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An Evolutionary Model of Economic Growth*

1. Introduction

The mathematical theory of economic growth and the modeling of the diffusion of innovations are currently viewed as two independent research areas in different phases of their respective "life cycles." The topics that have been vigorously developed in economic growth theory during the last 20 years (optimality of infinite trajectories, "turnpike" behavior) are apparently close to the point of exhaustion, while the study of innovations is still primarily in the empirical stage and has not produced finished models. The obvious organic interrelationship between growth processes and the emergence of new technologies suggests that the two research areas should be synthesized into one. In line with this view, we have modeled the innovation process in terms of the evolution of the efficiency distribution curve of different technologies [1]. Analysis of the corresponding differential-difference equation has shown that the curve should eventually acquire a standard form, independent of the initial conditions. This result is qualitatively consistent with empirical observations.

The next natural step in the development of this topic should examine the interaction of spontaneous innovation and imitation against the background of the quantitative growth of capacity. The main goal of the present paper is the implementation of this program for a one-commodity model. The first attempt to move in this direction was made by Iwai [2], who used a set of different, and in our opinion more artificial, assumptions.

Sec. 2 derives the generalized evolution equation of the efficiency distribution curve of capacities and presents a theorem on asymptotic solutions, analogous to the main theorem of [1]. Sec. 3 develops the proposed growth model and the basic results. The main result (Theorem 2) claims that under certain conditions the evolution of the distribution of capacities in the process of exponential growth approximately follows the globally stable evolution equation of Sec. 2. This provides a better substantiation for the observed stability of the form of the distribution curve. Sec. 4 discusses some modifications of the model. All the proofs are collected in Sec. 5.

2. Evolution of the Efficiency Distribution Curve of Technologies

Consider a production system (e.g., a sector of the economy) with many producers, each assigned to one or another efficiency levels. The notion of efficiency can be defined in different ways, e.g., as the value added or the profit per unit capacity. In this section, the particular form of the efficiency index is immaterial, and the only relevant assumption is that each producer strives to move to a level with a higher value of this index.

Denote by $F_n(t)$ the proportion of producers that at time t occupy efficiency levels with index not exceeding n. Here $t \in [0,\infty)$, and n may take any integer values. The sequence $F = \{F_n\}$ as a function of time describes the evolution of the distribution curve of the producers by efficiency levels. We assume that the proportion $\varphi(F_n)$ of producers move in unit time from level n to level n + 1; jumping over several efficiency levels is not allowed. The number of producers is assumed constant.[1] The proportion $F_n - F_{n-1}$ of all producers are found on level n, and this proportion decreases in unit time by the amount $\varphi(F_n)(F_n - F_{n-1})$. In this way, we arrive at a system of differential-difference equations that describe the evolution of the efficiency distribution curve of the producers

$$dF_n/dt = \varphi(F_n)(F_{n-1} - F_n). \qquad (1)$$

Natural initial conditions for (1) are the following:

$$0 \le F_n(0) < 1, \quad n = 1, \ldots, N,$$

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$$F_n(0) = 0, \ N \le 0, \qquad (2)$$
$$F_n(0) = 1, \ n > N.$$

In our study, however, we will use more general initial conditions of the form

$$0 \le F_n \le 1, \ F_n(0) = 0, \ n \le 0, \ \lim_{n \to \infty} F_n(0) = 1 \qquad (3)$$

Throughout the paper, we adopt the following assumption:
(i) the function φ is Lipschitzian, positive, nonincreasing on the interval [0, 1], and $\varphi(0) > \varphi(1)$.

The assumption that φ is decreasing corresponds to the notion that the intensity of imitation increases with the increase of the proportion $1\text{-}F_n$ of technologically advanced producers. The most efficient firms occupy levels where F_n is close to 1. They cannot borrow the technology from anyone else. Their efficiency, however, can increase through innovation, and we accordingly assume that $\varphi(1) > 0$. Let

$$\Phi(F) = \int_F^1 (\varphi(y))^{-1} dy, \quad F \in [0, 1].$$

Then Eq. (1) is rewritten in the form

$$d\Phi(F_n)/dt = F_n - F_{n-1}.$$

To each sequence of functions $F = \{F_n\}_{n=0}^{\infty}$ we associate the expression

$$B_F(t) = \sum_{n=1}^{\infty} \Phi(F_n(t) - t.$$

We now introduce an additional assumption.
(ii) The initial conditions are such that $B_F(0) < \infty$.

