_{k\in\alpha} y_k = \sum_{k\in\alpha} x_k^0, \quad y_k \geqslant 0, \quad k \in \alpha. \quad (4)$$

Then the resources are distributed in conjunction with the solution that has been found. Here the resources of the participants who do not enter into the set α remain unchanged. Through the transaction, there will be a new state of the system $x^1 = (x_1^1, \ldots, x_k^1, \ldots, x_m^1)$, in which a bargain among the participants of a certain other set takes place, and so on. In the process of these transformations the total utility $f(x)$ does not decline and the total quantity of resources remains equal to $\sum_{k=1}^{m} x_k^0$.

Of course, for certain participants the value of the utility function after the bargain may turn out to be less than the initial value. Therefore, we must assume that, simultaneously with the transfer of resources, monetary calculations are made ("side payments" in the terminology of game theory), so that as a result of the bargain not one of the participants loses. A system of financial calculations may be determined by many methods, but for our purposes a specific method need not exist.

We will now assume that the number of participants in each bargain does not exceed θ. We ask whether a sequence of bargains $\xi_\alpha(\cdot)$ exists for the given initial state x^0 such that for some method of choosing $x^s \in \xi_{\alpha^s}(x^{s-1})$ the total utility will reach a maximum. (The value of this maximum $f^*(x^0)$ is determined by the initial state alone.) The simplest examples show that such sequences may not exist.

Example. We will consider the system consisting of three participants with the following utility functions: $f_1(y_1) = \min(u_1; v_1)$, $f_2(y_2) = 0.4u_2$, $f_3(y_3) = 0.4v_3$. Here $y_k = (u_k, v_k)$, u_k, v_k are scalars, $k = 1, 2, 3$. Let the initial state $x^0 = (x_1^0, x_2^0, x_3^0)$ be as follows: $x_1^0 = (0.0)$; $x_2^0 = (1.0)$; $x_3^0 = (0.1)$; only bargains between all possible pairs of participants are feasible. It is easy to see that any of the sets $\xi_\alpha(x^0)$, $\alpha = \{1, 2\}, \{1,3\}, \{2,3\}$ contains x^0 and does not contain any other point. Thus, no sequence of feasible bargains leads the system from the state x^0. However, as is easy to show, $f(x^0) = 0.8 < f^*(x^0) = 1$. To achieve the maximum state $x^* = (1.1;\ 0.0;\ 0.0)$ we must have a simultaneous transfer of resource u from the second participant and of resource v from the third participant to the first, i.e., a bargain among three participants.

An example that is similar to this in concept is given in [1], which studies a model close to the one we have described above, except that only pair-wise bargains are considered. It follows from the results of [1] that with $n=1$ (the case of one resource) we may always reach an optimal state through pair-wise bargains.

Thus, in order to guarantee the existence of an optimizing sequence, it is necessary to allow interaction between a definite number of participants at each stage. It will be shown below that this number depends, generally speaking, not only on the number n of resources, but also on the differential properties of the utility function. In this connection, we will require the concept of quasi-summator functions, which we consider in the next section.

2. Quasi-Summator Functions

We will present certain definitions and results from [2] that will be used below.

Let $\varphi(y)$ be a concave function, $y \in R^n$. The vector $p \in R^n$ is called a supporting functional to $\kappa\varphi(y)$ at the point y, if $\varphi(y+h) \leqslant \varphi(y) + ph$ (1) for all $h \in R^n$. The set $P(y)$ of supporting functionals to $\kappa\varphi(y)$ at the point y is closed, convex, and bounded. If $\varphi(y)=\sum_{k=1}^{m} a_k\varphi_k(y)$, $a_k \geqslant 0$, we have $P(y)=\sum_{k=1}^{m} a_k P_k(y)$. For a function that is differentiable at the point y, the set $P(y)$ contains a single gradient vector. We will denote by $\varphi'(y, h)$, $h \in R^n$ the limit:

$$\varphi'(y,h)=\lim_{t\to+0}\frac{\varphi(y+th)-\varphi(y)}{t}. \tag{5}$$

If $\varphi(y)$ is concave, the limit (5) exists for all y, $h \in R^n$, such that

$$\varphi'(y,h)=\min_{p\in P(y)} ph. \tag{6}$$

Let $y=(y_1,\ldots,y_n)$, $\nu\subset N=\{1,2,\ldots n\}$, $\bar{\nu}=N\setminus\nu$. We will denote by $Pr_\nu y$ the vector with coordinates z_i, $i=1,2,\ldots,n$, satisfying the condition:

$$\begin{aligned} z_i &= y_i, \text{ if } i\in\nu;\\ z_i &= 0, \text{ if } i\in\bar{\nu}. \end{aligned} \tag{7}$$

The following definition, obviously, should be introduced first and represents a fundamental principle for this section.

