GROSS SUBSTITUTABILITY MAPPINGS IN ECONOMIC EQUILIBRIUM THEORY

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A survey of the results relating to the application of gross substitutability in economic equilibrium theory. The topics considered include existence and uniqueness of equilibrium, comparative statics, coalition stability, and stability of price-adjustment tatonnement processes. The main theorems cover the case of multivalued demand satisfying the gross substitutability condition and, in particular, are applicable to linear exchange models.

Mappings endowed with the property of gross substitutability occur in many branches of mathematical economics, in multivariable control theory, and also in other fields. The corresponding literature is quite extensive. This survey mainly covers the works with economic equilibrium orientation published after 1965. Earlier results are only treated partially, and most of them can be found in the monographs [36, 10, 19, 20].

An essential feature of our survey is the systematic treatment of multivalued mappings with gross substitutability recently introduced in [26, 28, 57]. This approach enabled us to incorporate in the general theory all the linear models which previously required special methods.

The following notation system is used in the article. If X is a set, then 2^X is the system of all its subsets, int X is the interior of X. $\mathbf{R}^n$ is the n-dimensional euclidean space, $\mathbf{R}_+^n$, $\mathbf{R}_-^n$ are sets of vectors in $\mathbf{R}^n$ with nonnegative and nonpositive components, respectively; $x=(x_i)$ is a vector with components x_i whose dimension is specified in each particular case. If $x, y \in \mathbf{R}^n$ then xy denotes the scalar product of the two vectors. If α is a scalar, then αx is the product of the vector by the corresponding number. If $M=\{1,\ldots,m\}$ is the set of integers from 1 to m and $d^k \in \mathbf{R}^n$, the symbol $(d^k, k \in M)$ denotes a naturally ordered system of vectors.

Definitions, theorems, formulas, etc., and subsections are identified by two-digit notation: the first number represents the corresponding section, and the second number is the sequence within the section.

1. DEFINITIONS, EXAMPLES, MAIN ASSUMPTIONS

1.1. We start with some basic definitions. Let $P \subset \mathbf{R}^n$ and the function $\mathscr{D}=(\mathscr{D}_i)$ associates with each vector $p \in P$ a certain vector $\mathscr{D}(p) \in \mathbf{R}^n$.

Definition 1.1. We say that the function $\mathscr{D}$ has the property of gross substitutability (g.s.), and call it a GS-function,* if $\mathscr{D}_i(p)$ is nondecreasing in p_j for all i, j, $i \neq j$.

*The notation is a mnemonic representing "gross substitute;" note that the symbols $\mathscr{D}$ used for the function and p for its argument are associated with their interpretation as "demand" and "prices."

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Definition 1.2. If $\mathscr{D}_i(p)$ is increasing in $p_j \forall i,\ j,\ i \neq j$, then $\mathscr{D}$ satisfies the property of strict gross substitutability and is called a strict GS-function.

We sometimes speak of strict gross substitutability in differential form, implying the condition

$$\frac{\partial \mathscr{D}_i(p)}{\partial p_j} > 0 \ \forall i,\ j,\quad i \neq j \text{ and } \forall p \in P. \tag{1.1}$$

Let $p \in P,\ q \in P,\ N = \{1, \ldots, n\}$. We use the notation

$$I(p, q) = \{i \mid p_i = q_i,\ i \in N\}. \tag{1.2}$$

Definition 1.3. A GS-function is called indecomposable if for all $p \in P,\ q \in P$ the condition $p \leqslant q$ implies the inequality

$$\min_{i \in I(p,q)} (\mathscr{D}_i(p) - \mathscr{D}_i(q)) < 0.$$

The notion of substitutes was introduced by Hicks [56] and the term "gross substitutability" is due to Mosak [68]. There is, however, no uniform terminology to this date. Sometimes gross substitutability (g.s.) is identified with the property in Definition 1.2, and g.s. in the sense of Definition 1.1 is called "weak gross substitutability." Strict gross substitutability is sometimes also called strong. Other terminology is also used.

The origin of the term "gross substitutability" is associated with the interpretation of $\mathscr{D}$ as the demand function dependent on the price vector p. This property a priori holds if the user regards all the goods as substitutes; then as the price of one good increases, the demand for all the other goods does not decrease.* This is, of course, not always so. In consumption theory, complementarity is defined in addition to substitutability (see Sec. 7). Thus, butter and margarine are examples of substitutes, while gasoline and automobiles are examples of complements.

Although gross substitutability is not a universal property, it has been studied by many authors. This is due to several factors. First, gross substitutability is a fairly frequent phenomenon (see Subsec. 1.2). Second, the results relating to gross substitutes are also applicable to complementary goods and to "mixed" systems (Sec. 7). Third, this property combined with certain additional assumptions ensures "correct" (i.e., consistent with economic intuition) behavior of the system. For models with gross substitutability we can prove convexity of the set of equilibrium prices or even their uniqueness (Secs. 2, 3), elucidate the variation of the equilibrium as a function of various exogenous parameters (Sec. 4), establish convergence of price adjustment (tatonnement) processes (Sec. 6), etc. It is by no means clear that all these results can be derived also for a wider class of cases.

In a number of important cases (see Subsec. 1.2), demand is not a single-valued function of the price vector. Therefore, the following generalization of gross substitutability was proposed in [26, 28].

Let $P \subset \mathbf{R}^n$ and let the mapping $\mathscr{D}$ associate witheach vector $p \in P$ a certain set $\mathscr{D}(p) \subset \mathbf{R}^n$.

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*The meaning of the adjective "gross" will be elucidated in Subsec. 1.2.

Definition 1.4. A mapping $\mathscr{D}$ has the property of gross substitutability, and is called a GS-mapping, if for any two vectors p, q from P such that $p \leqslant q$ and $I(p, q) \neq \varnothing$ (see (1.2)) and for any $d=(d_i) \in \mathscr{D}(p)$, $f=(f_i) \in \mathscr{D}(q)$, we have

$$\min_{i \in I(p,q)} (d_i - f_i) \leqslant 0. \tag{1.3}$$

If, moreover, for p ≠ q we have a strict inequality in (1.3), the GS-mapping is called indecomposable.

The following proposition is easily proved.

Proposition 1.1. Let $\mathscr{D}$ be a single-valued function. If $\mathscr{D}$ satisfies Definition 1.1, then it is a GS-mapping in the sense of Definition 1.4. Conversely, if $\mathscr{D}$ is continuous, defined on an open set P, and is a GS-mapping, then it has the property of gross substitutability in the sense of Definition 1.1.

Definition 1.3 and the notion of indecomposable GS-mapping are similarly related.

We will soon show (see Example 1.1) that the continuity condition in the second part of Poposition 1.1 cannot be relaxed. Thus, a single-valued GS-mapping and a GS-function are not equivalent concepts.

The gross substitutability property in the many-valued case is not additive in general (see Example 1.2). However, the GS-mappings arising in many equilibrium models belong to a certain subclass which is closed under algebraic addition. This subclass merits a special definition.

Definition 1.5 [26]. We say that $\mathscr{D}$ is an AGS-mapping if for any vectors $p=(p_i)$, $q=(q_i)$ from P such that $p \leqslant q$, $I(p, q) \neq \varnothing$ and for any $d=(d_i) \in \mathscr{D}(p)$, $f=(f_i) \in \mathscr{D}(q)$, we have

$$\sum_{i \in I(p,q)} p_i d_i \leqslant \sum_{i \in I(p,q)} q_i f_i. \tag{1.4}$$

If, moreover, for p ≠ q we have a strict inequality in (1.4), the AGS-mapping is called indecomposable.

Independently of [26, 28], Howitt [57] defined a class of mappings occupying an intermediate position between AGS-mappings and indecomposable AGS-mappings.

If $P \subset \operatorname{int} \mathbf{R}_+^n$, then an (indecomposable) AGS-mapping is obviously a(n) (indecomposable) GS-mapping.

GS- and AGS-mappings have a natural economic interpretation.

Let $\mathscr{D}(p)$ be the collection of all possible demand vectors corresponding to the price vector p; assume that the prices of some commodities have increased to q_i, $i \notin I(p, q)$. Then (1.3) implies that, for all the possible realizations of demand at the points p and q, there is a commodity with unchanged price the demand for which does not decrease; inequality (1.4) implies that in this case the cost of purchasing all the commodities with unchanged prices does not diminish.

In what follows we will require the following concept from [27] (see also [1]).

Definition 1.6. The mapping $\mathscr{F}: P \to 2^{\mathbf{R}^n}$, $P \subset \mathbf{R}^l$ is nondecreasing if for any vectors p, q, d, f such that $p \in P$, $q \in P$, $p \leqslant q$, $d \in \mathscr{F}(p)$, $f \in \mathscr{F}(q)$, there are vectors $d' \in \mathscr{F}(p)$, $f' \in \mathscr{F}(q)$ satisfying the inequalities

$$d \leqslant f', \ f \geqslant d'. \tag{1.5}$$

If for $p \neq q$ the inequalities in (1.5) are strict, then $\mathcal{F}$ is increasing.

We say that $\mathcal{F}$ is nonincreasing (decreasing) if the mapping $(-\mathcal{F})$ is nondecreasing (increasing).

Definition 1.7. The mapping $\mathcal{F}: P \to 2^{R^n}$, $P \subset \mathbf{R}^l$ is called normal (strictly normal) if for any $p \in P$ the mapping $\Phi_p(\lambda) = \mathcal{F}(\lambda p)$ is nondecreasing (increasing).

1.2. Examples of GS-Mappings. In equilibrium theory it is generally assumed that each consumer (or each group of consumers) is characterized by an objective function u(x) (where x is the n-dimensional vector of consumption goods), and the value of his demand function is the solution of the problem maximizing u(x) subject to the budget constraint

$$u(x) \to \max, \ px \leqslant \beta, \ x \geqslant 0, \tag{1.6}$$

where β is the consumer income.

Denote by $\mathcal{C}(p, \beta)$ the set of solutions of the problem (1.6). Our immediate objective is to discuss the conditions which ensure gross substitutability of $\mathcal{C}(p, \beta)$ for $P = \operatorname{int} R_+^n$ and a fixed $\beta > 0$ (the case $\beta = 0$ is self-evident). Since $\mathcal{C}(p, \beta)$ is positive homogeneous, it suffices to check gross substitutability for $\beta = 1$.

By the Slutsky equation [30, p. 255], each derivative $\partial \mathcal{C}_i / \partial p_j$ is the sum of two components, representing the "income effect" and the "substitution effect." The substitution effect is associated with simultaneous changes in prices and income, leaving the maximum utility level unchanged. Gross substitutability, on the other hand, is the outcome of both these effects, which is emphasized by the term originally introduced in [68] (for a discussion also see [20, p. 305]).

Given the Hessian matrix of the function u, we can compute the derivatives $\partial \mathcal{C}_i / \partial p_j$ [30]; in this way, we obtain necessary and sufficient conditions of gross substitutability for a smooth u. These conditions are very cumbersome and difficult to use. However, in the particular case when

$$u(x) = \sum_{i=1}^{n} u_i(x_i), \tag{1.7}$$

a simple criterion is available, which is applicable if the consumer income is a function of the prices.

THEOREM 1.1. Let the function (1.7) have no maximum on R_+^n, let $u_i(x_i)$ be continuous and concave, and $u_i'(x_i)x_i$ be defined* for $x_i > 0$ and nondecreasing for all i. Also assume that the function $\delta(p)$ is nondecreasing and nonnegative on $\operatorname{int} \mathbf{R}_+^n$. Then $\mathcal{F}(p) = \mathcal{C}(p, \delta(p))$ is an AGS-mapping on $\operatorname{int} R_+^n$.

This proposition follows directly from Slutsky's results [30] if $\delta(p) = 1$, u is twice continuously differentiable and strictly concave, and $\mathcal{C}(p, 1) > 0$ for all $p > 0$. Then $\mathcal{C}(p, 1)$ is a strict GS-function. It is noted in [18] that if strict positivity is relaxed, then $\mathcal{C}(p, 1)$ remains a GS-function, but is not necessarily strict. For linear u, Theorem 1.1

*Concavity implies that $u(tx + (1-t)y) \geqslant tu(x) + (1-t)u(y)$, $0 < t \leqslant 1$, $x, y \in \mathbf{R}_+^n$. By u'_i we denote the derivatives of u_i.

was proved in [26] (see also [57]). The formulation includes the case of multivalued demand with nonlinear u.

The proof of Theorem 1.1 follows from an analysis of optimality conditions for the problem (1.6). We can similarly show that if, in addition to the assumptions of Theorem 1.1, the functions $u_i'(x_i)x_i$ are strictly increasing and $\mathcal{F}(p)\subset \operatorname{int} \mathbf{R}_+^n$ for all $p>0$, then the AGS-mapping $\mathcal{F}(p)$ is indecomposable.

By Theorem 1.1, objective functions of the form $u(x)=\sum_{i=1}^{n}\mu_i x_i^{\alpha_i}$, where $0\leqslant\alpha_i\leqslant 1$, $\mu_i\geqslant 0$, $\max_{i=1,\ldots,n}\mu_i\alpha_i>0$, generate a demand which is an AGS-mapping. If $u_i'(x_i)\to+\infty$ for $x_i\to+0\forall i$, then $\mathscr{C}(p,\delta(p))$ is indecomposable.

Let us consider two further examples from [26], which illustrate the relationship between the concepts defined above.

<u>Example 1.1.</u> A single-valued discontinuous GS-mapping (in the sense of Definition 1.4) does not necessary satisfy Definition 1.1. Let the mapping $\mathscr{C}(p,1)$ be generated by the problem (1.6) with $u(x)=\sum_{i=1}^{3}x_i$ and $\delta(p)\equiv 1$. In order to obtain a suitable example, it suffices to consider the function $\mathcal{F}(p)\in\mathscr{C}(p,1)$ subject to two conditions: $\mathcal{F}(1,1,1)=(0,0,1)$ and $\mathcal{F}(2,1,1)=(0,1,0)$.

<u>Example 1.2.</u> The sum of two GS-mappings is not necessarily a GS-mapping. To show this, consider the sets of solutions of two extremal problems

$$x_1+2x_2+x_3\to\max,\quad p_1x_1+p_2x_2+p_3x_3\leqslant 16,\quad x_i\geqslant 0;$$
$$x_1+x_2+x_3\to\max,\quad p_1x_1+p_2x_2+2p_3x_3\leqslant 16,\quad x_i\geqslant 0$$

for p = (2, 2, 1) and p = (3, 2, 1).

Note that the second problem generates a GS-mapping which is not an AGS-mapping; another example can be derived from Lemma 1 in [25].

Some classes of utility functions (smooth but not necessarily representable in the form (1.7)) for which $\mathscr{C}(p,1)$ is a GS-function are indicated in [33]. Examples of production systems whose supply function (with minus sign) satisfies Definition 1.1 are also given in [33].

1.3. The case of linear dependence of income on prices is of special importance in equilibrium theory. Thus, $\delta(p)=pw$, where $w\in R_+^n$ (pw is the value of the resources w held by the trader). Transforming the Slutsky equation, Fisher [48] obtained necessary and sufficient conditions (in terms of $\mathscr{C}(p,\beta)$) ensuring g.s. of $\mathscr{C}(p,pw)$ for all $w\in R_+^n$. These conditions constitute relationships between income elasticities of demand, Allen–Uzawa elasticities of substitution, and the quantities $p_i\mathscr{C}_i/\beta$. The proof of these conditions requires lengthy mathematics.