It can be shown (see [2]) that in cases (i) and (ii) the solution $F = \{F_n(t)\}_{n=0}^{\infty}$ of the problem (1), (3) exists, is unique, and for any t satisfies the conditions $F_n(t) = 0$ for $n \le 0$, $F_n t) \to 1$ for $n {+} \infty$.

Moreover, $B_F(t) \equiv B_F(0)$, because B_F is the first integral of Eq. (1). If $F_n(0) \ge F_{n-1}(0)$ for all n, then $F_n(t) \ge F_{n-1}(t)$ for all n, t, so that (1) indeed describes an evolution of the distribution curve.

Theorem 1. Let condition (i) hold. Then for any solution $F = \{F_n\}$ of the problem (1), (3) that satisfies (ii) there exists a constant a such that

$$\sup_n |F_n(t) - F^*(n - ct - a)| \longrightarrow 0, \qquad t \to \infty,$$

where $F^*(x)$ is the solution of the equation

$$c(dF/dx) = \varphi\left[F(x)\right]\left\{(F(x) - F(x\text{-}1)\right\}, \quad c^{-1} = \Phi(0),$$

satisfying the conditions

$$0 \le F \le 1, \quad \sum_{n=-\infty}^{0} \left\{\Phi(0) - \Phi(F(n))\right\} < \infty, \quad \sum_{n=1}^{\infty} \Phi(F(n)) < \infty.$$

The function $F_n^*(t) = F^*(n - ct - a)$ for any a is the solution of Eq. (1). It depends only on a linear combination of the variables n and t and is often called a traveling wave, because its graph moves along the n-axis with constant velocity c. For a linear φ, the function $F^*(x)$ was derived in explicit form in [1]: it was found to be the logistic probability distribution.

The constant a in Theorem 1 is computed from the value of the first integral on the solution $F = \{F_n\}$ using the equation

$$B_F(0) = \sum_{n=-\infty}^{0} (\Phi(F^*(n\text{-}a) - \Phi(0))) + \sum_{n=1}^{\infty} \Phi(F^*(n\text{-}a)).$$

Theorem 1 shows that, as a result of the interaction between imitation and innovation, the shape of the efficiency distribution curve of the technologies eventually stabilizes; this curve moves with almost constant velocity along the abscissa axis; neither the shape nor the velocity asymptotically depend on the initial conditions. The model thus explains the stability of the shape of the empirically observed distribution curve.

For the case of a linear φ, Theorem 1 constitutes the main result of [1]. Its proof in the general form presented in this paper relies on entirely different constructs (see [3]) and is omitted for reasons of space.

3. Evolution Model of Economic Growth

So far, we have assumed that the number of firms remains constant and ignored the differences in productivity across firms. In this section, we describe a simple model that allows simultaneously for qualitative improvement of production capacity through imitation and innovation, as well as for quantitative growth.

Consider a certain sector, and let M_n be the total production capacity that returns a profit λ_n per unit capacity in unit time. The index λ_n will be used as the efficiency measure of the technology of level n. We assume that efficiency increases with the increase of n, i.e., $\lambda_{n+1} > \lambda_n$. We further assume that the profit is reinvested in the expansion of capacity, so that nonproductive distributions of the profit are ignored in our model.[2]

We define the distribution function $F = \{F_n\}$ of capacities by efficiency levels,

$$F_n = \Big(\sum_{k=0}^{n} M_k\Big)\Big(\sum_{k=0}^{\infty} M_k\Big)^{-1}, \quad n = 0, 1, 2, \ldots,$$

assuming that the expression in the right-hand side is meaningful.

We postulate that the profit $\lambda_n M_n$ earned on level n-m is split into two capital investment streams. The proportion $\varphi_0(F_n)$ of this profit is channeled to formation of capacities of the next level n+1, while the remaining part $(1-\varphi_0(F_n)\lambda_n M_n$ is spent to expand the production of level n. Here φ_0 is a given function. In the Schumpeterian paradigm, firms of level n can create capacities on level n+1 through a process of imitation; the intensity of imitation increases with the increase in the proportion $1-F_n$ of greater capacities. We accordingly assume that the function φ_0 is decreasing in F_n. If there is no imitation, then technological improvement is possible through innovation. Thus, $\varphi_0(1)$ must be positive.