Definition 4. We will call the function $\varphi(y)$, $y=(y_1,\ldots,y_n)$ a quasi-summator function on the set of variables $\{y_i,\ i\in\nu\}$ at the point y^0, if $\varphi'(y^0,h)$ exists, and for all $h\in R^n$ the equation

$$\varphi'(y^0,h)=\varphi'(y^0,Pr_\nu h)+\varphi'(y^0,Pr_{\bar{\nu}} h). \tag{8}$$

holds.

In the future we will use the simpler expression "$\varphi(y)$ is quasi-summator on ν" in addition to the expression "$\varphi(y)$ is quasi-summator on the set of variables $\{y_i,\ i\in\nu\}$." If $\varphi'(y^0,h)$ exists for any h, it follows from the definition that $\varphi(y)$ is quasi-summator at the point y^0 for N and the empty set Φ. The quasi-summator nature with respect to $\nu\subset N$ always involves a quasi-summatorability over $\bar{\nu}$. The function $\varphi(y)$, which is differentiable at the point y^0, is quasi-summator at this point for any subset of variables. Actually, in this case, $\varphi'(y^0,h)=\sum_{i=1}^{n} p_i h_i$, where $p=(p_1,\ldots,p_n)$ is the gradient of $\varphi(y)$ at the point y^0.

We will indicate one more important subclass of quasi-summator functions. Assume $\varphi(y) = \psi(g_1(y^1), \ldots, g_i(y^i), \ldots, g_l(y^l))$; $\tilde{z}_i = g_i(\tilde{y}^i)$, $y^i \in R^{k_i}$, $y = (y^1, \ldots, y^i, \ldots, y^l)$; that the function $\psi(z_1, \ldots, z_l)$ is differentiable at the point $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_l)$; and all the derivatives $g_i'(\tilde{y}^i, h^i)$, $i = 1, 2, \ldots, l$, exist for any $h_i \in R^{k_i}$. It is easy to derive directly from (5) the result that $\varphi'(\tilde{y}, h) = \sum_{i=1} \psi_i' g_i'(\tilde{y}^i, h^i)$, where $\psi' = (\psi_1', \ldots, \psi_i', \ldots, \psi_l')$ is a gradient $\psi(z_1, \ldots, z_l)$, at the point $\tilde{z}$. It follows from this that $\varphi(y)$ is quasi-summator on the set of coordinates of the component y^i for any $i = 1, 2, \ldots, l$.

Let $P \subset R^n$, $\nu \subset N = \{1, 2, \ldots, n\}$. We will introduce the notation:

$$P_\nu = Pr_\nu P = \{z \mid z = Pr_\nu y,\ y \in P\}. \tag{9}$$

Below, we will use the following obvious properties of the operation of set projection:

$$Pr_\nu(P + Q) = Pr_\nu P + Pr_\nu Q, \tag{10}$$

$$Pr_\nu Pr_\mu P = Pr_{\nu \cap \mu} P, \tag{11}$$

$$Pr_\Phi P = 0, \quad Pr_N P = P. \tag{12}$$

It is especially useful to bear in mind the relationship:

$$Pr_\mu P + Pr_\nu P \supset Pr_{\mu \cup \nu} P, \tag{13}$$

that is correct for $\nu \cap \mu = \Phi$.