The relationship between gross substitutability of the mappings $\mathscr{C}(p,\beta)$ and $\mathscr{C}(p,pw)$ becomes more transparent if $\mathscr{C}(p,1)$ is normal. By positive homogeneity of $\mathscr{C}(p,\beta)$ in (p,β), normality of $\mathscr{C}(p,1)$ implies that $\mathscr{C}(p,\beta)$ is nondecreasing in the income variable for any fixed p.

For single-valued demand functions, the normality property was investigated originally by Slutsky [30], who established that a utility function of the form (1.7) with $u_i'>0$, $u_i''<0$ generates a strictly normal demand. A linear utility function clearly generates a normal (but not strictly normal) demand.

The following proposition was given in [48] for the single-valued case and subsequently generalized in [27].

Proposition 1.2. If $\mathscr{C}(p, pw)$ is a GS-mapping on $\operatorname{int} \mathbf{R}_+^n$ for any $w \geqslant 0$, then $\mathscr{C}(p, 1)$ is also a GS-mapping. Conversely, if $\mathscr{C}(p, \beta)$ is positive homogeneous of degree zero in (p, β) and normal, then gross substitutability of $\mathscr{C}(p, 1)$ implies gross substitutability of $\mathscr{C}(p, pw)$, and indecomposability of $\mathscr{C}(p, 1)$ implies indecomposability of $\mathscr{C}(p, pw)$ for any $w \geqslant 0, w \neq 0$.

1.4. The need to study GS-mappings also arises in intersectoral balance models, which have obvious important applications. These models stimulated extensive study of matrices with nonnegative off-diagonal elements, the so-called Metzler matrices. The theory of these matrices was presented, e.g., in [20].

If $\varphi(p)$ is a nondecreasing vector function and $\lambda \in \mathbf{R}_+^1$, then $\varphi(p) - \lambda p$ has the property of gross substitutability. Therefore the problem of finding the nonnegative eigenvectors of a monotone operator in $\mathbf{R}^n$ is closely related with finding the zeros of a GS-function (the vectors on which the function vanishes). These problems are typical of nonlinear intersectorial balance models [1, 3, 4, 6, 7].

1.5. Examples of Equilibrium Models. In what follows, alongside the properties of GS-mappings, we will consider in an abstract form two relatively simple equilibrium models which have been studied by many authors (the value of these models in the theory of a planned economy is discussed, e.g., in [8, 1]).

Suppose that each of the m agents (consumers) is characterized by the objective function $u_k : \mathbf{R}_+^n \to \mathbf{R}^1$ and the income function $\delta_k : R_+^n \to R_+^1$, $k \in M = \{1, \ldots, m\}$. Denote by $\mathscr{C}^k$ the mapping which associates with each pair $p, \delta_k(p)$ the set of solutions of the problem (1.6) for $u = u_k$, $\beta = \delta_k(p)$. Then the mapping $\mathscr{C}^k(p, \delta_k(p))$ characterizes the dependence of the k-th agent demand on the prices p. Suppose that the supply vector $s \in \mathbf{R}_+^n$ is fixed.

Definition 1.8. The collection of vectors $(q, f^k, k \in M)$ is called an equilibrium if $q \in R_+^n$, $f^k \in \mathscr{C}^k(q, \delta_k(q)) \forall k \in M$ and $\sum_{k \in M} f^k = s$. Here q is called the equilibrium price vector, and the tuple $(f^k, k \in M)$ is called an equilibrium allocation of resources.

If $\delta_k(p) = pw^k$, where $w^k \in \mathbf{R}_+^n$, and $s = \sum_{k=1}^m w^k$, we obtain a pure exchange model [20, 36]. Denote this model by $\mathfrak{M}_1(W)$, where $W = (w^k, k \in M)$ is the collection of initial stocks. If $\delta_k(p) \equiv \beta_k$, then the corresponding construct is called a fixed income model, denoted by $\mathfrak{M}_2(B, s)$, where $B = (\beta_k) \in \mathbf{R}_+^m$.

Consider the excess demand mapping

$$\mathscr{D}(p) = \sum_{k=1}^m \mathscr{C}^k(p, \delta_k(p)) - s. \tag{1.8}$$

Clearly the set of equilibrium prices coincides with the set

$$\mathscr{L}^0 = \{p \mid 0 \in \mathscr{D}(p)\}. \tag{1.9}$$

In what follows we invariably assume that the model satisfies the following two conditions.

U1. Each objective function defined on $\mathbf{R}_+^n$ is continuous, strictly quasiconcave,* and does not attain a maximum on $\mathbf{R}_+^n$.

U2. For any commodity i, there is a trader k whose objective function is strictly increasing in the i-th variable.

From U2 it follows that the mapping $\mathcal{D}(p)$ is not defined on the boundary of $\mathbf{R}_+^n$.

The structure of the set $\mathcal{D}^0$ for a GS-mapping $\mathcal{D}$ will be discussed in the following sections.

In more general equilibrium models [10, 20, 36], supply is also a function of the prices. Many of the results remain valid in this case too.

Note that if all the objective functions satisfy U1, then the excess demand in the model $\mathfrak{M}_1$ satisfies the identity

$$pd=0 \ \forall d\in\mathcal{D}(p),\ \forall p, \tag{1.10}$$

which is known as the Walras law. In model $\mathfrak{M}_2$ the Walras law does not apply, but we have the relationship

$$pd=\sum_{k=1}^{m}\beta_k-ps \ \forall d\in\mathcal{D}(p),\ \forall p. \tag{1.11}$$

Note that the model $\mathfrak{M}_2(B, s)$ can be regarded as a particular case of the model $\mathfrak{M}_1(W)$. Indeed, for $w^k=\frac{1}{\sum_r\beta_r}\beta_k s,\ W=(w^k, k\in M)$, the set of equilibria in $\mathfrak{M}_1(W)$ coincides with the set of equilibria in $\mathfrak{M}_2(B, s)$ if the prices in $\mathfrak{M}_1$ are normalized by $ps=\sum_k\beta_k$. The model $\mathfrak{M}_2$ is of independent interest, but it also plays an important auxiliary role in the study of the more complex model $\mathfrak{M}_1$.

1.6. Let us now introduce certain assumptions which will be needed in what follows in applications to various mappings.

A1. The mapping defined on $\operatorname{int}\mathbf{R}_+^n$ is convex-valued, closed, and maps any compact set from $\operatorname{int}\mathbf{R}_+^n$, into a nonempty bounded set in the space $\mathbf{R}^n$.

A2. The mapping is positive homogeneous of degree α.

A3. If q is a zero of the mapping, and d = (d_i) is included in the image set of p, d ≠ 0, the condition $p\geqslant q$ implies that $\min_i d_i<0$ and the inequality $p\leqslant q$ implies that $\max_i d_i>0$.

A4. The mapping satisfies the Walras identity

$$pd=0 \ \forall d\in\mathcal{D}(p),\ \forall p.$$

It is easily seen that A4 implies A3. Let U1 be true. Then the excess demand $\mathcal{D}$ in the model $\mathfrak{M}_1$ satisfies all the assumptions A1-A4 (where A2 holds only for $\alpha=0$). The excess demand in $\mathfrak{M}_2$ has the properties A1 and A3, whereas A2 and A4 break down.

1.7. In conclusion of this section, we give a theorem from [26] which establishes a relationship between GS- and AGS-mappings.

*That is, $u(tx+(1-t)y)\geqslant\min\{u(x),\ u(y)\}\ \forall t\in(0,\ 1)$, and strong inequality holds if u(x) ≠ u(y). If u is concave (see footnote on p.2016), then it is also quasiconcave.

THEOREM 1.2. An (indecomposable) GS-mapping $\mathscr{D}$ satisfying A1 is an (indecomposable) AGS-mapping if and only if for any p and arbitrary elements d, d' from the set $\mathscr{D}(p)$ we have the equality pd = pd'.

The justification of Theorem 1.2 in [26] is based on two lemmas, which will be useful in what follows.

LEMMA 1.1. Let V be a convex compact set in $\operatorname{int}\mathbf{R}_+^n$ and let the mapping $\mathscr{D}$ satisfy A1. Then there is a pair of vectors r, h such that $r\in V,\ h\in\mathscr{D}(r)$ and $rh\geqslant vh\ \forall v\in V$.

To prove the lemma, it suffices to apply Kakitani's fixed point theorem to the direct product $\mathscr{P}(d)\times\mathscr{D}(p)$, where $\mathscr{P}(d)=\{p\mid p\in V,\ pd=\max_{v\in V} vd\}$.

Let $p=(p_i),\ q=(q_i)$. The following notation will be used throughout the rest of the article:

$$\min\{p, q\}=(\bar{p}_i), \text{where } \bar{p}_i=\min\{p_i, q_i\}, \tag{1.12}$$

$$\begin{gathered}\max\{p, q\}=(\bar{\bar{p}}_i), \text{where } \bar{\bar{p}}_i=\max\{p_i, q_i\}.\\ J_1(p, q)=\{i\mid p_i<q_i\},\quad J_2(p, q)=\{i\mid p_i\geqslant q_i\}.\end{gathered} \tag{1.13}$$

If $a=(a_i)\in\mathbf{R}^n$ and $J\subset N=\{1,\ldots,n\}$, then $a[J]$ is an n-dimensional vector with the components

$$a_i[J]=\begin{cases}a_i, & i\in J,\\ 0, & i\notin J.\end{cases} \tag{1.14}$$

The following result provides an important tool for the investigation of the properties of GS-mappings and is often used in the subsequent sections.

LEMMA 1.2 (on Combination). Let the GS-mapping satisfy A1 and let $d\in\mathscr{D}(p),\ f\in\mathscr{D}(q),\ \bar{p}=\min\{p, q\},\ \bar{\bar{p}}=\max\{p, q\}$. Then there are vectors $\bar{a}\in\mathscr{D}(\bar{p}),\ \bar{\bar{a}}\in\mathscr{D}(\bar{\bar{p}})$ such that

$$\bar{a}\leqslant d[J_1]+f[J_2],\quad \bar{\bar{a}}\geqslant d[J_2]+f[J_1],$$

where $J_1=J_1(p, q),\ J_2=J_2(p, q)$.

A proof based on Lemma 1.1 will be found in [26]. It is close to the proof of Theorem 2 in [28].

2. EQUILIBRIUM PRICE SETS

2.1. Consider the mapping $\mathscr{D}: P\to 2^{\mathbf{R}^n},\ P\subset\mathbf{R}_+^n$ and let

$$\mathscr{D}^0=\{p\mid 0\in\mathscr{D}(p)\}. \tag{2.1}$$

The vectors from $\mathscr{D}^0$ will be called the zeros of the mapping $\mathscr{D}$ or, if $\mathscr{D}$ is interpreted as excess demand, the equilibrium price vectors. In this section we give conditions for $\mathscr{D}^0$ to be nonempty and convex or to consist of a single element. We also consider the related question of the existence and the properties of the inverse mapping

$$\mathscr{D}^{-1}(y)=\{p\mid y\in\mathscr{D}(p)\},\quad y\in Y\subset\mathbf{R}^n. \tag{2.2}$$

For a linear GS-function $\mathscr{D}$ with $P=R_+^n,\ Y=R_-^n$, the necessary and sufficient conditions for the existence of $\mathscr{D}^{-1}$ are known (see, e.g., [20, p. 95]); these functions are specified by Metzler matrices [10, p. 255].

Now let $\hat{P}=\{p\mid\hat{p}\leqslant p\leqslant\hat{q}\},\ \mathscr{D}(p)=\varphi(p)-p$, where $\varphi:\mathbf{R}^n\to\mathbf{R}^n$, φ is nondecreasing. Assume that $\mathscr{D}(\hat{p})\geqslant 0,\ \mathscr{D}(\hat{q})\leqslant 0$. Then $\hat{P}\cap\mathscr{D}^0\neq\varnothing$ by the Birkhoff–Tarsky theorem (see, e.g., [21, p. 53] and alternative versions in the appendix to [1]). The proposition remains valid for an arbitrary continuous GS-function [79, 92]. For $\hat{p}\in \operatorname{int} \mathbf{R}_+^n$ it can be generalized to the multi-valued case using Lemmas 1.1 and 1.2. Let

$$\mathcal{D}^+=\{p\,|\,\mathcal{D}(p)\cap \mathbf{R}_+^n\neq\varnothing\},\quad \mathcal{D}^-=\{p\,|\,\mathcal{D}(p)\cap \mathbf{R}_-^n\neq\varnothing\}. \tag{2.3}$$

THEOREM 2.1. Let the GS-mapping $\mathcal{D}$ satisfy A1 and $\mathcal{D}^+\cap\mathcal{D}^-=\mathcal{D}^0$. If $\hat{p}\in\mathcal{D}^+$, $\hat{q}\in\mathcal{D}^-$, $\hat{p}\leqslant\hat{q}$, then $\mathcal{D}^0\neq\varnothing$ and there is a vector $r\in\mathcal{D}^0$, $\hat{p}\leqslant r\leqslant\hat{q}$.

Note that for AGS-mappings defined on $P=\operatorname{int}\mathbf{R}_+^n$, we always have $\mathcal{D}^+\cap\mathcal{D}^-=\mathcal{D}^0$. A close result is contained in [27].

2.2 Now suppose that $\mathcal{D}$ satisfies the Walras identity A4. If $P\subset\operatorname{int}\mathbf{R}_+^n$, then clearly $\mathcal{D}^+=\mathcal{D}^-=\mathcal{D}^0$ and the proposition of Theorem 2.1 is trivial. Note that on the entire $\mathbf{R}_+^n$ there is no continuous nonzero GS-function satisfying A4 (see, e.g., [20, p. 309]). A number of existence theorems of equilibrium prices for a continuous positive homogeneous GS-function satisfying A4 and defined on the cone $P\subset\mathbf{R}_+^n$ (not necessarily convex) which does not contain the origin and is closed in the relative topology of $\mathbf{R}_+^n$ are given in [20, Sec. 18.3] and in [59]. Certain conditions are assumed on the boundary of P.

Let $P=\operatorname{int}\mathbf{R}_+^n$ and let $\mathcal{D}$ satisfy A1, A4. Then the existence of an equilibrium vector can be proved without gross substitutability by using the following natural boundary condition (it was actually used in [20]): there are constants $\gamma>0$ and $\varepsilon>0$ such that, if $\|p\|\geqslant\gamma$, $T=\{i\,|\,p_i\leqslant\varepsilon\}\neq\varnothing$, then any vector $d\in\mathcal{D}(p)$ has at least one positive component with an index from T. This proposition is easily derived from Lemma 1.1. It is applicable to the model $\mathfrak{M}_1(W)$ if U1, U2 hold and for any i, j = 1, ..., n there is a trader with the i-th commodity whose objective function is strictly increasing in the j-th variable.

Necessary and sufficient conditions for the existence of equilibrium are known only for linear exchange models [50]. So far it is not clear how to extend these conditions to the nonlinear case, even with gross substitutability.

2.3. Let us now consider the uniqueness of the zero and the existence of the inverse mapping. First we focus on cases which are independent of A2 and A4.