Summarizing these arguments, we obtain the economic growth equation

$$dM_n/dt = (1-\varphi_0(F_n))\lambda_n M_n + \varphi_0(F_{n-1})\lambda_{n-1}M_{n-1}, \quad (4)$$

where the function φ_0 satisfies the assumptions (i) of Sec. 2, and the boundary and initial conditions have the form

$$M_0(t) \equiv 0, \quad M_n(0) \geq 0, \quad \sum_{n=1}^{\infty} M_n(0) > 0, \quad M_n(0) = 0, \quad n > N \quad (5)$$

where N is a positive integer.

We naturally assume that the efficiency is bounded. Therefore in what follows we use the following conditions.

(a) The sequence λ_n is positive, increasing, and convergent to a limit λ.

Assuming that the series $M = \sum_{k=1}^{\infty} M_k$ can be differentiated term by term, we rewrite (4) in the variables F_n. Note that

$$\frac{1}{F_n}\frac{dF_n}{dt} = \frac{1}{\sum_{k=1}^{n} M_k}\sum_{k=1}^{n}\frac{dM_k}{dt} - \frac{1}{M}\frac{dM}{dt}. \quad (6)$$

Taking $M_n/M \to \infty$ as $n \to \infty$, we obtain from (4)

$$(1/M)\, dM/dt = (1/M)\sum_{k=1}^{\infty} \lambda_k M_k = \sum_{k=1}^{\infty} \lambda_k (F_k - F_{k-1}). \quad (7)$$

Combining this result with (4) and (6) and noting that $M_0(t) \equiv 0$, we obtain

$$dF_n/dt = -\varphi_0(F_n)\lambda_n(F_n - F_{n-1}) - r_n \quad (8)$$

where

$$r_n = -\sum_{k=1}^{n} \lambda_k (F_k - F_{k-1}) + F_n \sum_{k=1}^{\infty} \lambda_k (F_k - F_{k-1}) \quad (9)$$

Apply the Abel transformation,

$$\sum_{k=1}^{n} \lambda_k (F_k - F_{k-1}) = \lambda_n F_n + \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) F_k.$$

Clearly $\lim_{n\to\infty} F_n = 1$. Therefore

$$\sum_{k=1}^{n} \lambda_k (F_k - F_{k-1}) = \sum_{k=n}^{\infty} (\lambda_{k+1} - \lambda_k) + \lambda_n - \sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k) F_k.$$

Substituting these expressions for the sums in (9), we obtain after simple manipulations

$$r_n = (1 - F_n)\sum_{k=1}^{n-1} (\lambda_{k+1} - \lambda_k) F_k + F_n \sum_{k=n}^{\infty} (\lambda_{k+1} - \lambda_k)(1 - F_k). \quad (10)$$

Conditions (5) are obviously equivalent to the following:

$$0 \leq F_1(0) \leq \ldots \leq F_n(0),$$
$$F_0(t) \equiv 0, \quad F_n(0) = 1, \; n > N. \quad (11)$$

The Cauchy problem (8), (11) is the subject of the following analysis. We will first state a proposition which confirms that the problem is well-posed.

Proposition 1. Let conditions (i) and (a) hold. Then the problem (8), (10), (11) is uniquely solvable. Its solution $F = \{F_n\}$ satisfies the relationships

$$0 \le F_1(t) \le \ldots \le 1, \qquad (12)$$

$$F_n(t) \to 1 \text{ as } n \to \infty \text{ for any } t,$$

$$F_n(t) \to 0 \text{ as } t \to \infty \text{ for any } n.$$

The proofs of all the propositions are collected in Sec. 5.

Proposition 2. Under the conditions of Proposition 1, the problem (4), (5) is uniquely solvable on the set of functions $\{M_n(t)\}$ for which the series $\sum_{n=1}^{\infty} M_n(t)$ is convergent and allows term-by-term differentiation.

We can naturally assume that if the proportion of capacities on all low-efficiency levels $\le n$ becomes small, then the corresponding capacities are liquidated and the released resources are channeled to the development of more efficient assets. This process can be considered in the framework of our model by defining the function Υ_0 so that its value at zero is greater than 1. Then, as it follows from Proposition 3, not only the proportions but also the absolute capacities will rapidly decay to zero over time.