Theorem 1. Let P be a set of supporting functionals to the concave function $\varphi(y)$, $y \in R^n$ at the point y^0. In order that $\varphi(y)$ be quasi-summator on $\nu \subset N$ at this point, it is necessary and sufficient that the following relationship hold:

$$P = P_\nu + P_{\bar{\nu}}. \tag{14}$$

Proof. According to (6), equation (8) is equivalent to the

following relationship: $\min_{p \in P} ph = \min_{p \in P} pPr_\nu h + \min_{p \in P} pPr_{\bar{\nu}} h$. It is easy to check that $\min_{p \in P} pPr_\nu h + \min_{p \in P} pPr_\nu h = \min_{p \in P_\nu} ph + \min_{p \in P_{\bar{\nu}}} ph$ $= \min_{p \in P_\nu + P_{\bar{\nu}}} ph$. Thus, to prove the theorem it is enough to show equivalence of relations (14) and (15):

$$\min_{p \in P} ph = \min_{p \in P_\nu + P_{\bar{\nu}}} ph. \tag{15}$$

The last relationship must be fulfilled identically for $h \in R^n$. It is obvious that (15) follows from (14) and that $P_\nu + P_{\bar{\nu}} \supset P$. We will show that if (15) holds, $P_\nu + P_{\bar{\nu}} \subset P$. (2) If it is not so, a vector $\tilde{p} \in (P_\nu + P_{\bar{\nu}}) \setminus P$ exists. We will consider the hyperplane that strictly separates $\tilde{p}$ from the closed convex finite set P. Suppose that q is its directional vector, and $qp > q\tilde{p}$ for all $p \in P$. But then, $\min_{p \in P_\nu + P_{\bar{\nu}}} qp \leqslant q\tilde{p} < \min_{p \in P} qp$, which contradicts (15). The theorem is proved.

Corollary. If the concave function $\varphi(y)$, $y \in R^n$ is quasi-summator at the point y^0 on the sets $\nu, \mu \subset N = \{1, 2, \ldots, n\}$, it is quasi-summator on the intersection and the union.

Proof. By assumption and Theorem 1, the following equations hold: $P = P_\nu + P_{\bar{\nu}} = P_\mu + P_{\bar{\mu}}$. Therefore, utilizing (10)-(13) we get $P = Pr_\mu(P_\nu + P_{\bar{\nu}}) + Pr_{\bar{\mu}}(P_\nu + P_{\bar{\nu}}) = P_{\mu \cap \nu} + P_{\mu \cap \bar{\nu}} + P_{\bar{\mu} \cap \nu} + P_{\bar{\mu} \cap \bar{\nu}} \supset P_{\mu \cap \nu} + P_{\overline{\mu \cap \nu}}$. But then $P_{\mu \cap \nu} + P_{\overline{\mu \cap \nu}}$ and, by Theorem 1, $\varphi(y)$ is quasi-summator on $\mu \cap \nu$.

Since $\varphi(y)$ is quasi-summator on $\bar{\mu}$ and $\bar{\nu}$, it immediately follows that it is quasi-summator on the $\bar{\mu} \cap \bar{\nu}$ and, consequently, on $\overline{\bar{\mu} \cap \bar{\nu}} = \mu \cup \nu$ as well.

Corollary 2. Suppose the sets ν_i, $i = 1, 2, \ldots, l$, form a subdivision of the set $N = \{1, 2, \ldots, n\}$. (3) In order that the concave function $\varphi(y)$, $y \in R^n$ be quasi-summator on each ν_i at the point y^0, it is necessary and sufficient that the relationship

$$P = \sum_{i=1}^{l} P_{\nu_i}. \tag{16}$$

hold.

Here, as above, P is a set of supporting functionals to the function $\varphi(y)$ at the point y^0.

Proof. From (16) and (13), it is easy to derive the relationship $P \supset P_{\nu_i} + P_{\bar{\nu}_i}$. But then, in conformity with (12) and (13), $P = P_{\nu_i} + P_{\bar{\nu}_i}$. Sufficiency now follows from Theorem 1. Necessity will be shown by induction. According to Theorem 1, this assertion is true for $l = 2$. Let it be true for $l - 1$.

Then, using Corollary 1, we may write: $P = \sum_{i=1}^{l-2} P_{\nu_i} + P_{\nu_{l-1} \cup \nu_l}$.

But, since $P = P_{\nu_l} + P_{\bar{\nu}}$ (Theorem 1), we obtain, by using (10)-(13),

$$P = Pr_{\nu_l}\left(\sum_{i=1}^{l-2} P_{\nu_i} + P_{\nu_{l-1} \cup \nu_l}\right) + Pr_{\bar{\nu}_l}\left(\sum_{i=1}^{l-2} P_{\nu_i} + P_{\nu_{l-1} \cup \nu_l}\right) =$$

$$P_{\nu_l} + \sum_{i=1}^{l-2} P_{\nu_i} + P_{\nu_{l-1}},$$

which was required.