Assuming a differentiable GS-function $\mathcal{D}$ defined on a domain of the form $P=\{p\,|\,\hat{p}<q<\hat{q}\}$, Gale and Nikaido [51; 20, p. 365] derived necessary and sufficient conditions ensuring the existence of a nondecreasing inverse function: at any point from P, the Jacobian matrix $\mathcal{D}'$ of the function $\mathcal{D}$ should satisfy the inequalities

$$\max_i a_i b_i>0\quad \forall a=(a_i)\in\mathbf{R}^n \text{ and } b=(b_i)=\mathcal{D}'a. \tag{2.4}$$

Conditions guaranteeing existence and univalence of the inverse of a GS-function, without assuming smoothness, were given in [78-80, 84, 89]. The most general (and very similar) results were independently derived by Sandberg [79] and Yun [89]. In what follows we will prove a proposition close to Theorem 3 from [79] which is also applicable to the multi-valued case.

Let us introduce another assumption, weaker than A4.

A5. If $d\in\mathcal{D}(p), d'\in\mathcal{D}(p)$ for some p and $d\leqslant d'$, then d = d'.

For AGS-mappings (Definition 1.5), A.5 is satisfied automatically for positive p.

Take a fixed set $Y\subset R^n$.

THEOREM 2.2. Suppose that the GS-mapping $\mathscr{D}$ satisfies A1, A5, and the following condition holds:

(*) for any $r \in \operatorname{int} R_+^n$, $y \in Y$ there are p, q, d, f such that $p \leqslant r \leqslant q$, $d \in \mathscr{D}(p)$, $f \in \mathscr{D}(q)$, $d \geqslant y \geqslant f$.

Then a nonincreasing inverse mapping $\mathscr{D}^{-1}$ is defined on Y. It is single-valued if the following condition holds:

(**) if $d \in \mathscr{D}(p)$, $f \in \mathscr{D}(q)$, $q \geqslant p$, $f \geqslant d$, then q = p.

If (**) does not hold for some p, q, d, f, and $d \in Y$ or $f \in Y$, then $\mathscr{D}^{-1}$ is not single-valued.

The proof of Theorem 2.2 is based on the following lemma from [26].

LEMMA 2.1. If the GS-mapping $\mathscr{D}$ satisfies A1, then the set $\mathscr{D}^+$ (see (2.3)) is closed under the operation max, and $\mathscr{D}^-$ is closed under the operation min. If A3 holds, then $\mathscr{D}^0$ is closed under max and min.

Lemma 2.1 readily follows from combination Lemma 1.2.

Proof of Theorem 2.2. By Theorem 2.1 the mapping $\mathscr{D}(p) - y$ has a zero for any $y \in Y$; therefore $\mathscr{D}^{-1}$ is defined. Let $a, b \in Y$, $a \leqslant b$, $r \in \mathscr{D}^{-1}(a)$, $v \in \mathscr{D}^{-1}(b)$. Denote

$$\bar{p} = \min\{r, v\}, \quad \bar{\bar{p}} = \max\{r, v\}, \quad \mathscr{F}(p) = \mathscr{D}(p) - b. \tag{2.5}$$

Applying Lemma 2.1 to the mapping $\mathscr{F}(p)$, we find $\bar{g} \in \mathscr{F}(\bar{p})$, $\bar{g} \leqslant 0$. By assumption, there are vectors q, f such that $f \in \mathscr{D}(q)$, $q \leqslant \bar{p}$, $f \geqslant b$. By Theorem 2.1, the set $\{p \mid q \leqslant p \leqslant \bar{p}\}$ includes the vector v' such that $0 \in \mathscr{F}(v')$; therefore $v' \leqslant r$, $v' \in \mathscr{D}^{-1}(b)$. We similarly find $r' \in \mathscr{D}^{-1}(a)$, $r' \geqslant v$. We have thus proved that $\mathscr{D}^{-1}$ is nondecreasing.

Now suppose that $r \in \mathscr{D}^{-1}(b)$, $v \in \mathscr{D}^{-1}(b)$, $b \in Y$, $r \neq v$. Consider $\bar{p}$, $\bar{\bar{p}}$ and $\mathscr{F}(p)$ as defined in (2.5). Applying Lemma 2.1 to $\mathscr{F}(p)$, we obtain $\bar{a} \in \mathscr{D}(\bar{p})$, $\bar{\bar{a}} \in \mathscr{D}(\bar{\bar{p}})$, $\bar{a} \leqslant b \leqslant \bar{\bar{a}}$. But $\bar{p} \leqslant \bar{\bar{p}}$, therefore condition (**) does not hold. Conversely, suppose that (**) does not hold for some r, v, a, b, so that $a \in \mathscr{D}(r)$, $b \in \mathscr{D}(v)$, $v \geqslant r$, $v \neq r$, $b \geqslant a$, and let, say, $b \in Y$. By assumption, there are q, f such that $f \in \mathscr{D}(q)$, $q \leqslant r$, $f \geqslant b$. Therefore the function $\mathscr{D}(p) - b$ has a zero on the set $\{p \mid q \leqslant p \leqslant r\}$ other than v, i.e., the mapping $\mathscr{D}^{-1}$ is not single-valued. Q.E.D.

As an application of Theorem 2.2, consider the model $\mathfrak{M}_2(B, s)$ (Subsec. 1.5) for $s > 0$, assuming that U1 and U2 hold (Subsec. 1.5). Let $Y = \{y \mid y > -s\}$. In this case, Theorem 2.2 is not particularly useful to prove the existence of equilibrium, since (*) is not obvious. However, the existence of equilibrium for any $s > 0$ follows from the corresponding theorem (see Subsec. 2.2) for $\mathfrak{M}_1(W)$ for $w^k = (\sum_r \beta_r)^{-1} \beta_k s$, and therefore condition (*) holds. Conditions A5 and (**) follows from budget equalities (see (1.11)). Therefore, the equilibrium prices are unique and do not increase with the increase in s (more general results will be derived in Subsec. 4.3).

2.4. Let the GS-function $\mathscr{D}$ be defined on the entire R_+^n and be continuous. A number of economic models [3, 6] and also a number of multivariable control problems [15, 16, 9] are reducible to problems of maximizing a certain nondecreasing function on $\mathscr{D}^+$ or minimizing such a function on $\mathscr{D}^-$. The solution of these problems is independent of the particular criterion, since the set $\mathscr{D}^-$, if nonempty, includes a minimum point (i.e., a vector p* such that $p^* \leqslant p \, \forall p \in \mathscr{D}^-$), and the set $\mathscr{D}^+$, if nonempty and bounded, includes a maximum point. This fact easily follows from the closure of $\mathscr{D}^-$ under the operation min and the closure of $\mathscr{D}^+$ under max (see Lemma 2.1, which, strictly speaking, applies to a mapping defined on $\text{int}\mathbf{R}_+^n$). As noted in [3], the existence of a minimum point in $\mathscr{D}^-$ was first noted by É. B. Ershov.

The minimum point of the set $\mathscr{D}^-$ satisfies the conditions $p_i\mathscr{D}_i(p)=0,\ i=1,\ldots,n$. The maximum point of $\mathscr{D}^+$ always reduces $\mathscr{D}$ to zero. Therefore Theorem 2.2 can be applied as a criterion to identify the maximum point. On the other hand, the following proposition, actually established in [3] (see also [9]), suggests yet another criterion for the uniqueness of the zero.

THEOREM 2.3. Let $\mathscr{D}$ be a continuous GS-function on $\mathbf{R}_+^n$ with concave components, and $\mathscr{D}(0)>0$. If $\mathscr{D}^-$ is nonempty, then the minimum point of the set $\mathscr{D}^-$ is the unique zero of the function $\mathscr{D}$ and it coincides with the maximum point of the set $\mathscr{D}^+$.

Proof. Let p^* be a minimum point. Gross substitutability and the condition $\mathscr{D}(0)>0$ imply that $p^*>0,\ \mathscr{D}(p^*)=0$. If $\mathscr{D}$ has another zero, then p^* is not a maximum point in $\mathscr{D}^+$. Then there is $p\in\mathscr{D}^+,\ p\geqslant p^*,\ p\neq p^*$. Take $\varepsilon>0$ so that $q=p^*-\varepsilon(p-p^*)\geqslant 0$. From the concavity of $\mathscr{D}$ is follows that $\mathscr{D}(q)\leqslant 0$. But $q\leqslant p^*,\ q\neq p^*$, which contradicts the minimality of p^*.

2.5. If the GS-function is positive homogeneous of zero degree and satisfies the Walras identity A4, then the set of its zeros is convex. This result, due to McKenzie [64], is also given in [38] and in [10, 20]. In [26] it is extended to the multivalued case and an arbitrary degree of homogeneity. The proof of [26] is actually preserved when A4 is replaced with A3.

THEOREM 2.4. If the GS-mapping $\mathscr{D}$ satisfies the assumptions A1-A3, then $\mathscr{D}^0$ is convex.

The proof follows immediately from Lemma 2.1 and the following geometric fact: a cone† in $\mathbf{R}_+^n$ which is closed under max and min is convex [26].

Note that a positive homogeneous indecomposable GS-mapping has at most one zero (up to a scalar factor). This follows directly from the definitions.

2.6. In conclusion of this section, let us prove an inequality which is very important in stability analysis of price adjustment processes (see Subsec. 6.4). For the single-valued case, it was proved in [35, 38]. It also implies convexity of $\mathscr{D}^0$ [38], although, unlike Theorem 2.4, this assumes the Walras identity.

THEOREM 2.5. Let the GS-mapping $\mathscr{D}$ satisfy A1, A2, A4. If $q\in\mathscr{D}^0$ and $d\in\mathscr{D}(p)$, then $qd\geqslant 0$, and if $p\notin\mathscr{D}^0$, then $qd>0$.

Proof. Consider the partition $\Omega(p,q)$ of the coordinate set $N=\{1,\ldots,n\}$ into classes, each with constant ratio p_i/q_i:

$$\Omega(p,\ q)=\{N_1,\ldots,\ N_t,\ldots,\ N_l\},\quad N_t=\{i\,|\,p_i/q_i=\gamma_t\},\quad \gamma_t>\gamma_{t+1}.$$

Let $p^k=(p_i^k)=\min\{p,\ \gamma_k q\},\ H^k=\{i\,|\,p_i<\gamma_k q_i\}=\bigcup_{t>k}N_t$. Since $0\in\mathscr{D}(\gamma_k q)$, then by combination Lemma 1.2, there are vectors g^k such that

$$g^k\in\mathscr{D}(p),\quad g^k\leqslant d\,[H^k]. \tag{2.6}$$

†The set K is a cone if $K=\lambda K\ \forall\lambda>0$. The zero vector (the origin) may belong to K, but not necessarily so.

Since $p^k[H^k]=p[H^k]$, the Walras law and (2.6) imply that

$$0=p^k g^k \leqslant p^k d[H^k]=pd[\bigcup_{t>k} N_t]. \tag{2.7}$$

If $\beta_t=pd[N_t]$, then (2.7) leads to the inequalities

$$\sum_{t>k}\beta_k \geqslant 0, \quad k=1,\ldots, l-1. \tag{2.8}$$

Therefore

$$0=\gamma_1^{-1}\sum_{t=1}^{l}\beta_t \leqslant \gamma_1^{-1}\beta_1+\gamma_2^{-1}\sum_{t=2}^{l}\beta_t \leqslant \ldots \leqslant \sum_{t=1}^{l}\gamma_t^{-1}\beta_t=qd, \tag{2.9}$$

i.e., the first proposition of the theorem is true.

Now let qd = 0. In this case, (2.9) implies that (2.8), (2.7) hold as equalities. Therefore, by (2.6),

$$\mathscr{D}(p^k)\ni g^k, \quad g^k=d[\bigcup_{t>k} N_t], \quad k=1,\ldots, l-1. \tag{2.10}$$

Since $p^l=\gamma_l q$, then $p^l \in \mathscr{D}^0$. We will prove that if $p^{k+1}\in\mathscr{D}^0$, then also $p^k\in\mathscr{D}^0$, $1\leqslant k<l$. Clearly, $p^k=\max\{p^{k+1}, p^k\}$, $\tilde{H}^k=\{i \mid p_i^{k+1}<p_i^k\}=\bigcup_{t\leqslant k} N_t$. From (2.10) we have that $g^k[\tilde{H}^k]=0$. Since $0\in\mathscr{D}(p^{k+1})$, Lemma 1.2 implies that there is a vector $f\in\mathscr{D}(p^k)$, $f\geqslant 0$. By the Walras law, f = 0, i.e., $p^k\in\mathscr{D}^0$.

By induction $p^1\in\mathscr{D}^0$. But since $p^1=p$, this completes the proof. Q.E.D.

3. EQUILIBRIUM PRICES AND EQUILIBRIUM ALLOCATIONS

3.1. Let

$$\mathscr{D}(p)=\sum_{k\in M}\mathscr{D}^k(p), \quad M=\{1, 2,\ldots, m\}. \tag{3.1}$$

In the model $\mathfrak{M}_1(W)$ (see subsec. 1.5) and in a number of other models, the mapping $\mathscr{D}^k$ in interpreted as the excess demand of the k-th agent.

Let

$$\mathbf{E}=\left\{(p,\ d^k,\ k\in M) \mid d^k\in\mathscr{D}^k(p),\ \ \sum_{k\in M} d^k=0\right\},$$

$$\mathbf{D}=\left\{(d^k,\ k\in M) \mid \exists p: d^k\in\mathscr{D}^k(p),\ \ \sum_{k\in M} d^k=0\right\}.$$

E is the "equilibrium set," $\mathscr{D}^0$ (see (2.1)) is interpreted as the set of equilibrium price vectors, and **D** is defined as the set of equilibrium allocations of the resources.

Let $\mathbf{E}\neq\varnothing$. The following proposition was actually proved in [26].

<u>THEOREM 3.1.</u> Let each GS-mapping $\mathscr{D}^k$ satisfy A1 and A2 for $\alpha=0$, and let A3 hold for their sum $\mathscr{D}$. Then

$$\mathbf{E}=\mathscr{D}^0\times\mathbf{D}, \tag{3.2}$$

and the sets $\mathscr{D}^0$ and **D** are convex.

If $\mathscr{D}^k$ are single-valued, then by (3.2) **D** includes at most one element.

A4 is assumed in [26], but the proof remains valid under the weaker assumption A3. As a result, Theorem 3.1 implies, say, convexity of p-optimal allocations, which is proved in [25, Theorem 2] by a different method.

Theorem 3.1 is particularly useful for pure exchange models (see Corollary 3.1).

3.2. Consider the uniqueness of the equilibrium in the models $\mathfrak{M}_1$ (Subsec. 1.5). If the objective functions in $\mathfrak{M}_1$ are linear, then, as shown by Gale [50], all the equilibria are "equally profitable" for each trader. In order to generalize this result, we need the following definition [50, 26].

Definition 3.1. Two equilibria (see Definition 1.8) $(p, d^k, k\in M)$ and $(q, f^k, k\in M)$ are equivalent if $u_k(d^k)=u_k(f^k)\ \forall k\in M$. If all the equilibria are equivalent, we say that we have a preference-unique equilibrium.