Proposition 3. Under the conditions of Proposition 1, $\lim_{t\to\infty} \ell n M_n(t)/t = \lambda$. If $\varphi_0(0) \le 1$, then $\lim_{t\to\infty}(\ell n M_n(t)/t) = \lambda_n(1 - \varphi_0(0))$. If $\varphi_0(0) \le 1$, then $\lim_{t\to\infty} \ell n M_n(t)/t = \lambda_1(1 - \varphi_0(0))$.

We will need yet another assumption that restricts the rate of convergence of λ_n to the limit λ.

(b) The sequence n satisfies the inequality

$$\sum_{k=1}^{\infty} k(\lambda - \lambda_k) < \infty$$

Theorem 2. Let conditions (i), (a), (b) hold and let $F = \{F_n\}$ be the solution of the problem (8), (10), (11). Then there exists a constant d such that

$$\sup_n |F_n(t) - F^*(n - ct - d)| \to 0, \quad t \to \infty, \qquad (13)$$

where F^* is the function introduced in Theorem 1 for $\varphi = \lambda\varphi_0$,

$$c^{-1} = \Phi(0) = \int_0^1 (\varphi(y))^{-1} dy.$$

Theorem 2 asserts that eventually the residual term r_n in (8) can be ignored and λ_n can be replaced with its limit value λ, all this without causing substantial changes in the evolution of the distribution curve.

Hence it follows that the evolution of the distribution curve for a "mature" industry (when the increase of efficiency is slow) is approximately described by an equation of the form (1); this is the asymptotic equation of the trajectories in our model.

The proof of the main theorem, Theorem 2 (see Sec. 5), is fairly complicated and not particularly intuitive. This natural result may appear quite surprising if we look at the residual term r_n: it is by no means clear if this term is small or not. Let us explain why this is so. Let n_0 be a sufficiently large number. Partition each sum in (10) into two summands: the sum of the terms with $k \le n_0$ and the tail with $k \ge n_0$. It can be proved that $F_n(t) \to 0$ as $t \to \infty$. Therefore, for sufficiently large t, the first summands will be close to zero, and smallness of the second summands ensures convergence of λ_n.

The decrease of r_n to zero of course does not necessarily imply that Theorem 2 holds. We additionally must ensure that r_n as a function of time is integrable on the half-line. Condition (b) guarantees integrability, but perhaps it can be relaxed.

4. On Some Modifications of the Model

The growth model described in previous sections is purely theoretical and highly simplified. We hope that it will provide the basis for the construction of more realistic, empirically testable models. This, however, requires considerable additional effort. First, the notion of capacity should be formalized. The most natural definition of capacity is the sum of fixed assets or the sum of fixed assets and working capital. The actual distribution of profit is of course more complicated than the simple scheme assumed in the model, and $\lambda_n M_n$ only represents the part of the profit allocated to capacity formation and renovation. The hypothesis that sectoral growth directly depends on profits is not obvious and requires further verification and, possibly, refinement.

If profitability is used as an efficiency measure, then at a first glance we run into an inconsistency between the modeled and the actual behavior of the growth trajectories in a number of industries with diminishing profitability. Two modifications of the model can be proposed, which are apparently free from this inconsistency. The first modification allows for depreciation, so that the right-hand side of (4) includes a term proportional to M_{n+1} (part of capacities are downgraded from level n+1 to level n). The second modification allows for the effect of the varying ratio between labor and capital assets in the industry. The assets typically increase faster than the number of employed persons, which may be one of the possible reasons for the reduction of profitability. In this case, technological progress partially compensates for the relative "shortage" of labor. A possible generalization of Eq. (1) allowing for this factor has the form

$$dM_n/dt = g(L/M)[(1-\varphi(F_n))\lambda_n M_n + \varphi(F_{n-1})\lambda_{n-1}M_{n-1}] \qquad (14)$$

where $L = L(t)$ is the number of employed persons in the industry (a given function of time) and $g(\cdot)$ is a nondecreasing function of a single variable. Now λ_n is a "conditional profitability," with the ratio L/M remaining fixed, and the "true" profitability is $\rho_n(t) = \lambda_n g(L(t)/M(t))$. Under natural assumptions, an analog of Theorem 2 holds for the trajectories of Eq. (14). The shift of the distribution curve to the right along the λ-axis is not in conflict with its shift to the left along the ρ-axis. Note that the discretization of the efficiency parameter (i.e., the choice of the correspondence $\lambda_n = \lambda(n)$) is also important for ensuring consistency between the model and statistical data.