3. Theorem for the Existence of an Optimizing Sequence

We turn now to consideration of the problem formulated in Section 1. Below we show the sufficient conditions for the existence of an optimizing sequence of states and a method for constructing it.

We assume that $x_k = (x_{1k}, \ldots, x_{jk}, \ldots, x_{nk})$, $k = 1, 2, \ldots, m$; $N = \{1, 2, \ldots, n\}$. Earlier, the space R^n was interpreted as the space of resource sets; in this connection the set $\nu \subset N$ will be identified with the set of all resources enumerated by numbers from ν.

Definition 5. We will call the set (of resources) $\nu \subset N$, $\nu \neq \Phi$ a complex at the point $x^0 = (x_1^0, \ldots, x_k^0, \ldots, x_m^0)$ relative to the function $f(x) = \sum_{k=1}^{m} f_k(x_k)$, $x_k \in R^n$, if: (a) all the functions $f_k(x_k)$, $k = 1, 2, \ldots, m$ are quasi-summator on $\{x_{jk} | j \in \nu\}$ at the point x_k^0; (b) not one of the proper subsets of the set ν is not possessed of property (a).

At any point at least one complex exists since the functions $f_k(x_k)$ are quasi-summator on N. If ν and μ are two different complexes at the point x^0, then $\nu \cap \mu = \Phi$. Otherwise, according to corollary 1, the set $\nu \cap \mu$ would possess property (a) which would contradict 5. If ν is a complex, the set $\bar{\nu}$ either is itself a complex, or else contains a certain complex. Thus, the following assumption is justified.

Lemma 1. At any point the totality of complexes defines a subset of the set N.

The concept of complex that we have introduced is connected with another concept used in economic practice. "The value" of a small change in resource entering complex ν, generally speaking, depends on the increases in the other resources in ν, and does not depend on the change in the resources relating to other complexes. It will be shown below that the structure of efficient local behavior of resource redistribution is determined largely by the character of the partition of the set of resources into complexes.

Before we consider the proof of the basic theorems, we will introduce a few other concepts which will be useful in what follows.

Definition 6. A feasible transformation $\xi_\alpha(x)$ is called efficient at the point $x \geqslant 0$, if $f(\xi_\alpha(x)) > f(x)$. If $\alpha \in A$ and $f(\xi_\alpha(x)) = \max_{\beta \in A} f(\xi_\beta(x))$, we will call the transformation $\xi_\alpha(x)$ most efficient at point x. Definitions 1 and 6 also define the notions of "efficient bargain" and "most efficient bargain" (at point x).

Definition 7. The point $x \geqslant 0$ is called a deadlock if not a single feasible transformation in it is efficient.

The optimal point, of course, is a deadlock; the converse, generally speaking, is not true (see the example in Section 1).

Theorem 2. Let $f_k(x_k)$, $k = 1, 2, \ldots, m$, be concave; $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ — a deadlock; and θ — the greatest number of resources containing the complex $\theta < m$. If bargains between any $\theta + 1$ participants are feasible, $\tilde{x}$ is an optimal point.

Proof. From the assumptions and the definition of a deadlock point, it follows that for any $\alpha \in A$ the totality of vectors $\{\tilde{x}_k | k \in \alpha\}$ forms a solution to the problem:

$$\sum_{k \in \alpha} f_k(y_k) \to \max, \quad \sum_{k \in \alpha} y_k = \sum_{k \in \alpha} \tilde{x}_k; \quad y_k \geqslant 0, \quad k \in \alpha. \quad (17)$$

Let P_k be the set of supporting functionals to the function $f_k(y_k)$ at the point $\tilde{x}_k$. In order that the set $y_k = \tilde{x}_k$, $k \in \alpha$, be a solution to the problem (17), it is necessary and sufficient that vectors $p_k{}^\alpha \in P_k$, $w_k{}^\alpha \geqslant 0$ and λ^α exist satisfying the conditions in [2]:

$$p_k{}^\alpha = \lambda^\alpha - w_k{}^\alpha, \quad w_k{}^\alpha \cdot \tilde{x}_k = 0, \quad k \in \alpha. \quad (18)$$