The existence of nonequivalent equilibria involves substantial difficulties in the application of equilibrium models as a tool of economic analysis. It is therefore important to establish the equivalence conditions. Equality (3.2) is clearly one of such conditions. Therefore, Theorem 3.1 leads to the following generalization of the proposition from [50].

COROLLARY 3.1 [26]. Let each objective function in the model $\mathfrak{M}_1(W)$ satisfy U1 and let U2 hold (Subsec. 1.5). If the individual excess

$$\mathscr{D}^k(p)=\mathscr{C}^k(p,\ p\bar{w}^k)-w^k \tag{3.3}$$

is a GS-mapping for any k, then the equilibrium is preference-unique.

It is shown in [26] that, given properties U1 and U2, the equality (3.2) is necessary for the equivalence of equilibria in $\mathfrak{M}_1$. This proposition, not using gross substitutability, is also valid for an essentially wider class of models (which includes, in particular, $\mathfrak{M}_2(B, s)$).

Let us introduce some assumptions relating to the individual demand $\mathscr{C}^k(p, 1)$ for unit income.

S1. $\mathscr{C}^k(p, 1)$ satisfies A1, $\mathscr{C}^k(p, 1)\subset R_+^n\ \forall p$.

S2. $\mathscr{C}^k(p, \beta)=\mathscr{C}^k(p/\beta, 1)\ \forall\beta>0;\ \mathscr{C}^k(p, 0)=\{0\}$.

S3. $pc^k=1\ \forall p$ and $\forall c^k\in\mathscr{C}^k(p, 1)$.

S4. $\mathscr{C}^k(p, 1)$ is a GS-mapping.

S5. $\mathscr{C}^k(p, 1)$ is normal (see Definition 1.7).

Note that S2, S4, and S5 imply the GS-property for (3.3) (Proposition 1.2).

Properties S1-S3 follow from assumption U1 (Subsec. 1.5) for the individual utility functions. It can be shown that the conditions of Theorem 1.1 are sufficient for S1-S5.

We recall that the equilibrium price vector in the model $\mathfrak{M}_1(W)$, $W=(w^k, k\in M)$, is the zero of the sum of mappings (3.3), and in the model $\mathfrak{M}_2(B, s)$, where $B=(\beta_k, k\in M)$, it is the zero of the mapping $\sum_{k=1}^{m}\mathscr{C}^k(p, \beta_k)-s$. Only positive prices are assumed.

The following results are only based on the above-listed properties of the individual demand functions; it is not required that these functions are generated by a maximization problem.

Sufficient conditions for uniqueness of equilibrium prices were derived in [45] for a linear exchange model.* A more general result is given in [31]. In order to present it, we need the following definition.

*In [45] these conditions are erroneously declared to be also necessary. A counterexample is given in [31, p. 13].

Definition 3.2 [31]. We say that the model $\mathfrak{M}_1(W)$ is decomposable into l submodels $\mathfrak{M}_1(W_t)$ if 1) $W_t=(w^k, k\in M_t)$, where the sets M_t, $t=1,\ldots, l$ constitute a partition of the trader set M; 2) $w_i^k=0$ for $k\in M_t$, $i\notin N_t$, where N_t, $t=1,\ldots, l$ constitute a partition of the set of resources $N^+=\left\{i \,\middle|\, \sum_{k\in M} w_i^k>0\right\}$; 3) the collection $(c^k, k\in M)$ of the demand vectors c^k of the traders k is an equilibrium allocation in the model $\mathfrak{M}_1(W)$ if and only if the collections $(c^k, k\in M_t)$ are equilibrium allocations in the models $\mathfrak{M}_1(W_t)$ for all t = 1, 2, ..., l.

We assume that the model always has the trivial decomposition for l = 1. If there are no other decompositions, the model is called indecomposable.

In a decomposable model, the traders from different groups M_t clearly possess different kinds of resources and, in any equilibrium, exchange is only conducted among traders of the same group. Therefore $c_i^k = 0$ for $k\in M_t$, $i\notin N_t$ for any equilibrium allocation $(c^k, k\in M)$ of the model $\mathfrak{M}_1(W)$.

If $\mathbf{P}_1(W)$, $\mathbf{P}_1(W_t)$ are equilibrium price sets in $\mathfrak{M}_1(W)$ and $\mathfrak{M}_1(W_t)$, respectively, then by Definition 3.2

$$\mathbf{P}_1(W)=\bigcap_{t=1}^{l}\mathbf{P}_1(W_t). \tag{3.4}$$

In the following theorems we use the notation from (1.14).

THEOREM 3.2 [31]. Let conditions S1-S5 be true, $\mathbf{P}_1(W)\neq\varnothing$. Then there are a decomposition of the model $\mathfrak{M}_1(W)$ into l submodels and vectors $a^t=(a_i^t)\in R_+^n$, $t=1,\ldots, l$, such that $a_i^t>0$ for $i\in N_t$ and

$$\mathbf{P}_1(W)=\bigcap_{t=1}^{l}\{p\,|\,p\geqslant\lambda a^t,\ p\,[N_t]=\lambda a^t\,[N_t],\ \lambda\in\operatorname{int}\mathbf{R}_+^1\}. \tag{3.5}$$

The proof essentially uses the theorem of the structure of the equilibrium set in the model $\mathfrak{M}_2$ with fixed income (see Subsec. 3.3, Theorem 3.5) and the construction described in Sec. 5 (see the properties of the function $\mathscr{H}$ in Subsec. 5.3).

Suppose that the partition N_t of the resource set corresponds to the decomposition of the model $\mathfrak{M}_1(W)$ introduced in Theorem 3.2. Then, as claimed by the theorem, the equilibrium price proportions on each set N_t are uniquely determined. The next theorem follows directly from Theorem 3.2.

THEOREM 3.3 [31]. Let conditions S1-S5 be true and let $\mathbf{P}_1(W)\neq\varnothing$. If the model $\mathfrak{M}_1(W)$ is indecomposable, then the equilibrium prices are unique (up to a scalar factor) on the set of nonzero resources of the model, i.e., for any p, $q\in\mathbf{P}_1(W)$ there is a number $\lambda>0$ such that

$$p\,[N^+]=\lambda q\,[N^+],\quad N^+=\left\{i\,\middle|\,\sum_{k\in M} w_i^k>0\right\}.$$

In a decomposable model, the matrix with the columns w^k is reduced to block-diagonal form. Therefore, either of the following conditions is sufficient for uniqueness of the equilibrium prices: 1) $\exists k\in M$, $w^k>0$; 2) $\forall i, j$, $i\neq j$, $\exists k\in M$, $w_i^k w_j^k>0$. Each of the following conditions ensures uniqueness of the equilibrium prices on N^+: 3) $\exists i$, $w_i^k>0$ $\forall k\in M$; 4) $\forall k\in M$, $r\in M$, $k\neq r$, $\exists i$, $w_i^k w_i^r>0$.

In cases then conditions 1)-4) do not hold, the indecomposability of the model can be checked using analogous conditions in terms of equilibrium allocations. Note that unique equilibrium prices are also possible in decomposable models.

A corresponding example from [31] includes three commodities and three traders with the objective functions $u_1(x)=u_2(x)=x_1+x_2+x_3,\ u_3(x)=x_1+x_2+2\sqrt{x_3}$ and initial stocks $w^1=(1, 0, 0)$, $w^2=(0, 1, 0), w^3=(0, 0, a)$, where $a>0$.

It is easily seen that for $a\leqslant 1$ the price vectors from the set $\{p\mid p_3\geqslant p_1=p_2\geqslant p_3\sqrt{a}\}$ are equilibrium prices. Since traders 1 and 2 may exchange their commodities, the only possible decomposition is that with $M_1=\{1, 2\}, M_2=\{3\}$. For $a<1$ the model is decomposable, since the prices are not unique. It is also decomposable for $a=1$. Using (3.4), we can easily check that there are no other equilibrium prices. Thus, for $a=1$, the equilibrium price proportions are uniquely determined.

For $a>1$, the model is indecomposable, which is easily seen by constructing an equilibrium allocation* with the vector $c^3>0$.

3.3. Now let us consider a model with fixed incomes $\mathfrak{M}_2(B, s)$. As we have noted above, given the conditions U1 and U2 (Subsec. 1.5), the existence of equilibrium in $\mathfrak{M}_2(B, s)$ for $B>0, s>0$ follows from well-known theorems (see, e.g., [20]). If the incomes of some traders or the stocks of some commodities may take zero values, the following conditional existence theorem is useful.

Let

$$Z(B)=\{k\mid \beta_k=0\},$$

where $B=(\beta_k)\in R_+^m$. The set of equilibrium prices in the model $\mathfrak{M}_2(B, s)$ will be denoted by $P_2(B, s)$.

THEOREM 3.4 [27]. Let assumptions S1-S5 be true. If

$$Z(B^*)\supset Z(B)\supset Z(\tilde{B}),\quad s^*\geqslant s\geqslant \tilde{s},$$

and $P_2(B^*, s^*)\neq\varnothing,\ P_2(\tilde{B}, \tilde{s})\neq\varnothing$, then also $P_2(B, s)\neq\varnothing$.

The proof is based on Theorem 2.1.

In conclusion of this section, we present a theorem on the structure of the equilibrium set in the model $\mathfrak{M}_2(B, s)$.

THEOREM 3.5. Let conditions S1-S4 hold in the model $\mathfrak{M}_2(B, s)$. Then the equilibrium set $\mathbf{E}$ (if it is nonempty) is the direct product of the convex set of equilibrium price vectors $\mathbf{P}$ and the convex set of equilibrium allocations $\mathbf{C}$,

$$\mathbf{E}=\mathbf{P}\times\mathbf{C}.$$

There is also a vector $a=(a_i)\in R_+^n$ such that $a_i>0$ for $i\in N^+=\{i\in N\mid s_i>0\}$ and

$$\mathbf{P}=\{p\mid p\geqslant a,\ p_i=a_i \text{ for } i\in N^+\}.$$

Under the assumptions U1 and U2, this theorem was proved in [26], where a somewhat more general model (including production) was considered. In this case, S4 can be replaced with

*In [31, p. 19] it is erroneously claimed that there are no equilibria for $1<a<3$. In fact we have the following equilibrium in this case: $p_1=p_2=p_3,\ c^1=(1-\alpha, 0, \alpha),\ c^2=(0, 1-\beta, \beta),\ c^3=(\alpha, \beta, 1)$, where $\alpha>0, \beta>0, \alpha+\beta=a-1$.

gross substitutability of the joint demand. A proof of Theorem 3.5 not relying on the existence of objective functions generating the individual demand was derived in [31].

4. COMPARISON OF EQUILIBRIA

4.1. Comparison of equilibria, or comparative statics, is a subdivision of economic equilibrium theory focusing on the response of equilibrium economic systems to external forces. A typical problem can be stated in the following terms. Consider a system in equilibrium, with excess demand which is positive homogeneous of degree zero and satisfies the Walras identity (Subsec. 1.6). One of the goods, say the n-th, is selected as the numeraire and its price is fixed. Suppose that as a result of changes in the objective functions, the consumer incomes, or the technology, the excess demand for the i-th good increased while the excess demand for the numeraire decreased (this is a so-called binary change, the simplest type of disturbance consistent with the Walras identity). How do the equilibrium prices change?

This topic was posed by Hicks [56], who assumed that, if all the goods are substitutes, then as a result of such a disturbance 1) the price of the i-th good increases; 2) all the other prices do not decrease; 3) the proportionate increase in any of the prices does not exceed that in the price of the i-th good. These three assertions are often referred to as Hicksian laws. They may appear to be "economically obvious," yet their proof requires very strong assumptions. Local results in the smooth case were obtained by Mosak [68] (see also [46]). The general case of binary changes for single-valued excess return satisfying g.s. (before and after the disturbance) and some additional assumptions was studied by Morishima [66, 19]. He proves the second and the third Hicksian laws with the aid of the indecomposability condition. These and some additional results in comparative statics are also given in [36, 21].

In what follows we consider the properties of GS-mappings which in certain cases allow predicting the signs of the changes in the equilibrium prices for various disturbances. We will also prove a number of additional propositions for the multivalued case, in particular the co-called LeChatelier–Samuelson principle [76, 77, 19].

4.2. The following theorem (a generalization of Theorem 1 [27]) implies the first Hicksian law, and in the case of unique equilibrium prices, also the second and the third laws.

Recall that $\mathcal{D}^0$ denotes the set of zeros of the mapping $\mathcal{D}$ (see (2.1)).

THEOREM 4.1. Let the GS-mapping $\mathcal{D}$ satisfy A1-A3, $d=(d_i)\in\mathcal{D}(p)$, $d\neq 0$ and $q=(q_i)\in\mathcal{D}^0$. Then there are indexes j, s and a vector $q^*=(q_i^*)\in\mathcal{D}^0$ such that $d_j<0$, $d_s>0$ and we have

$$q_j^*/p_j\leqslant q_i^*/p_i\leqslant q_s^*/p_s \text{ for all i} \tag{4.1}$$

$$q_i^*=q_i \text{ for all i such that } d_i\neq 0 \tag{4.2}$$

Proof. Since $\mathcal{D}$ is homogeneous, A3 implies that $N^-=\{i\,|\,d_i<0\}\neq\varnothing$, $N^+=\{i\,|\,d_i>0\}\neq\varnothing$. Let

$$\lambda=\max_{i\in N^-} p_i/q_i,\ q'=\max\{\lambda q,\ p\},\ \mu=\min_{i\in N^+} p_i/q_i,\ q''=\min\{p,\ \mu q\}.$$

Clearly, $q_j'=\lambda q_j=p_j$, $q_s''=p_s=\mu q_s$ for some $j\in N^-$, $s\in N^+$. Since $0\in\mathscr{D}(\lambda q)$, then by Lemma 1.2, using the definition of λ, there is $f\in\mathscr{D}(q')$ such that $f\geqslant d[I]\geqslant 0$, $I=\{i\,|\,\lambda q_i<p_i\}$. By A3, f = 0 and therefore $d[I]=0$. Thus,

$$q'\in\mathscr{D}^0,\ p_i\leqslant\lambda q_i=q_i' \text{ for } d_i\neq 0.$$

We similarly find that

$$q''\in\mathscr{D}^0,\ p_i\geqslant\mu q_i=q_i'' \text{ for } d_i\neq 0.$$

By Lemma 2.1, $\hat{q}=\max\{q,\lambda^{-1}q'\}\in\mathscr{D}^0$, $q^*=\min\{\hat{q},\mu^{-1}q''\}\in\mathscr{D}^0$. Clearly, (4.2) holds. It is easily checked that $\mu^{-1}p\geqslant q^*\geqslant\lambda^{-1}p$. Q.E.D.

COROLLARY 4.1. If under the conditions of Theorem 4.1 $p\notin\mathscr{D}^0$, then $q_j/p_j<q_s/p_s$ for some j, s such that $d_j<0$, $d_s>0$.

The Hicksian laws follow from Theorem 4.1 if we assume that $\mathscr{D}$ is the perturbed excess demand, and p are the old equilibrium prices; the properties of the original excess demand are not used directly. Actually, however, they determine the class of disturbances under which $\mathscr{D}$ satisfies the conditions of Theorem 4.1.