5. Proofs

We start with a lemma which is frequently used in what follows.

Lemma 1. Let $F = \{F_n(t)\}$ by the solution of the problem (8), (10), (11) given the conditions (i) and (a). Then for any τ, $0 < \tau < 1$, we have the bound

$$\sup_{n \le \tau\nu t} F_n(t) \le (\gamma/\sqrt{\nu t})e^{-\epsilon\nu t}$$

where $\nu = \lambda_1\varphi(1)$, and the positive constants γ and ϵ depend only on τ).

Proof. Consider the functions $\{\check{F}_n(t)\}$ that satisfy the initial conditions (11) and the linear equation of the form

$$d\check{F}_n/dt = -\nu(\check{F}_n - \check{F}_{n-1}). \qquad (15)$$

We will first prove the inequality

$$F_n(t) \le \check{F}_n(t). \qquad (16)$$

To this end, consider the functions $V_n = F_n - \check{F}_n$. By (8), (11), and (15), they satisfy the equalities

$$dV_n/dt = \nu(V_n - V_{n-1}) - \gamma_n, \quad V_n(0) = 0, \qquad (17)$$

where γ_n are some nonnegative functions. Fixing $t > 0$, we define the sequence $W_k(t) = \max_{n \le k} V_n(t)$, where k is an arbitrary natural number. From (17) we directly obtain the inequality $dW_k/dt \le 0$. Therefore, $W_k(t) \le W_k(0) = 0$. From this bound we obtain $V_n(t) \le W_n(t) \le 0$. Hence follows inequality (16).

Let us prove the bounds of the lemma for the functions $\check{F}_n(t)$. The solution $\{\check{F}_n(t)\}$ of the linear problem (11), (15) can be obtained in explicit form as

$$\check{F}_n(t) = e^{-\nu t}\sum_{k=0}^{n-1} (\nu t)^k F_{n-k}(0)(k!)^{-1}.$$

From this representation we obtain the following inequalities form $n \le \tau\nu t$:

$$\check{F}_n(t) \le \exp(-\nu t)\sum_{k=0}^{n-1} (\nu t)^k (k!)^{-1}$$
$$\le \exp(-\nu t)(\nu t)^n (n!)^{-1}(1-(n/\nu t))^{-1}$$
$$\le \exp(-\nu t)(\nu t)^n (n!)^{-1}(1-\tau)^{-1}.$$

Now, using Stirling's formula, we obtain

$$\check{F}_n(t) \le (1-\tau)^{-1} \times \exp(\tau\nu t - \nu t)\exp(-\tau\nu t \ln\tau)(2\pi t)^{1/2}$$
$$\le (1-\tau)^{-1}\exp(\nu t[\tau(1-\ln\tau) - 1]) \times (2\pi\tau\nu t)^{-1/2}$$
$$\le \gamma(\nu t)^{-1/2}\exp(-\epsilon\nu t),$$

where $\gamma = (1-\tau)^{-1}(2\pi\tau)^{-1/2}$; $\epsilon = 1 - \tau(1-\ln\tau)$.

This combined with (16) proves Lemma 1.

Proof of Proposition 1. Take a natural number ℓ and consider an auxiliary finite system of equations

$$dF_n/dt = -\varphi(F_n\lambda_n(F_n-F_{n-1}) - r_n^{(1)}, \quad n = 1,\ldots,\ell, \tag{18}$$

where

$$r_n^{(\ell)} = (1-F_n)\sum_{k=1}^{n-1}(\lambda_{k+1}-\lambda_k)F_k + F_n\sum_{k=n}^{\ell-1}(\lambda_{k+1}-\lambda_k)(1-F_k). \tag{19}$$

As before, take (11) as the initial conditions and assume that $l \geq N$. Let $\{F_n\}^n_{n=1} = F$ be the solution of (11), (18), (19). We will show that $F_n(t) < 1$ for all $t > 0$, $n \leq l$.

Assume that this is not so, then by (11) there is a time $t_0 > 0$ and an index n_0 such that $F_{n0}(t_0) < 1, t < t_0, F_{n0}(t_0) = 1, F_{n0-1}(t) < 1, t \leq t_0$. But by (18), (19) in this case $r^1_{n0}(t_0) \geq 0$, $dF_{n0}(t_0/dt < 0$, which is impossible.