Let ν_i, $i = 1, 2, \ldots, l$, be a partitioning of the set N into complexes at the point $\tilde{x}$. We introduce the notation:

$$W_k = \{w_k | w_k \geqslant 0, \quad w_k \tilde{x}_k = 0\}, \quad k = 1, 2, \ldots, m, \quad (19)$$

$$L_{ki} = Pr_{\nu_i} P_k + Pr_{\nu_i} W_k, \quad i = 1, 2, \ldots, l, \quad k = 1, 2, \ldots, m. \quad (20)$$

The sets L_{ki}, obviously, are convex; they may be considered to be concentrated on the subspace of dimension of θ. From the relations (18)-(20), we conclude that for any feasible $\alpha \bigcap_{k \in \alpha} L_{ki} \supseteq Pr_{\nu_i} \lambda^\alpha$.

Since all the α containing the $\theta + 1$ subscripts are feasible, we may apply Khelli's theorem. (4) We obtain $\bigcap L_{ki} \neq \Phi$ for any i (we recall that $M = \{1, 2, \ldots, m\}$). Let $\lambda_i \in$ define $\in \bigcap^{k \in M} L_{ki}$, and then $\lambda_i \in Pr_{\nu_i} P_k + Pr_{\nu_i} W_k$ for all $k \in M$. We will define $\lambda^M = \sum_{i=1}^{k \in M} \lambda_i$. Obviously, $W_k = \sum_{i=1}^{l} Pr_{\nu_i} W_k$; and, through corollary 2, $P_k = \sum_{i=1}^{l} Pr_{\nu_i} P_k$ as well. Thus, for any k, vectors $p_k{}^M \in P_k$, $w_k{}^M \in W_k$ exist, such that $\lambda^M = p_k{}^M + w_k{}^M$. This relationship is equivalent to (18), with $\alpha = M$ and is a sufficient condition for the optimality of $\tilde{x}$. The theorem is proved.

Corollary 3. Suppose that for any k the function $f_k(x_k)$ is differentiable at the point $\tilde{x}_k$ and all the possible bargains between pairs of participants are feasible. If $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_m)$ is a deadlock point, it is optimal.

Proof. Since $f_k(x_k)$, $k = 1, 2, \ldots, m$, is quasi-summator at the point $\tilde{x}_k = (\tilde{x}_{1k}, \ldots, \tilde{x}_{nk})$ for any set $\{x_{jk}\}$, that includes one variable, every resource forms a complex and $\theta = 1$, which it was required to show.

Corollary 4. Assume that m is greater than n. If all the possible bargains between the $(n + 1)$-th participants are feasible, any deadlock point is optimal. (5)

Proof. Let $\tilde{x}$ be a deadlock point and θ be the greatest number of resources contained in its complex. If A does not contain all the subsets $\alpha \subset M$ of capacity $\theta + 1$, we will consider a new set of subscripts of feasible bargains: $A_1 = \{\alpha | \alpha \neq \Phi, \alpha \subset \beta, \beta \in A\}$. Since $\theta \leqslant n$, A_1 contains all the sets of the $\theta + 1$ participants. If the bargain $\xi_\beta(x)$ is not efficient at a given point, then for any $\alpha \subset \beta$ the bargain $\xi_\alpha(x)$ and subsequent ones will not be efficient in the same point. Therefore, the point $\tilde{x}$ is a deadlock for larger sets of feasible bargains as well. Therefore, our assertion is proved by the corollary of theorem 2.

We will now show that in the conditions of theorem 2 an optimizing sequence exists for any initial point there.