Theorem 4.1 relies on condition A3, which is weaker than the Walras identity and is applicable to arbitrary demand disturbances (and not only to binary disturbances). This is essential in the analysis of price adjustment (see Sec. 6) and in the proof of coalition stability of equilibria (Sec. 5).

Morishima [66, 19] proved the second and third Hicksian laws in strong form; i.e., as a result of an increase in the excess demand for the n-th good (the numeraire), all the equilibrium prices (except the n-th) strictly increase and the proportionate increase in the price of the i-th good is higher than that in any other price. Morishima assumes that the excess demand is a continuous GS-function satisfying A2, A4 and additionally the strong indecomposability condition:

If $p\leqslant p'$, $p\neq p'$ and the set $I=\{i\,|\,p_i=p_i'\}$ includes at least two elements, then there are j, $s\in I$ such that

$$\mathscr{D}_j(p)\neq\mathscr{D}_j(p'),\ \mathscr{D}_s(p)\neq\mathscr{D}_s(p').$$

Opoitsev [21, p. 72] proves the third Hicksian law in strong form using a somewhat different condition on the function $\mathscr{D}(p)=(\mathscr{D}_i(p))$, which also guarantees uniqueness of the equilibrium prices:

$$\mathscr{D}_i(p) \text{ decreases in } p_i\,\forall i \text{ and increases in } p_n\,\forall i\neq n.$$

Some of these results follow from Theorems 4.2 and 4.3 (Subsec. 4.3).

4.3. Now consider the case when the changes in the excess demand for all the goods have the same sign. This case is impossible if the Walras law is assumed to hold, but it is of definite interest for models of type $\mathfrak{M}_2$.

THEOREM 4.2 [29]. Let the GS-mapping $\mathscr{D}$ satisfy A1, A3 and let $q\in\mathscr{D}^0$. If $d=(d_i)\in\mathscr{D}(p)$, $d\geqslant 0$, then $\max\{p,q\}\in\mathscr{D}^0$. Moreover, $q_i>p_i$ for $d_i>0$, and if $\mathscr{D}$ is indecomposable and $d\neq 0$, then $q>p$. From the condition $d\leqslant 0$ it follows that $\min\{p,q\}\in\mathscr{D}^0$ and $q_i<p_i$ for $d_i<0$, and if $\mathscr{D}$ is indecomposable and $d\neq 0$, then $q<p$.

Proof. By combination Lemma 1.2, for $\hat{q}=\max\{p, q\}$ there is a vector $a\in\mathcal{D}(\hat{q})$ such that $a\geqslant d[I]\geqslant 0,\ I=\{i\,|\,p_i\geqslant q_i\}$. A3 implies that $a=0$, so that $d[I]=0$. Thus, $\hat{q}\in\mathcal{D}^0$ and $q_i>p_i$ for $d_i>0$. For an indecomposable $\mathcal{D}$, the relations $\hat{q}\geqslant p,\ d\neq 0,\ d\geqslant 0$ imply $\hat{q}>p$, but then $q>p$. The rest of the theorem is proved along the same lines.

The following proposition sharpens Theorem 4.2 under the additional assumptions of normality and strict normality of the excess demand (see Definition 1.7).

THEOREM 4.3. Let the GS-mapping $\mathcal{D}$ satisfy A1, A3, be normal, and $\mathcal{D}^0\neq\varnothing$. If $d\in\mathcal{D}(p),\ d\geqslant 0,\ N^+=\{i\,|\,d_i>0\}\neq\varnothing$, then for some $q\in\mathcal{D}^0$ we have

$$q\neq p,\quad 1\leqslant q_i/p_i\leqslant\max_{t\in N^+}q_t/p_t\ \forall i. \tag{4.3}$$

If in addition $\mathcal{D}$ is strictly normal, then (4.3) holds for any $q\in\mathcal{D}^0$, and

$$q_i/p_i<\max_{t\in N^+}q_t/p_t\ \ \forall i\notin N^+. \tag{4.4}$$

Similarly, if $d\in\mathcal{D}(p),\ d\leqslant 0,\ N^-=\{i\,|\,d_i<0\}\neq\varnothing$, then for some $q\in\mathcal{D}^0$ we have

$$q\neq p,\quad 1\geqslant q_i/p_i\geqslant\min_{t\in N^-}q_t/p_t\ \forall i. \tag{4.5}$$

In case of strict normality, (4.5) holds for any $q\in\mathcal{D}^0$, and the right inequality is strict $\forall i\notin N^-$.

Proof. Consider the case $d\geqslant 0,\ N^+\neq\varnothing$. Then by Theorem 4.2 there is $v\in\mathcal{D}^0,\ v\geqslant p,$ $\lambda=\max_{i\in N^+}v_i/p_i>1$. Clearly, (4.3) holds for $q=\min\{\lambda p, v\}$. By Lemma 1.2, using normality and the definition of λ, there is a vector $\bar{a}\in\mathcal{D}(q)$ such that $\bar{a}\leqslant f[I]\leqslant d[I]=0,\ \ f\in\mathcal{D}(\lambda p),$ $I=\{i\,|\,\lambda p_i<v_i\}$. By A3, $\bar{a}=0$, i.e., $q\in\mathcal{D}^0$.

Now suppose that $\mathcal{D}$ is strictly normal, q is an arbitrary vector from $\mathcal{D}^0$. If the inequality $p\leqslant q$ does not hold, then by Theorem 4.2 $\alpha=\min_{i\notin N^+}\{q_i/p_i\}<1$. Then $\alpha p\leqslant q,$ $I=\{i\,|\,\alpha p_i=q_i\}\neq\varnothing$ and there is a vector $g\in\mathcal{D}(\alpha p),\ g>d\geqslant 0$. Since this contradicts the GS-property, we have $p\leqslant q$.

If (4.4) does not hold, then $\mu=q_r/p_r\geqslant q_i/p_i$ for some $r\notin N^+$ and all i. Since $\mu>1$ (by Theorem 4.2), then by strict normality there is $h\in\mathcal{D}(\lambda^{-1}q),\ h>0$. But this contradicts the GS-property, since $p\geqslant\lambda^{-1}q,\ I'=\{i\,|\,p_i=\lambda^{-1}q_i\}\neq\varnothing$ and $I\cap N^+=\varnothing$ (by Theorem 4.2). Thus, (4.4) holds and this completes the first part of the proof. For the case $d\leqslant 0$, $N^-\neq\varnothing$, the proof is entirely analogous.

Let the perturbed demand $\mathcal{D}$ satisfy the conditions of Theorem 4.3, and in the initial equilibrium p only the first component of the vector $d\in\mathcal{D}(p)$ is nonzero and positive. Also assume that the new vector of equilibrium prices is unique. Then by Theorem 4.3, the prices of all the commodities do not decrease, and the relative increase of the first price is maximal; with strict normality, the proportionate increase of the first price exceeds that of any other price.

Let us now consider the behavior of the equilibrium prices in the model $\mathfrak{M}_2(B, s)$ (see Subsec. 1.5) with changing incomes and supply. The proof of the following proposition is based on Theorem 4.2.

As before, we use $P_2(B, s)$ to denote the set of equilibrium prices in $\mathfrak{M}_2(B, s)$.

THEOREM 4.4. Suppose that conditions S1-S5 hold, and

$$B \geqslant B', \quad s \leqslant s', \quad q=(q_i) \in P_2(B, s), \quad q'=(q_i') \in P_2(B', s').$$

Then $q_i \geqslant q_i'$ for any i such that $s_i > 0$. Moreover, let $B \neq B'$ and suppose that at least one of the following conditions holds:

(a) the demand function of some trader k such that $\beta_k > \beta_k'$ is strictly normal;

(b) the demand function of some trader k with positive income $\beta_k > 0$ is indecomposable.

Then $q > q'$.

Proof. If $\mathscr{D}(p)$ is the excess demand in the model $\mathfrak{M}_2(B, s)$, then, by normality of the traders, there is $f \in \mathscr{D}(q'), \ f \geqslant 0$. Since $\bar{q} = \max\{q, q'\} \in P_2(B, s)$ by Theorem 4.2, we have $qs = \sum_{k=1}^{m} \beta_k = \bar{q}s$. Therefore $q_i = \bar{q}_i$ for $s_i > 0$, i.e., $q_i \geqslant q_i'$.

If $B \neq B'$, then $f \neq 0$. Under condition (a), the vector f may be chosen strictly positive. Therefore the second proposition of the theorem follows from Theorem 4.2. Q.E.D.

COROLLARY 4.2. If $q, q' \in P_2(B, s)$ and $s_i > 0$, then $q_i = q_i'$.

Note that propositions close to Theorem 4.4 are given in [32] (for the single-valued case) and in [27].

Remark. Suppose that the GS-function $\mathscr{D}(p) = (\mathscr{D}_i(p))$ is defined and continuous on $\operatorname{int} \mathbf{R}_+^n$, satisfies A2, A4, and Morishima's strong indecomposability condition [19] (see Subsec. 4.2). Then Theorem 4.2 implies the second Hicksian law in strong form. Indeed, let the price of the n-th good (the numeraire) be fixed. Consider the function $\tilde{\mathscr{D}}(p)$ with (n − 1) components $\tilde{\mathscr{D}}_i(p) = \mathscr{D}_i(p), \ i = 1, \ldots, n-1$. By the Walras law, the functions $\mathscr{D}(p)$ and $\tilde{\mathscr{D}}(p)$ have the same set of zeros (up to a scalar factor). Furthermore, $\tilde{\mathscr{D}}(p)$ may be regarded as defined on $\operatorname{int} \mathbf{R}_+^{n-1}$; it is easily seen to satisfy condition A3, and it is also normal and has the g.s. property. Strong indecomposability of $\mathscr{D}(p)$ implies indecomposability of $\tilde{\mathscr{D}}(p)$. Thus, Theorem 4.2 applies to $\tilde{\mathscr{D}}$ and implies the strong form of the Hicksian law for a system with excess demand $\mathscr{D}$.

Also note that if the GS-function $\mathscr{D}(p)$ satisfies A2 and A4, then Opoitsev's additional condition [21] (see Subsec. 4.2) leads to strict normality of $\tilde{\mathscr{D}}(p)$, whence by Theorem 4.3 we conclude that the strong form of the third Hicksian law is valid for $\mathscr{D}(p)$.

4.4. Suppose that in an equilibrium system the demand for the first good increased (due to a decrease in the demand for the n-th good, the numeraire). Let the prices of the goods from the set L, $L \ni n$, be fixed and assume that there are prices equating demand and supply for all the goods $i \notin L$; for the goods from L the equilibrium is sustained by "external deliveries." By a known theorem of comparative statics, often called the Le Chatelier–Samuelson principle, expansion of the set L (under certain assumptions on the demand function) cannot increase the difference between the prices of the first good in the new and the old equilibrium states. The increase of the first price is maximized if all the prices (except the n-th price) are "elastic" and minimized if they are all fixed. If in this process the expansion of the set L is associated with a strict decrease of the first price, we have a strong form of the Le Chatelier–Samuelson principle.

The extension of the Le Chatelier principle to economic systems is due to Samuelson [76, 77]. He considered two types of models. In the first case, the model is described by an extremal problem, and in the second case the excess demand GS-function is directly specified. The formulation of the principle is essentially different for the two cases. The first case was developed in [60, 61, 88] and is not considered here.

Under g.s., the formulation of the principle includes comparison of the increments of all the free prices, and not only the prices of the commodity the demand for which changed; the qualitative behavior of all the free equilibrium prices is the same.

Samuelson [77] sketched a proof of the strong form principle for smooth demand functions and strict g.s. Morishima [19, p. 25] relaxed the smoothness assumption. In order to obtain the strong form of the theorem, he used the strict g.s. condition. The Le Chatelier–Samuelson principle was also proved in [21, p. 73] under intermediate assumptions between g.s. and strict g.s.

All the above-mentioned authors assume single-valued excess demand; in [19, 21] it is additionally postulated that the demand is positive homogeneous of zero degree and that the Walras identity holds. In what follows the Le Chatelier–Samuelson principle is proved for the multivalued case under the following additional assumption.

A6. For any p, q, d, f such that $d\in\mathcal{D}(p)$, $f\in\mathcal{D}(q)$ $p\leqslant q$, we have $pd\geqslant qf$.

Clearly, A4 implies A6. But A6 is satisfied also in fixed income models $\mathfrak{M}_2$, for which the Walras identity does not hold.

Let $p\in\operatorname{int}\mathbf{R}_+^n$ and $K\subset N=\{1,\ldots,n\}$. Denote

$$Q_{\mathcal{D}}(K,p)=\{q\in\operatorname{int}\mathbf{R}_+^n\,|\,q[N\setminus K]=p[N\setminus K],\ \exists d\in\mathcal{D}(q): d[K]=0\}. \tag{4.6}$$

Vectors from $Q_{\mathcal{D}}(K,p)$ will be called partial equilibrium prices (for commodities from K, on condition that the prices of the commodities from L = N\K are fixed at the level p_i).

THEOREM 4.5 (Le Chatelier–Samuelson Principle). Let the GS-mapping $\mathcal{D}$ satisfy A1, A6 and consider a given vector p and given sets $K_{t+1}\subset K_t\subset N$, $t=1,2,\ldots,\tau$ such that $Q_{\mathcal{D}}(K_t,p)\neq\varnothing$. If $d\in\mathcal{D}(p)$, $d[K_1]\geqslant 0$, then there are vectors $q^t\in Q_{\mathcal{D}}(K_t,p)$ such that $p\leqslant q^{t+1}\leqslant q^t$. If $d[K_1]\leqslant 0$, then for some $p^t\in Q_{\mathcal{D}}(K_t,p)$ we have $p^t\leqslant p^{t+1}\leqslant p$.

The proof of Theorem 4.5 requires a detailed study of partial equilibrium, which is the subject of the next subsection. As a byproduct, we will derive the conditions of existence and uniqueness of partial equilibrium prices and thus sharpen the proposition of Theorem 4.5.

4.5 We start by enumerating some important properties of partial equilibrium.

THEOREM 4.6. Let the GS-mapping $\mathcal{D}$ satisfy A1, A6, $K\subset N$. Then for $Q_{\mathcal{D}}(K,p)\neq\varnothing$ the following propositions are true:

1) $Q_{\mathcal{D}}(K,p)$ is closed under the operations max and min;

2) if $q, q'\in Q_{\mathcal{D}}(K,p)$ and $d\in\mathcal{D}(q)$, $d[K]=0$, then $d\in\mathcal{D}(q')$;

3) if $\mathcal{D}$ is indecomposable and K ≠ N, then $Q_{\mathcal{D}}(K,p)$ includes one vector only;

4) if $q\in Q_{\mathcal{D}}(K,p)$, $d\in\mathcal{D}(p)$ and $d[K]\leqslant 0$, then $\min\{q,p\}\in Q_{\mathcal{D}}(K,p)$, $q_i\leqslant p_i$ for $d_i<0$;

5) if $q\in Q_{\mathcal{D}}(K,p)$, $d\in\mathcal{D}(p)$, $d[K]\geqslant 0$, then $\max\{q,p\}\in Q_{\mathcal{D}}(K,p)$, $q_i\geqslant p_i$ for $d_i>0$;

6) if $q\in Q_{\mathcal{D}}(K,p)$, $q'\in Q_{\mathcal{D}}(K',p)$, $K'\subset K$, then $\min\{q,q'\}\in Q_{\mathcal{D}}(K',p)$ for $q\geqslant p$ and $\max\{q,q'\}\in Q_{\mathcal{D}}(K',p)$ for $q\leqslant p$.