We will now show that $F_n(t)$ is nondecreasing in $f_n = F_n - F_{n-1}$ for any t. Let $f_n = F_n - F_{n-1}$. From (18), (19) we obtain

$$df_n/dt = -\varphi_0(F_n)\lambda_n f_n + \varphi_0(F_{n-1})\lambda_{n-1}f_{n-1} - f_n(\lambda_N - \lambda_n) + f_n\sum_{k=1}^{\ell-1}(\lambda_{k+1}-\lambda_k)F_k, \tag{20}$$

where $f_0(t) \equiv 0$.

If $f_n(t) < 0$ for some t, then seeing that $f_n \geq (0)$, there exists an interval (t_0, t_1) such that $f_n(t_0) = 0$, $f_n(t) < 0$ for $t \in (t_0, t_1)$. Then for some a we have from (20)

$$df_n/dt \geq af_n + \epsilon_n(t), \quad \epsilon_n(t) = \varphi_0(F_{n-1}(t))\lambda_{n-1}f_{n-1}(t).$$

Let $f_n = e^{at}V_n$. From the last inequality we obtain

$$dV_n/dt \geq \epsilon_n(t)e^{-at} \tag{21}$$

Let n = 1. Since $\epsilon_1(t) \equiv 0$, then $dV_1(t)/dt \geq 0$ for all $t \in (t_0, t_1)$; therefore, $f_1(t) \geq 0$. We have thus obtained a contradiction for n = 1. Since $f_1(t) \geq 0$, then $\epsilon_2(t) \geq 0$, and from (21) we obtain $V_2(t) \geq 0$, $f_2(t) \geq 0$, which contradicts the starting assumption for n = 2, and so on. Thus, F_n is nondecreasing in n. Since $F_0(t) \equiv 0$, then $F_n(t)$ is nonnegative for any n, t.

Let $\{F_n^{(\ell)}\} = F^{(\ell)}$ be the solution of (18), (19). In view of the above, the derivatives $dF^{(\ell)}/dt$ are uniformly bounded in ℓ, t. Letting ℓ go to infinity, we find functions $F_n(t)$ that satisfy (8), (11) and the inequalities $0 \leq F_1(t) \leq \ldots 1$.

Fix t and let $\theta(t) = \lim_{n\to\infty} F_n(t)$. From (8), (10) we have

$$d\theta/dt = -(1-\theta)(\sum_{k+1}^{t}(\lambda_{k+1}-\lambda_k)F_k.$$

Viewing F_k as given functions of time, we can consider the last equation as linear in $\theta(t)$ with the initial condition $\theta(0) = 1$. This equation has a unique solution, $\theta(t) = 1$. That $F_n(t) \to 0$ as $t \to \infty$ follows from Lemma 1.

The proof of uniqueness of the solution of the problem (8), (10), (11) is standard. Let $F = \{F_n\}$ and $\bar{F} = \{\bar{F}_n\}$ be two solutions. Then

$$F(t) - \bar{F}(t) = \int_0^t(G(F) - G(\bar{F}))dt,$$

where G(F) is the right-hand side of the system (8). By (i) the function G satisfies the Lipschitz condition with the norm of F defined as $\sup_{n,t} F_n(t)$. Hence for small we have $F(t) = \bar{F}(t)$. Continuing in the same way, we prove uniqueness of the solution on the entire line.

Remark. The Lipschitzian property of the function φ, in addition to its continuity, was needed only in the proof of uniqueness of the solution.

Proof of Proposition 2. By Proposition 1, formulas (9) and (10) are equivalent. From the equations $dM/dt = M\sum_{k=1}^{\infty}\lambda_k(F_k - F_{k-1})$, $M_n = M(F_n - F_{n-1})$, we obtain M and M_n. Since $M_n/M \to 0$, then, as is easily checked, the series $M(t) = \sum_{n=1}^{\infty} M_n(t)$ can be differentiated term by term, and the problem (8), (9), (11) for $\{F_n\}$ is equivalent to the problem (4), (5), for $\{M_n\}$. Proposition 2 therefore follows from Proposition 1.