Theorem 3. Let $x^s \in \xi_{\alpha}{}^s (x^{s-1})$, $s = 1, 2, \ldots$, $\alpha_s \in A$. If (a) a set of feasible bargains is chosen so that any deadlock point is optimal and (b) the sequence ξ_{α_s} contains a finite subsequence of more efficient bargains (definition 6), $f(x^s) \to f^*(x^0)$ as $s \to \infty$. (6)

Proof. The sequence $f(x^s)$ does not decrease monotonically and that is why it converges. Let $f(x^s) \to \bar{f} < f^*(x^0)$ and $\alpha_{s_i} = \beta(i)$ be a subsequence with subscripts of the most efficient bargains such that $x^{s_i-1} \to \bar{x}$. Since $f(x^{s_i-1}) \to f(\bar{x}) < f^*(x^0)$, an efficient bargain with subscript $\gamma \in A$ exists at the point $\bar{x}$. We get the required contradiction now from the following change of relationships:

$$\bar{f} < f(\xi_\gamma(\bar{x})) = \lim_{i\to\infty} f(\xi_\gamma(x^{s_i-1})) \leqslant \lim_{i\to\infty} f(\xi_{\beta(i)}(x^{s_i-1})) = \lim_{i\to\infty} f(x^{s_i}) = \bar{f}.$$

Here we use the continuity of $f(\xi_\gamma(x))$ as a function of x. The theorem has been proved.

On the basis of theorems 2 and 3, we get the following sufficient conditions for the existence of an optimizing sequence.

Theorem 4. Let $x^0 = (x_1{}^0, \ldots, x_k{}^0, \ldots, x_m{}^0)$, $x^0 \geqslant 0$, and for any y satisfying the conditions $y \geqslant 0$, $\sum_{k=1}^{m} y_k = \sum_{k=1}^{m} x_k{}^0$, the greatest number of resources in the complex does not exceed θ. If all the bargains among the $\theta + 1$ participants are feasible, an optimizing sequence of states proceeding from point x^0 exists.

Instead of a condition of feasibility for all the bargains between the $\theta + 1$ participants, we may require that somewhat less stringent conditions be fulfilled: for any $\alpha \subset M$, containing $\theta + 1$ elements, we find a $\beta \in A$ such that $\alpha \subset \beta (*)$. If the condition (*) is not fulfilled, the assertion of theorem 4 is not true, generally speaking. An example for $\theta = n = 2$ was considered in Section 1. It may easily be generalized to the case of arbitrary n and θ.

One other corollary of the foregoing assertions should be mentioned. We will say that the point $x \geqslant 0$ is isolated if it is not contained in any optimizing sequence of states. A deadlock point may not be either optimal or isolated. It follows

from theorem 3 that when there are no nonoptimal deadlock points, isolated points are also absent. Corollary 3 shows that the set of nonoptimal deadlock points belongs to the set of points of undifferentiability of the function $f(x)$ (if only pairwise bargains are feasible) and, consequently, has Lebesque measure zero. However, more detailed consideration of the example in Section 1 shows that the measure of the set of isolated points is not necessarily equal to zero. Thus, as a result of "local" changes in the function $f_k(x_k)$, "global" results may arise.

4. Random Bargain Sequences

The method of search for the optimizing sequence shown in theorem 3 requires that the most efficient bargains be effected in a finite sequence of steps. The question arises of whether fulfillment of this condition can be ensured without presupposing the existence of centralized information in the system. It is natural to assume that the participants enter into bargains according to some random mechanism. One of the possible statements of the problem is as follows.

On the set $\Omega = \{\omega\}$ of sequences of the form $\omega = (\alpha_1, \alpha_2, \ldots \alpha_s \ldots)$, $\alpha_s \in A$, we define a measure μ such that for any finite sets $\beta_i \in A_i$ and α_{s_i}, $i = 1, 2, \ldots, t$, the following equation holds: $\mu\{\omega \mid \alpha_{s_i} = \beta_i,\ i = 1, 2, \ldots, t\} = \prod_{i=1}^{t} p_{\beta_i}$, where $\sum_{\beta\in A} p_\beta = 1$, $p_\beta \geqslant p > 0$ for all $\beta \in A$.