The propositions of this theorem are simple corollaries of the following technical lemma.

LEMMA 4.1. Let the GS-mapping $\mathcal{D}$ satisfy A1, A6 and let $f\in\mathcal{D}(q)$, $f[K]=0$ for $K\subset N$, $L=N\setminus K$. Then if

$$d \in \mathscr{D}(p), \quad d[K] \leqslant 0, \quad p[L] \geqslant q[L], \tag{4.7}$$

we have $f \in \mathscr{D}(\bar{q})$, where $\bar{q} = \min\{p, q\}$, and $q_i \leqslant p_i$ whenever $d_i < 0$. If

$$d' \in \mathscr{D}(p'), \quad d'[K] \geqslant 0, \quad p'[L] \leqslant q[L], \tag{4.8}$$

then $f \in \mathscr{D}(\bar{\bar{q}})$, where $\bar{\bar{q}} = \max\{p', q\}$, and $q_i \geqslant p'_i$ whenever $d_i > 0$.

Proof. By combination Lemma 1.2, there is $g \in \mathscr{D}(\bar{q})$ such that

$$g \leqslant d[I_1] + f[I_2], \quad I_1 = \{i \mid p_i < q_i\}, \quad I_2 = N \setminus I_1.$$

By (4.7), $d[I_1] \leqslant f[I_1] = 0$, i.e., $q \leqslant f$. If $g \neq f$, then $\bar{q}g < \bar{q}f = qf$, which contradicts A6. Therefore $g = f$, $d[I_1] = 0$, whence follows the proposition of the lemma for the case (4.7). The second proposition is similarly proved. Q.E.D.

Proof of Theorem 4.6. Propositions 1. 4, and 5 of Theorem 4.6 are obvious by Lemma 4.1. Proposition 2 follows from Lemma 4.1 since $q' = \max\{q', \bar{q}\}$, where $\bar{q} = \max\{q' q\}$. Proposition 3 follows from the definition of an indecomposable GS-mapping using propositions 1 and 2 of the theorem. Proposition 6 is easily checked by applying Lemma 4.1. Q.E.D.

Now using properties 4-6 of partial equilibrium, we can easily prove the Le Chatelier–Samuelson principle.

Proof of Theorem 4.5. For $d[K_1] \geqslant 0$, proposition 5 of Theorem 4.6 indicates that there are vectors $v^t \in Q_{\mathscr{D}}(K_t, p)$ such that $v^t \geqslant p$. Let $q^1 = v^1$ and construct $q^{t+1} = \min\{q^t, v^{t+1}\}$. By proposition 6 of Theorem 4.6, $q^{t+1} \in Q_{\mathscr{D}}(K_{t+1}, p)$. Clearly, $q^t \geqslant q^{t+1} \geqslant p$. The first proposition is thus proved. The proof of the second proposition follows the same lines. Q.E.D.

The following theorem gives conditions for the existence of partial equilibrium.

Theorem 4.7. Let the GS-mapping $\mathscr{D}$ satisfy A1, A5. If for given $\tilde{p}, K$ there exist vectors p, q, d, f such that

$$p \leqslant \tilde{p} \leqslant q, \quad p_i < q_i \text{ for } i \notin K, \tag{4.9}$$

$$d \in \mathscr{D}(p), \quad d[K] \geqslant 0, \quad f \in \mathscr{D}(q), \quad f[K] \leqslant 0, \tag{4.10}$$

and then there is a vector r such that

$$p \leqslant r \leqslant q, \quad r \in Q_{\mathscr{D}}(K, \tilde{p}). \tag{4.11}$$

The proof is based on Lemmas 1.1 and 1.2 (it is not given here).

Under the additional assumption of normality, Theorem 4.7 implies that nonemptiness of $\mathscr{D}^0$ leads to existence of partial equilibrium with arbitrarily fixed prices of an arbitrary proper subset of goods, since for any $\tilde{p}$ there are p, q, d, f satisfying (4.9) and (4.10) in this case.

5. COALITION STABILITY OF ECONOMIC EQUILIBRIUM

5.1. Fairly recently, Gale [49] and then Aumann and Peleg [40] established that an exchange equilibrium may be unstable in a certain sense. Their analysis is conveniently interpreted in terms of international trade (trade between countries or economic regions).

Suppose that in each region there are several firms which exchange goods among themselves and with other firms at equilibrium prices. Suppose that the firms in the first region redistributed their initial stocks and continue exchange transactions within the framework of the entire system. Given the new allocation of the initial stocks, the old prices no longer equate demand and supply. Suppose that price adjustment leads to a new equilibrium.

Is it possible that the firms in the first region benefit from the adjustment compared with the original state? Note that according to the core theory (see, e.g., [20, p. 285]), they can gain only by trading with "foreign" firms, some of which must lose.

An affirmative answer to this question indicates that the initial equilibrium is unstable under redistribution of the initial stocks. Examples of instability in this sense were demonstrated in [49, 40]. In [54] it was shown that "the majority" of pure exchange models are unstable and therefore stability must be ensured by very special conditions. Sufficient conditions of stability were derived in [27]. The corresponding result directly using the notion of GS-mapping is formulated below in a slightly generalized form.

5.2. Consider the model $\mathfrak{M}_1(W)$, $W=(w^k, k\in M)$, $w^k\in R_+^n$. A coalition is an arbitrary nonempty subset of consumers $\tilde{M}\subset M$.

Definition 5.1. The allocation of goods $(\tilde{c}^k, k\in M)$ is called $\tilde{M}$-admissible in the model $\mathfrak{M}_1(W)$ if it is an equilibrium allocation in the model $\mathfrak{M}_1(\tilde{W})$ for some $\tilde{W}=(\tilde{w}^k, k\in M)$ such that $\tilde{w}^k\in R_+^n$,

$$\sum_{k\in M}\tilde{w}^k\leqslant\sum_{k\in M}w^k,\quad \sum_{k\in\tilde{M}}\tilde{w}^k\leqslant\sum_{k\in\tilde{M}}w^k,\quad \tilde{w}^k\geqslant w^k,\ \text{if}\quad k\notin\tilde{M}. \tag{5.1}$$

By (5.1), a coalition may not only redistribute the initial resources between its members, but also destroy them partially or completely and transfer them to traders outside the coalition. Note that it is possible to gain from partial destruction of the stocks (a phenomenon observed in practice in capitalistic firms), as demonstrated by an example in [40]. The case when a firm can benefit by free transfer of part of the resources to a partner was studied by Balasko [41] in a two-commodity, two-trader model.

Definition 5.2. Let $(p, c^k, k\in M)$ be an equilibrium in the model $\mathfrak{M}_1(W)$. This equilibrium is weakly coalition stable if for any coalition $\tilde{M}$ and any $\tilde{M}$-admissible allocation $(\tilde{c}^k, k\in M)$ there is an index $r\in\tilde{M}$ such that $u_r(c^r)\geqslant u_r(\tilde{c}^r)$. If either $u_r(c^r)>u_r(\tilde{c}^r)$ for some $r\in\tilde{M}$ or $u_k(c^k)\geqslant u_k(\tilde{c}^k)$ for all $k\in\tilde{M}$, the equilibrium is called coalition stable.

Pareto optimality of equilibrium implies that preference uniqueness (Definition 3.1) is a necessary condition of weak coalition stability.

THEOREM 5.1. Let assumptions S1-S5 (Subsec. 3.2) hold. Then any equilibrium in $\mathfrak{M}_1(W)$ is weakly coalition stable. If, moreover, (A) all the mappings $\mathscr{C}^k(p,1)$ are strictly normal or (B) $\mathscr{C}^t(p,1)$ is indecomposable and $w^t\neq 0$ for some t, then any equilibrium is coalition stable.

If neither condition (A) nor (B) is satisfied, the second proposition of Theorem 5.1 is not necessarily true [27].

Note that Theorem 5.1 admits traders with zero initial stock vectors w^k. This corresponds to a case when the coalition enlists new consumers, which earlier traded only potentially due to lack of resources, or even creates an "artificial" consumer (a new firm) with special behavior to which part of the stocks are transferred. Theorem 5.1 shows that in this case the coalition members may benefit only if the conditions of gross substitutability and normality are violated.

5.3. The proof of Theorem 5.1 uses a construction which is also useful in other cases (in particular, in proving Theorem 3.2). Fix $W=(w^k\ k\in M)$, $w^k\geqslant 0$. Let $w=\sum_{k=1}^{m} w^k$ and suppose, as before, that $P_2(B,w)$ is the set of (positive) equilibrium prices in the model $\mathfrak{M}_2(B,w)$, $B=(\beta_k,k\in M)$. Let the conditions S1-S5 hold. If $p,p'\in P_2(B,w)$, then by Theorem 3.5, $pw^k=p'w^k\ \forall k$. Denote the value of this scalar product by $P_2(B,w)w^k$. Define the function $\mathcal{H}=(\mathcal{H}_k):\mathrm{int}\,\mathbf{R}_+^m\to\mathbf{R}^m$,

$$\mathcal{H}_k(B)=P_2(B,w)\,w^k-\beta_k. \tag{5.2}$$

Let $\mathcal{H}^0=\{B\,|\,\mathcal{H}(B)=0\}$ and $\mathbf{E}_1(W)$, $\mathbf{E}_2(B,w)$ be the equilibrium sets in the models $\mathfrak{M}_1(W)$ and $\mathfrak{M}_2(B,w)$, respectively.

LEMMA 5.1. Let $w^k\neq 0\ \forall k$ and $\mathbf{E}_1(W)\neq\varnothing$. If conditions S1-S5 hold, then

1) $\mathcal{H}(B)$ is defined on $\mathrm{int}\,R_+^n$ and is continuous;
2) $\mathcal{H}(\lambda B)=\lambda\mathcal{H}(B)\ \forall\lambda>0$;
3) $\sum_{k=1}^{m}\mathcal{H}_k(B)\equiv 0$;
4) $\mathcal{H}(B)$ has the property of gross substitutability;
5) $\mathbf{E}_1(W)=\bigcup_{B\in\mathcal{H}^0}\mathbf{E}_2(B,w)$.

The existence of $\mathcal{H}(B)$ on $\mathrm{int}\,\mathbf{R}_+^n$ follows from Theorem 3.4, and property 4 follows from Theorem 4.4. Continuity is easily checked using Theorem 4.4. The remaining propositions follow from the definitions.

Let $\xi=(\hat{p},\hat{c}^k,k\in M)$ and $\eta=(\tilde{p},\tilde{c}^k,k\in M)$ be equilibria in models $\mathfrak{M}_1(\hat{W})$ and $\mathfrak{M}_1(\tilde{W})$, respectively, and let the allocation $(\tilde{c}^k,k\in M)$ be $\tilde{M}$-admissible in $\mathfrak{M}_1(\hat{W})$ for the coalition $\tilde{M}$. In what follows we prove the proposition of Theorem 5.1 for the model $\mathfrak{M}_1(\hat{W})$, additionally assuming that $\hat{w}^k\neq 0\ \forall k$.

Denote

$$\hat{\beta}_k=\hat{p}\hat{w}^k=\hat{p}\hat{c}^k,\quad \tilde{\beta}_k=\tilde{p}\tilde{w}^k=\tilde{p}\tilde{c}^k;\quad \hat{B}=(\hat{\beta}_k),\quad \tilde{B}=(\tilde{\beta}_k);$$
$$\hat{w}=\sum_{k=1}^{m}\hat{w}^k,\quad \tilde{w}=\sum_{k=1}^{m}\tilde{w}^k.$$

In order to establish weak stability, it suffices to show that $\hat{p}\tilde{c}^r\leqslant\hat{\beta}_r$ for some $r\in\tilde{M}$. Without loss of generality, let $\tilde{w}^k\neq 0\ \forall k$. Consider the function $\mathcal{H}(B)$ defined in (5.2) for $w^k=\tilde{w}^k$. Clearly, $\tilde{B}\in\mathcal{H}^0$. If $\hat{B}\notin\mathcal{H}^0$ then applying Theorem 4.1 to the function $\mathcal{H}$, we find a vector $B'=(\beta_k')$ and an index r such that

$$B'\in\mathcal{H}^0,\quad B'\geqslant\hat{B},\quad \beta_r'=\hat{\beta}_r,\quad r\in M^-=\{k\,|\,\mathcal{H}_k(\hat{B})<0\} \tag{5.3}$$

(by property 3, $M^-\neq\varnothing$). If $\hat{B}\in\mathcal{H}^0$, then let $B'=\hat{B}$. By Proposition 1.2 and the conditions S, Theorem 3.1 on the direct product is applicable to the model $\mathfrak{M}_1(\tilde{W})$. By this theorem,

$$p'\tilde{c}^k=p'\tilde{w}^k=\beta'_k,\quad k\in M,$$

where $p'\in P_2(B',\tilde{w})$. Since

$$\hat{p}\in P_2(\hat{B},\hat{w}),\quad B'\geqslant\hat{B},\quad \tilde{w}\leqslant\hat{w},$$

by Theorem 4.4, we obtain

$$\hat{p}\tilde{c}^k \leqslant p'\tilde{c}^k = \beta'_k, \quad k \in M. \tag{5.4}$$

If $B' = \hat{B}$, then by (5.4), for any k, the vector $\tilde{c}^k$ is no better than $\hat{c}^k$ for the trader k. For $B' \neq \hat{B}$ this is true for some $r \in M^-$ (see (5.3)). If either condition (A) or (B) holds, then by Theorem 4.4 $\hat{p}\tilde{c}^r < \hat{\beta}_r$, so that $\tilde{c}^r$ is strictly worse than $\hat{c}^r$. It remains to show that $r \in \dot{M}$. Suppose that this is not so; then $\tilde{w}^r \geqslant \hat{w}^r$. Again applying Theorem 4.4, we obtain

$$0 > \mathscr{H}_r(\hat{B}) = P_2(\hat{B}, \tilde{w})\tilde{w}^r - \hat{\beta}_r \geqslant \hat{p}\hat{w}^r - \hat{\beta}_r = 0,$$

which is impossible. We have thus proved the theorem for $\hat{w}^k \neq 0$.

In the general case, the function $\mathscr{H}$ is defined on a cone determined by the equilibria ξ, η and the Theorem 3.4. Since some components of $\hat{B}$ may be zero, Theorem 4.1 does not apply, but (5.3) may be proved directly and all the following results remain true.

5.4. As proved in [24], increasing the individual income in $\mathfrak{M}_2(B, s)$ may reduce the objective function value in the new equilibrium, but the conditions of g.s. and normality rule out this phenomenon. These conditions also ensure coalition stability under redistribution of income in $\mathfrak{M}_2(B, s)$ [27].

6. TATONNEMENT PROCESSES

6.1. A considerable body of mathematical economic research deals with the stability of the price-adjustment tatonnement process specified by the system of differential equations

$$\frac{dp}{dt} = \mathscr{F}(p). \tag{6.1}$$

Here the right-hand side is linked by certain relationships with the excess demand function $\mathscr{D}$, e.g., it is proportional to the excess demand or satisfies the conditions

$$\operatorname{sign} \mathscr{F}_i(p) = \operatorname{sign} \mathscr{D}_i(p), \quad i = 1, \ldots, n, \ \forall p, \tag{6.2}$$

where

$$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Equation (6.1) is considered as a model of price behavior under conditions of perfect competition. It is intended to reflect the fact that in a market with many traders, the price of a good generally increases if demand exceeds supply and decreases in the opposite case.