This definition corresponds to an independent choice at each step s of a certain feasible set of participants with probabilities p_β, not depending on the superscript s. We will also introduce the notation:

$$U(\omega, x^0) = \{u \mid u = (x^0, x^1, \ldots, x^s \ldots),\ x^s \in \xi_{\alpha_s}(x^{s-1}),\ s = 1, 2, \ldots\}, \quad (21)$$

$$F(\omega, x^0) = \inf_{u \in U(\omega, x^0)} \lim_{s\to\infty} f(x^s). \quad (22)$$

We observe that $\lim f(x^s)$ necessarily exists for any sequence of states $\overset{\to\infty}{x} \in U(\omega, x^0)$, since in this case $f(x^s)$ is bounded and does not fall monotonically. We ask whether we may assert that the equation

$$F(\omega, x^0) = f^*(x^0) \tag{23}$$

is met with probability 1. Since $\lim_{s\to\infty} f(x^s) \leqslant f^*(x^0)$, the observance of equality (23) means that for almost all ω $\lim_{s\to\infty} f(x^s) = f^*(x^0)$, under any selection of vectors x^s from the set of solutions of the corresponding extremal problem.

If the result of any bargain is determined uniquely ($\xi_\alpha(x)$ are one-to-one mappings), the answer to this question follows easily from the results of Section 3. Actually, in this case $U(\omega, x^0)$ is a one-to-one function of ω and x^0. We will fix the initial state. Then we may define uniquely over ω a sequence $l(\omega)$ of subscripts of the most efficient bargains $l(\omega) = (l_1, l_2, \ldots, l_s, \ldots)$.

We will consider the events $B_s = \{\omega \mid \alpha_s \neq l_s\}$, $s = 1, 2, \ldots$. The random element $\alpha_s(\omega)$ does not depend upon $\alpha_1(\omega), \ldots, \alpha_{s-1}(\omega)$, while, $l_s(\omega)$, on the other hand, is uniquely determined by their values.

Therefore, $\mu(B_s) = \sum_{\alpha \in A} \mu\{\omega \mid l_s = \alpha\}\, \mu\{\omega \mid \alpha_s \neq \alpha\} \leqslant 1 - p$.

We assume that

$$\mu\Big(\bigcap_{s=1}^{h-1} B_s\Big) \leqslant (1-p)^{h-1}.$$

Then

$$\mu\Big(\bigcap_{s=1}^{h} B_s\Big) = \mu\{\omega \mid \alpha_s \neq l_s,\quad s = 1, 2, \ldots, h\} = \sum_{\alpha \in A} \mu\{\omega \mid \alpha_s \neq l_s,$$

$$s = 1, 2, \ldots, h-1,\ l_h = \alpha\}\ \mu\{\omega \mid \alpha_h \neq \alpha\} \leqslant (1-p)\,\mu\Big(\bigcap_{s=1}^{h-1} B_s\Big) \leqslant (1-p)^{h}$$

From the foregoing inequality, it follows that almost all ω contain the infinite sequence of most efficient bargains. Now with the help of theorems 2 and 3 we can establish the truth of the following statement.

Theorem 5. Let the conditions of theorem 4 be fulfilled and every feasible bargain be a one-to-one mapping. Then $f(x^s) \to f^*(x^0)$ with probability $\mu = 1$.

For the case of one resource the proof of theorem 5 follows directly from the results of [1]. (7)

The uniqueness of the mappings $\xi_\alpha(x)$ is guaranteed, for example, in the case of strict concavity of the functions $f_k(x_k)$.

It may be shown that in the conditions of theorem 4, equation (23) holds with the probability 1 for concave differentiable (but not necessarily strictly concave) functions $f_k(x_k)$.

A similar statement remains unproved for the general case. In connection with this, it is interesting to consider interaction schemes close to the ones studied above, for which a similar problem has been positively solved.

Let $x = (x_1, \ldots, x_k, \ldots, x_m)$ be the states of the system, and a — a numerical parameter greater than zero. We will introduce a definition of an elementary transformation:

$$\xi_\alpha(x, a) = \begin{cases} \Psi_\alpha(x, a), & \text{if} \quad \Psi_\alpha(x, a) \neq \Phi, \\ x, & \text{if} \quad \Psi_\alpha(x, a) = \Phi; \end{cases} \tag{24}$$

$$\Psi_\alpha(x, a) = \Big\{ y \mid y \geqslant 0,\ f(y) \geqslant f(x) + a,\ \sum_{k \in \alpha} y_k = \sum_{k \in \alpha} x_k,\quad y_k = x_k \text{ при } k \in \alpha \Big\} \tag{25}$$

Here, as above, $y = (y_1, \ldots, y_k, \ldots, y_m)$, $f(y) = \sum_{k=1}^{m} f_k(y_k)$, and α is a subset of participants. It is obvious that any sequence of the form $x^s \in \xi_{\alpha_s}(x^{s-1}, a)$, $s = 1, 2, \ldots$ is reached in a finite number of steps. Let x^a be any limiting state. It is clear that it need not be optimal, but in the conditions of theorem 4 it turns out to be close to the optimal for small values of a.