In Soviet economic literature, a different interpretation of the processes (6.1) is accepted, corresponding to the interpretation of equilibrium models as schemes that ensure solution matching in a planned economy. According to this interpretation, the equation (6.1) describes a rule which enables the planning organ to arrive at a balanced plan by a process of selecting the equilibrium prices [8]. It is significant that such a process can be organized utilizing only the current values of the excess demand: it does not require detailed information on utility functions and local technologies.

Discrete analogs of the equations (6.1) may be used as algorithms to compute equilibrium prices and programs.

The stability of the solutions of equation (6.1) in the economic context was first considered by Samuelson [75]. In [55, 69] it was shown that, given a differentiable GS-function $\mathcal{D}=\mathcal{F}$ (satisfying certain additional assumptions), the process is locally stable. The first global results were derived by Arrow and Hurwitz [37] and by Arrow, Block, and Hurwitz [35] for strict gross substitutability; the nonstrict case was considered in [38, 64, 86]. A survey of these and some other results can be found in [70], [36, Chaps. 11-13], [10, Chap. 9], [20, Chap. VI]. Rader [74] considered a special equilibrium model with production and showed that local asymptotic stability is ensured if the demand function (and not the excess demand) has the g.s. property, since the supply function S(p) automatically satisfies the monotonicity condition $(p-q)(S(p)-S(q))\geqslant 0\ \forall p, q$.

The rate of convergence of the price adjustment process near the equilibrium is estimated in [43].

In all the above-cited studies, the function $\mathcal{D}$ was assumed positive homogeneous of degree zero, and the Walras identity was usually postulated. The global results were obtained by the second Lyapunov method.

Recently, Howitt [57] proved global stability for a differential inclusion with the right-hand side defined by an AGS-mapping with an additional condition ensuring uniqueness of the equilibrium prices (up to normalization). In what follows, we will prove two theorems generalizing this result (see Subsec. 6.4).

In the single-valued case, many authors [37, 35, 86, 10] studied in addition to (6.1) also the so-called normalized (or normed) process in which the numeraire price is fixed. Other modifications of the process (6.1) were also considered. If the equilibrium vector is not assumed to be strictly positive, the process must be prevented from reaching the boundary of R_+^n. To this end, a certain condition of "reflection from the boundary" is added to (6.1) [38, 64, 86]. A more basic modification is associated with the assumption that prices change in response to expected, and not actual, demand. This introduces additional complications in the model. Some results of this kind will be found in [39, 10]. So far they remain without further development.

6.2. For the excess demand $\mathcal{D}(p)$, which is not necessarily single-valued, the continuous-time tatonnement process may be defined by the differential inclusion

$$\frac{dp}{dt}\in\mathcal{F}(p), \qquad (6.3)$$

where the mapping $\mathcal{F}$ satisfies the following sign constancy condition:

F1. For any vectors $p=(p_i)$, $f=(f_i)$ such that $f\in\mathcal{F}(p)$, there is $d=(d_i)\in\mathcal{D}(p)$ for which $\operatorname{sign} f_i=\operatorname{sign} d_i$ for all i.

The inclusion (6.3) defines a normalized process with fixed numeraire price (the n-th price) if $\mathcal{F}$ satisfies the following condition:

F2. For any $p=(p_i)$, $f=(f_i)$ such that $f\in\mathcal{F}(p)$, we have $f_n=0$ and $\operatorname{sign} f_i=\operatorname{sign} d_i$, $i\neq n$, for some $d=(d_i)\in\mathcal{D}(p)$.

Suppose that $\mathcal{F}$ is defined on the set $V\subset R^n$, $p^0\in V$ and T is a positive number. The function $p:[0,T]\to V$ is called a solution of the differential inclusion (6.3) on the interval [0, T] with the initial conditions $p(0)=p^0$ if it is absolutely continuous, $p(0)=p^0$ and

$\frac{dp(t)}{dt} \in \mathscr{F}(p(t))$ for almost all t. The function p(t) is a solution of (6.3) on $[0, +\infty)$ for $p(0)=p^0$ if it is a solution of (6.3) on [0, T] for any $T>0$ with the same initial conditions. In this case, p(t) is also called a path of the process (6.3).

A point $q \in V$ is called an equilibrium if $0 \in \mathscr{D}(q)$. As before, $\mathscr{D}^0 = \{q \mid 0 \in \mathscr{D}(q)\}$. The following definition of tatonnement stability is a generalization to the case of multivalued excess demand.

Definition 6.1 ([57], see also [86, 44]). The process (6.3) is quasistable if for any $p^0 \in V$ there is a path originating from p^0, each path is bounded, and all its limiting points* belong to $\mathscr{D}^0$. The process is stable if it is quasistable and every path has a single limiting point.

Note that this terminology differs from the conventional terminology of mathematical stability theory.

In what follows we assume that $\mathscr{D}$ satisfies conditions A1, A2, and A3 (see Subsec. 1.6). Condition A1 is also extended to $\mathscr{F}$.

This property is observed, say, in a mapping $\mathscr{F}$ defined in the following way:

$$\mathscr{F}(p) = \Gamma(p)\mathscr{D}(p) = \{f \mid f = \Gamma(p)d,\ d \in \mathscr{D}(p)\}, \tag{6.4}$$

where $\Gamma(p)$ is a diagonal matrix which is a continuous function of p; $\gamma_i(p)$ are the diagonal elements of this matrix. If all the functions $\gamma_i(p)$ are positive, then F1 holds; if $\gamma_n(p) \equiv 0$, $\gamma_i(p) > 0,\ i \neq n$, then $\mathscr{F}$ satisfies F2 and defines a normalized process.

Stability theorems for the process (6.3) will be proved in Subsec. 6.4. First, however, we have to establish the properties and the existence of paths of this process.

For the vectors $p=(p_i),\ q=(q_i) \in \operatorname{int} R_+^n$ we define

$$\lambda(p, q) = \max_{i \in V} p_i/q_i, \quad \mu(p, q) = \min_{i \in V} p_i/q_i, \tag{6.5}$$

where $N = \{1, 2, \ldots, n\}$.

The following lemma plays an important role in proofs of stability of the tatonnement process (6.3).

LEMMA 6.1. Let the GS-mapping $\mathscr{D}$ satisfy A1, A2, A3, and let one of the conditions F1, F2 hold for the mapping $\mathscr{F}$. If $q \in \mathscr{D}^0$ and the function p(t) is a solution of the differential inclusion (6.3) on the interval [0, T] for some $T>0$, then the function $\lambda(p(t), q)$ is nonincreasing, and $\mu(p(t), q)$ is nondecreasing on [0, T].

If q is a unique vector (up to a scalar factor) included in $\mathscr{D}^0$, and the difference $\lambda(p(t), q) - \mu(p(t), q)$ is constant on [0, T], then $p(t) = \alpha q$ for all $t \in [0, T]$ for $\alpha = \lambda(p(0), q) = \mu(p(0), q)$.

Proof. Denote $l(t) = \lambda(p(t), q),\ m(t) = \mu(p(t), q)$. For any $t \in [0, T]$ we have

$$\begin{gathered} l(t)q,\ \ m(t)q \in \mathscr{D}^0,\ \ l(t)q \geqslant p(t) \geqslant m(t)q, \\ I_l(t) = \{i \mid p_i(t) = l(t)q_i\} \neq \varnothing,\ \ I_m(t) = \{i \mid p_i(t) = m(t)q_i\} \neq \varnothing. \end{gathered} \tag{6.6}$$

Using the identities (where a, b are real numbers)

$$\max\{a, b\} \equiv (a+b+|a-b|)/2, \quad \min\{a, b\} \equiv (a+b-|a-b|)/2,$$

---

*P is a limiting point of a path if $p(t^\nu) \to \bar{p}$ for some sequence $t^\nu \to +\infty$.

we can easily check that $l(t), m(t)$ are absolutely continuous. Therefore, almost everywhere on [0, T], there exist arbitrary functions $l(t)$, $m(t)$, $p(t)$ (we denote them by $\dot{l}(t)$, $\dot{m}(t)$, $\dot{p}(t)$), and by (6.6) the following relationships hold for any $j\in I_l(t)$ and $s\in I_m(t)$:

$$\dot{l}(t)=q_j^{-1}\lim_{h\to+0} h^{-1}(l(t+h)q_j-l(t)q_j)\leqslant q_j^{-1}\lim_{h\to+0} h^{-1}(p_j(t+h)-p_j(t))=q_j^{-1}\dot{p}_j(t), \tag{6.7}$$

$$\dot{m}(t)=q_s^{-1}\lim_{h\to+0} h^{-1}(m(t+h)q_s-m(t)q_s\geqslant q_s^{-1}\lim_{h\to+0} h^{-1}(p_s(t+h)-p_s(t))=q_s^{-1}\dot{p}_s(t). \tag{6.8}$$

By Theorem 4.1 (6.6) and the inclusion $\dot{p}(t)\in\mathcal{F}(p(t))$ imply (in virtue of either of the conditions F1, F2) that for almost every $t\in[0,T]$ there are $j\in I_l(t)$, $s\in I_m(t)$ such that $\dot{p}_j(t)\leqslant 0$, $\dot{p}_s(t)\geqslant 0$. Therefore $\dot{l}(t)\leqslant 0$, $\dot{m}(t)\geqslant 0$ almost everywhere on [0, T]. By absolute continuity of $l(t)$ and $m(t)$, this leads to the first proposition of the lemma.

If $\mathcal{D}^0$ is a ray, Theorem 4.1 and relationships (6.6), (6.7), (6.8) imply (by either F1 or F2) that $\dot{l}(t)-\dot{m}(t)<0$ for almost all t such that $p(t)\notin\mathcal{D}^0$. Therefore, under the additional assumptions of the lemma, for almost all t we have $p(t)\in D^0$ $p(t)=l(t)q=m(t)q$. Thus, using the properties of the functions $p(t), l(t)$, and $m(t)$, we obtain the second proposition of the lemma. Q.E.D.

In Lemma 6.1 we assume that the differential inclusion (6.3) has a solution. In the following subsection we give the conditions for the existence of paths of this process.

6.3. Consider the compact set

$$K=\{p\,|\,\alpha\leqslant\mu(p,q)\leqslant\lambda(p,q)\leqslant\beta\}, \tag{6.9}$$

where λ and μ are defined in (5.6), $\beta>\alpha>0$ and $q\in\mathcal{D}^0$. Let $C_u([0, T]; K)$ be the space of continuous functions on [0, T] with values in K, endowed with the uniform convergence topology.

The following proposition can be proved following the scheme suggested in [44].

<u>THEOREM 6.1.</u> Let the GS-mapping $\mathcal{D}$ satisfy A1, A2, A3, and let A1 and one of the conditions F1, F2 hold for the mapping $\mathcal{F}$. Then, for every positive compact set K of the form (6.9) and every point p^0 from this set, there are paths of the process (6.3) originating from p^0 . Also (1) none of these paths leaves K; (2) the set $S_T(p^0)$ of the initial sections of the paths originating from p^0 with $t\in[0,T]$ is a nonempty compact set in $C_u([0, T]; K)$; (3) the mapping S_T is upper semicontinuous on K.

This theorem implies the existence of positive paths of the process (6.3) originating from any given point in $\operatorname{int}\mathbf{R}_+^n$.

The theorem is proved by constructing an auxiliary bounded mapping $\mathcal{G}$ which is defined (unlike $\mathcal{F}$) on the entire $\mathbf{R}^n$ and coincides with $\mathcal{F}$ on some positive compact set whose interior includes K (see [57]). For the auxiliary process

$$\frac{dp}{dt}\in\mathcal{G}(p) \tag{6.10}$$

paths exist by the Castaign–Valadier theorem (see Theorem A1 in [44, pp. 292-293]). Their initial sections are solutions of (6.3). By Lemma 6.1 the paths of processes (6.10) and (6.3) originating in K coincide for all t and do not leave K. Properties (2) and (3) follow from the Castaign–Valadier theorem.

6.4. We now proceed to analyze the stability of (6.3). By [44], the Lyapunov function of the process (6.3) on a closed set $V \subset \mathbf{R}^n$ is defined as the continuous function $\mathcal{G}: V \to \mathbf{R}^1$ such that (1) for each path p(t) of the process included in V, the function $\mathcal{L}(p(t))$ has a limit for $t \to +\infty$; (2) if there is a path p(t) originating in V such that for some $T>0$ the function $\mathcal{L}(p(t))$ is constant on [0, T], then $\mathcal{F}(p(0)) \ni 0$.

If all the conditions of Lemma 6.1 hold, then $\mathcal{L}(p) = \lambda(p, q) - \mu(p, q)$ is a Lyapunov function of the process (6.3) for any positive compact set K of the form (6.10). Therefore, the quasistability theorem of [44] can be applied to prove the following stability theorem for tatonnement processes (6.3) (both normalized and unnormalized) for a system with gross substitutability of the excess demand.

THEOREM 6.2. Let the GS-mapping $\mathcal{D}$ satisfy A1, A2, A3, and let A1 and one of the conditions F1 and F2 hold for the mapping $\mathcal{F}$. Then, if there is an equilibrium vector $q \in \mathcal{D}^0$ and it is unique (up to a scalar factor), the process (6.3) is stable on $\operatorname{int} \mathbf{R}_+^n$.

Proof. Since $\mathcal{L}(p) = \lambda(p, q) - \mu(p, q)$ is a Lyapunov function of the process (6.3) on the compact set K (6.10), Theorem 6.1 from [44] implies (by virtue of propositions (1), (2), and (3) of Theorem 6.1 from Subsec. 6.3) that any limiting point $\bar{p}$ of any path p(t) belongs to $\mathcal{F}^0 = \{p \mid \mathcal{F}(p) \ni 0\}$. By the conditions of Theorem 6.2, $\mathcal{F}^0 \subset \mathcal{D}^0$. Thus, the process (6.3) is quasistable.

By Lemma 6.1, the limits of the functions $\lambda(p(t), \bar{p})$ and $\mu(p(t), \bar{p})$ exist as $t \to +\infty$. Since $\bar{p} = \lim_{\nu \to +\infty} p(t^\nu)$ for some sequence $t^\nu \to +\infty$, then from the definition of λ and μ it follows that $\lim_{t \to +\infty} \lambda(p(t), \bar{p}) = \lim_{t \to +\infty} \mu(p(t), \bar{p}) = 1$. Since

$$\lambda(p(t), \bar{p})\, \bar{p} \geqslant p(t) \geqslant \mu(p(t), \bar{p})\, \bar{p},$$

we conclude that $p(t) \to \bar{p}$ for $t \to +\infty$. Q.E.D.

Note that many authors prove stability using propositions similar to Lemma 6.1 and Theorem 6.1, with the aid of the same Lyapunov function $\mathcal{L}(p) = \lambda(p, q) - \mu(p, q)$ (see, e.g., [35, 57]).

The results of Subsec. 3.2 are examples in which pure exchange models with multivalued excess demand (satisfying the conditions of Theorem 6.2) have unique equilibrium prices. This is not always so, however.