Theorem 6. Suppose that in the sequence α_s each symbol is repeated an infinite number of times, and the set A is such that any deadlock point is optimal. Then $f(x^a) \to f^*(x^0)$ as a goes to zero, no matter what the selection of final states x^a.

Proof. It is obvious that for any $\alpha \in A$:

$$0 \leqslant f(\xi_\alpha(x^a)) - f(x^a) \leqslant a.$$

The mapping $\xi_\alpha(x)$, as before, is given by (1) and (2). Let $a \to 0$, $a > 0$. Then

$$f(\xi_\alpha(x^a)) - f(x^a) \to 0. \tag{26}$$

If $\bar{x}$ is a limiting point of the sequence x^a, then, as follows from (26) and the continuity of $f(\xi_\alpha(x))$, we have the inclusion: $\bar{x} \in \xi_\alpha(\bar{x})$ and $A\,\alpha \in A$. Thus, $\bar{x}$ is a deadlock point and this means it is conditionally optimal, from which follows the required convergence.

Thus, in the conditions of theorem 4, any random mechanism that, with probability 1, gives rise to sequences containing a finite number of each of the feasible bargains of type (24) achieves a distribution of resources which for small values of a is close to the optimal distribution.

Comment. For every $\alpha \subset \{1, 2, \ldots m\}$ we will consider the following convex programming problem:

$$f(y_1, \ldots, y_k, \ldots, y_m) \to \max, \tag{27}$$

$$\sum_{k=1}^{m} g_k(y_k) \geqslant 0, \tag{28}$$

$$y_k \in Q_k, \quad k = 1, 2, \ldots, m, \tag{29}$$

$$y_k = \tilde{x}_k, \quad k \bar{\in} \alpha. \tag{30}$$

Here, y_k are vectors of size r_k; $g_k(y_k)$, $k = 1, 2, \ldots m$ are vector functions having n scalar components; $\sum_{k=1}^{m} g_k(\tilde{x}_k) \geqslant 0$, $\tilde{x}_k \in Q_k$ $k = 1, 2, \ldots m$, $m > n$.

The following statement, generalizing corollary 4, may be proved by the same method as theorem 2.

Theorem 7. Let the function $f(y_1, \ldots, y_k, \ldots, y_m)$ be concave and quasi-summator on y_k, $k = 1, 2, \ldots m, m$, at the point $(\tilde{x}_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_m)$, the vector functions $g_k(y_k)$ be concave; and the sets Q_k be closed and convex. If the vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_m)$ is a solution to problems (27)-(30) for any a containing $n+1$ elements, $\tilde{x}$ maximizes function (27) under the constraints of (28) and (29).

For the case of a differentiable function $f(y_1, \ldots, y_k, \ldots, y_m)$, and for linear vector functions $g_k(y_k)$ and $Q_k = \{y_k \mid y_k \geqslant 0\}$ this assertion is proved in [5], which appeared after this paper went into press.

Notes

1) ph is the inner product of the vectors p and h.

2) The following considerations, in essence, reproduce the proof of lemma 1 in [2].

3) I.e., $\nu_i \cap \nu_j = \Phi$, $i, j = 1, \ldots, l$, $\bigcup_{i=1}^{l} \nu_i = N$.

4) Khelli's theorem. Let K be a family of sets in θ-dimensional vector space in which K is finite or each set of K is compact. If all the $\theta+1$ sets of family of K have a common point, the intersection of all the sets of the family of K is not empty [3, 4].

5) The idea of considering the interaction between the $(n+1)$-th participants for the case of n resources was given to the author by B. S. Mitiagin in discussing [1].

6) We recall that $f^*(x^0)$ is the greatest value of the function $f(v)$ in problem (3) with $x = x^0$ (and, conseqnently, with $x = x^s$).

7) We emphasize that the proof given in [1] is justified only in conditions of uniqueness of the mappings $\xi_\alpha(x)$ (this is not stipulated in [1]).

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