If the equilibrium prices are not unique, we have the following theorem for a GS-mapping satisfying condition A4 (the Walras identity, Subsec. 1.6), which is stronger than A3. It implies partial stability of the normalized and unnormalized tatonnement processes.

THEOREM 6.3. Let the GS-mapping $\mathcal{D}$ satisfy conditions A1, A2, A4, and let the mapping $\mathcal{F}$ be defined by relationships (6.4) with constant nonnegative diagonal matrix Γ whose diagonal elements γ_i are positive for i ≠ n. Then, if $\mathcal{D}^0 \neq \varnothing$, the process (6.3) is stable on $\operatorname{int} \mathbf{R}_+^n$.

Proof. By Theorem 2.5 (see Subsec. 2.6), we have

$$qd \geqslant 0 \text{ for } q \in \mathcal{D}^0,\ d \in \mathcal{D}(p),\ p \in \operatorname{int} \mathbf{R}_+^n, \qquad (6.11)$$
$$\text{for } qd > 0 \text{ and } p \notin \mathcal{D}^0.$$

By A4, $\mathcal{F}^0=\mathcal{D}^0$. For some $q^*\in\mathcal{D}^0$, construct the function

$$\mathcal{L}(p)=\sum_{i=1}^{\nu}\gamma_i^{-1}(p_i-q_i^*)^2, \text{ where } \nu=\begin{cases} n, & \gamma_n>0,\\ n-1, & \gamma_n=0.\end{cases} \qquad (6.12)$$

For each path p(t) of the process (6.3) the following derivatives exist for any T > 0 almost everywhere on [0, T]:

$$\frac{d\mathcal{L}(p(t))}{dt}=-2q^*d^t, \text{where } d^t=(d_i^t)\in\mathcal{D}(p(t)),\ \gamma_i d_i^t=\frac{dp_i(t)}{dt}.$$

Therefore, by (6.11), $\mathcal{L}(p)$ is a Lyapunov function of the process (6.3).

The rest of the proof is entirely analogous to the proof of Theorem 6.2.

Note that the function (6.12) decreases along the paths. Therefore, under the conditions of Theorem 6.3, for $\gamma_i=1,\ i\neq n$, any path of the process (6.3) displays monotone convergence (in the euclidean norm) to the equilibrium.

Note that for systems with single-valued excess demand (assuming gross substitutability and the Walras identity) and nonunique equilibrium prices, the function (6.12) was also successfully applied to prove stability of the tatonnement process considered in Theorem 6.3 [35, 38]. The application of this function, as in Theorem 6.3, was based on inequalities (6.11), which were known for the single-valued case (see [35, 38]).

6.5. The process (6.1) has not been studied for multivalued inhomogeneous GS-mappings. However, the single-valued case was considered by a number of authors [13, 14, 21, 79, 89]. We reproduce here the fairly general result from [79].

THEOREM 6.4. Let the GS-function $\mathcal{F}(p)$ be defined and continuous on the set $P=\{p\mid v\leqslant p\leqslant r\}$, where $v,\ r\in\mathbf{R}_+^n,\ \mathcal{F}(r)\geqslant 0\geqslant\mathcal{F}(v)$. If the path p(t) of equation (6.1) converges to some point $p^*=(p_i^*)\in P$ independently of the initial state $p(0)\in P$, then

$$\min_i(q_i-p_i^*)\mathcal{F}_i(q)<0\quad \forall q=(q_i)\in P,\quad q\neq p^*. \qquad (6.13)$$

Conversely, if (6.13) holds for some $p^*\in P$, then p* is the unique equilibrium and $p(t)\to p^*$ for $t\to\infty$ $\forall p(0)\in P$.

In [79] it is also shown that under the conditions of Theorem 6.4, (6.13) is satisfied if $\forall p,\ q\in P$ such that $q\geqslant p,\ q\neq p$ there is a coordinate i for which $f_i(q)<f_i(p)$.

Note that if in the fixed income model $\mathfrak{M}_2(B, s)$ the individual demand functions $\mathscr{C}^k(p,\ \beta)$ satisfy assumptions S1-S5 (see Subsec. 3.2) and are differentiable and strictly positive, the following inequalities hold:

$$(p-q)(\mathscr{C}^k(p,\beta)-\mathscr{C}^k(q,\beta))<0\quad \forall p,\ q\in\operatorname{int}\mathbf{R}_+^n,\quad \beta\in\operatorname{int}\mathbf{R}_+^1. \qquad (6.14)$$

From this fact, proved in [24], it follows that the excess demand satisfies condition (6.13). Using the existence of equilibrium and normality and applying Theorem 6.4, we can easily show that the process (6.1) is stable. The following generalization of Theorem 2.5 and the relationship (6.14) is apparently also true: if the GS-mapping $\mathcal{D}$ satisfies A1, A3 and is normal, then $(p-q)d\leqslant 0$ for all p, q, d such that $d\in\mathcal{D}(p),\ p\in\operatorname{int}R_+^n,\ q\in\mathcal{D}^0$, and for $p\notin\mathcal{D}^0$ a strict inequality holds. However, so far this proposition remains unproved.

6.6. Tatonnement processes in discrete time were first considered by Uzawa [85]. One of the processes proposed in [85] is specified by the relationships

$$p_i(t+1)=\max\{0, p_i(t)+\rho\mathcal{D}_i(p(t))\}, \quad i=1,\ldots,n-1,$$
$$t=0,1,\ldots, \quad p_n\equiv 1, \quad p(0)=p^0\geqslant 0, \quad p^0\neq 0. \tag{6.15}$$

Uzawa does not exclude the case of zero prices for some commodities. The price vector $q\geqslant 0$ is called equilibrium if $\mathcal{D}(q)\leqslant 0,\ q\mathcal{D}(q)=0$.

THEOREM 6.5 [85]. Suppose that the function $\mathcal{D}(p)=(\mathcal{D}_i(p))$ is defined in some neighborhood in R_+^n, satisfies A2, A4, and is twice continuously differentiable; also suppose that the matrix with the general term $a_{jk}=\sum_{i=1}^{n} p_i \frac{\partial^2\mathcal{D}_i}{\partial p_j \partial p_k}$ is nonsingular at some equilibrium point. If the equilibrium vector q satisfies the condition

$$q\mathcal{D}(p)>0 \quad \forall p\neq q, \tag{6.16}$$

and ρ is sufficiently small, then $\lim_{t\to\infty} p(t)=q$ for any $p^0\in R_+^n$.

Under the conditions of Theorem 6.5, the inequality (6.16) follows from gross substitutability and indecomposability (see Theorem 2.5).

The system (6.15) defines a normalized tatonnement process. The unnormalized price adjustment process with strict g.s. was considered in [10, p. 310] (the proof of its stability contains many gaps).

If the GS-function $\mathcal{D}$ is not smooth, then the process (6.15) apparently may diverge for any fixed ρ. The convergence of the following unnormalized variable step process was proved in [23] (for the multivalued case):

$$p(t+1)=\max\left\{h;\ p(t)+\rho(t)\frac{d(t)}{\|d(t)\|}\right\}, \quad d(t)\in\mathcal{D}(p(t)),$$
$$p(0)=p^0\geqslant h, \tag{6.17}$$

where h is a fixed (small) positive vector. The process (6.17) is defined for $d(t)\neq 0$; for $d(t)=0$ we assume that it stops.

THEOREM 6.6 [23]. Suppose that the mapping $\mathcal{D}$ satisfies A1, A4, the sets $\mathcal{D}(p)$ are jointly bounded from below $\mathcal{D}^0\cap\{p\,|\,p\geqslant h\}\neq\varnothing$, and (6.11) holds. Also let

$$\rho(t)\geqslant 0, \quad \sum_{t=1}^{\infty}\rho(t)=\infty, \quad \sum_{t=1}^{\infty}\rho^2(t)<\infty.$$

Then the sequence (6.17) converges to the set $\mathcal{D}^0$.

Note that the proof in [23] actually implies convergence to a point in $\mathcal{D}^0$.

Using Theorem 2.5, we can easily restate Theorem 6.6 for GS-mappings.

Processes close to (6.15) are also considered in [5] (see, in particular, page 139) and in [21, Chap. 9]. The recent algorithms for the solution of variational inequalities (see the survey [2]) are also close to the above procedures: after an appropriate modification of the mapping $\mathcal{D}$ they allow to apply a constant step process. However, the available convergence proofs of such algorithms are based on conditions which differ from those assumed in Theorem 6.6.

Uzawa [85] proposed a process of successive price changes in order to find the zeros of the GS-function (similar to the well-known Gauss-Seidel method). Let $p(t) = (p_i(t))$ be the price vector in the t-th large iteration and suppose that $p_i(t + 1)$ have already been found for i = 1, ..., j − 1. Then, according to Uzawa, the value of $p_j(t+1)$ is determined in the j-th small iteration by solving the equation

$$\mathscr{D}_j(p_1(t+1), \ldots, p_{j-1}(t+1), v, p_{j+1}(t), \ldots, p_n(t)) = 0 \tag{6.18}$$

for the variable v.

THEOREM 6.7 [85]. If a strict GS-function $\mathscr{D}$ is defined and continuous on $\operatorname{int} R_+^n$, satisfies A2, A4, and $\mathscr{D}^0 \neq \varnothing$, then the process converges to a unique equilibrium vector.

Uzawa proved this theorem by using the Lyapunov function $\mathscr{L}(p) = \lambda(p, q) - \mu(p, q)$, where λ, μ are defined in (6.5), $q \in \mathscr{D}^0$.

In [11] it was shown that if the GS-function is not strict, the process of successive price changes may diverge. It was suggested to select the index j of equation (6.18) at random in each step, in accordance with a prespecified probability distribution $\pi_i(t)$, $i = 1, \ldots, n$, where $\sum_{i=1}^{n} \pi_i(t) = 1$, $\pi_i(t)$ are nonzero.

When computing the root of equation (6.18), the coordinate j is incremented if $\mathscr{D}_j > 0$, and decremented if $\mathscr{D}_j < 0$. Among the solutions of equation (6.18) we select the one which is "the first to occur," as it requires the least change of the j-th price. It was proved in [11, 17] that the modified process converges under the conditions of Theorem 6.7 even if gross substitutability is postulated instead of strict gross substitutability. There is no need to find an exact solution of (6.18) at each step. If $\hat{p}_j(t+1)$ is the solution of (6.18) closest to $p_j(t)$, then we may set $p_j(t+1) = p_j(t) + \alpha_j(t)(\hat{p}_j(t+1) - p_j(t))$, where $1 \geqslant \alpha_j(t) \geqslant \rho > 0$. The convergence of a more general process, allowing simultaneous changes in several variables, was proved in [17] under the same basic assumptions.

6.7. Among the algorithms available for computing the equilibrium vector under g.s. conditions, we should mention an exceptionally simple and efficient procedure which constitutes a modification of the simple iterative method. It was developed in [3, 4, 6] for the nonlinear intersectoral balance model. Let us consider a somewhat generalized version of this procedure.

If $\mathscr{D}$ is a continuous GS-function on R_+^n and the set $\{p \mid p \geqslant 0, \mathscr{D}(p) \geqslant 0\}$ is bounded from above, then it contains a maximum point q, such that $\mathscr{D}(q) = 0$ (Subsec. 2.4). Suppose that for some $\tilde{k}_i > 0$ the function $\mathscr{D}$ satisfies the condition

$$\mathscr{D}_i(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n) - \mathscr{D}_i(p_1, \ldots, p_{i-1}, p_i + \delta, p_{i+1}, \ldots, p_n) \leqslant \tilde{k}_i \delta, \quad i = 1, \ldots, n,$$

for any $\delta \geqslant 0$ and any p such that $q \leqslant p \leqslant \tilde{q}$, $\tilde{q}$ is an arbitrary vector. Then the following process converges to the vector q:

$$p_i(t+1) = \min\left\{p_i(t),\ p_i(t) + \frac{1}{k_i}\mathscr{D}_i(p(t))\right\}, \qquad k_i \geqslant \tilde{k}_i, \quad i = 1, \ldots, n, \quad p(0) = \tilde{q}, \tag{6.19}$$

and all the coordinates p(t) are nonincreasing. In some cases, e.g., in fixed income models, the initial vector $\tilde{q}$ can be found without difficulty.

An algorithm to find a minimum point of the set $\mathscr{D}^-$ which ensures that all the coordinates are nondecreasing is constructed along the same lines as (6.19) [3, 6, 9].

7. CONCLUDING REMARKS

The assumption of gross substitutability is highly restrictive in the economic sense. Therefore various attempts have been made to extend the results obtained for gross substitutes systems to a broader class of cases.

The simplest generalization is the following. We say that the mapping $\mathscr{D}$ has the property of gross complementarity (g.c.) if $(-\mathscr{D})$ is a GS-mapping. The term "g.c." for single-valued demand was introduced by Mosak [68], and a similar concept was used by Hicks [56]. Conditions A1, A2, A4 (Subsec. 1.6), A5 (Subsec. 2.3), and some other assumptions used above remain valid under the change of sign. All the corresponding results are therefore applicable in the g.c. case (A3 should be replaced with A4, however).

Morishima [67] divided all the goods (except the numeraire) into two groups, so that strict g.s. applies within each group and strict g.c. applies for goods in different groups. Special conditions are imposed on the numeraire. For smooth excess demand functions satisfying a number of additional assumptions, Morishima [67] proved uniqueness of the equilibrium vector and stability of the process (6.1). A "market" with g.s. and g.c. simultaneously was considered also in [21].

Although the existence of equilibrium was proved under highly general assumptions (see, e.g., [36, 20, 22]), necessary and sufficient conditions are only known for linear models [50]. It is not clear if existence theorems of the type of Theorem 3.4 remain valid for a wider class of cases without the g.s. condition.

A survey of equilibrium uniqueness results is given in [51, 20, 36, 13, 21]. Relationship (6.11) and other versions of the "revealed preference" condition [35, 38, 24] (see also [8, p. 138]) lead to convexity of the set of equilibrium prices or to uniqueness of equilibrium, and also to stability of the processes (6.1) (for $\mathscr{F}=\mathscr{D}$). Another condition, more general than strict g.s. in the smooth case, stipulates that the Jacobian matrix of the excess demand has a dominant diagonal at each point [63, 72] (see also [10, 36]). This condition also ensures uniqueness and stability of equilibrium. Two finite-increment forms of the dominant diagonal condition are given in [13, 14]. They are apparently applicable to the multivalued case as well, although this question has not been studied.

The tatonnement processes considered in Sec. 6 are not the only possible price adjustment processes. The analysis of some of these processes is based on conditions of the type (6.11), although entirely different assumptions may be used. A survey of the corresponding results will be found in [36, Chap. 13] and also in [86, 52]. Among the recent studies on this subject, we should mention the article by Smale [83].

Unfortunately, very little is known about the specific properties of utility functions which ensure that the excess demand satisfies various particular conditions.

Possible generalizations of the theorems of comparative statics were studied in [62, 53, 73]. The results indicate that the g.s. condition cannot be substantially relaxed.

So far, Theorem 3.1 and Corollary 3.1 (equivalence of nonunique equilibria) remain without generalization. The same applies to Theorem 5.1 of coalition stability.

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