Proof of Proposition 3. By (4), (6)

$$(1/M)(dM/dt) = (1/M)\sum_{k=1}^{\infty}\lambda_k M_k \leq \lambda.$$

Fix $\epsilon = 0$ and let n be such that $\lambda_n \geq \lambda - \epsilon$. Then

$$(1/M)(dM/dt) \geq (1/M)\sum_{k=n+1}^{\infty}\lambda_k M_k$$

$$\geq (\lambda-\epsilon)(1-\sum_{k=1}^{n} f_k), \qquad f_k = M_k/M.$$

By Lemma 1, there exists $T(\epsilon)$ such that $\sum_{k=1}^{n} f_j \leq \epsilon$ for $t \geq T(\epsilon)$. Thus, $\lambda \geq d\ell nM/dt \geq \lambda-\epsilon(1+\lambda)$ whence we obtain the first assertion of Proposition 3.

In order to prove the second assertion, rewrite the system of the first n equations in (4) in the form

$$dM_k/dt = (1-\varphi(0))\lambda_k M_k + \varphi(0)\lambda_{k-1}M_{k-1} + (\varphi(0)-\varphi(F_k))\lambda_k M_k$$
$$+ (\varphi(F_{k-1})-\varphi(0)\lambda_{k-1}M_{k-1}, \qquad k = 1,\ldots,n,$$

where $\varphi = \varphi_0$.

By Lemma 1, the functions $F_k(t)$ and thus the functions $(\varphi(0)-\varphi(F_k))\lambda_k$ are exponentially decreasing, and therefore the asymptotic behavior of the functions $M_k(t)$ is defined by a system with constant coefficients (Theorem 8.1 [4])

$$d\overline{M}^{(n)}/dt = A^{(n)}\overline{M}^{(n)}, \qquad \overline{M}^{(n)} = (\overline{M}_1,\ldots,\overline{M}_n).$$

The only nonzero elements in the matrix $A^{(n)}$ are the diagonal elements $a_{kk}^{(n)} = (1-\varphi(0))\lambda_k = \mu_k$ and the elements $a_{k(k-1)}^{(n)} = \varphi(0)\lambda_{k-1}$. Clearly, the numbers μ_k are the eigenvalues of $A^{(n)}$, and therefore the last component of the solution $\overline{M}^{(n)}$ is representable in the form $\overline{M}_n = \sum_{k=1}^{n} a_k e^{\mu_k t}$ (for simplicity, we assume that all λ_k are distinct). It is easy to calculate that

$$a_n = \sum_{k=0}^{n-1}(-1)^k \overline{M}_{n-k}(0)\frac{\varphi^k(0)\lambda_{n-1}\cdots\lambda_{n-k}}{(\mu_{n-1}-\mu_n)(\mu_{n-2}-\mu_n)\cdots(\mu_{n-k}-\mu_n)},$$

$$a_1 = \overline{M}_1(0)\frac{\varphi^{n-1}(0)\lambda_1\cdots\lambda_{n-1}}{(\mu_1-\mu_2)(\mu_1-\mu_3)\cdots(\mu_1-\mu_n)}.$$

If $\varphi(0)<1$, then $\mu_k>\mu_{k-1}$, so that $a_n>0$ and $\overline{M}_n a_n^{-1}e^{-\mu_n t} \to 1$. If $\varphi(0)>1$, then $a_1>0$, and therefore $\overline{M}_n a_1^{-1}e^{-\mu_1 t} \to 1$. Since $\ell nM_n(t)/\ell n\,\overline{M}_n(t) \to 1$ as $t\to\infty$, this leads to the second assertion of Proposition 3.

The proof of Theorem 2 relies on a number of lemmas.

Lemma 2. Under the conditions of Lemma 1, the functions $\{r_n(t)\}$ (10) have bounds of the form $r_n(t) = O(t^{-1/2}e^{-\epsilon t\nu})$ if $n \leq \tau\nu t$ and

$$r_n(t) \leq (\lambda-\lambda_n) + ((\lambda-\lambda_{[\tau\nu t]}) + O(t^{-1/2}e^{-\epsilon t\nu}))(1-F_n),$$

if $n \geq \tau\nu t$, where $0<\tau<1$ and ν is the constant from Lemma 1.

If $x \to 0$ or $x \to \infty$, then $O(x)$ will denote a function such that $O(x)x^{-1} \leq$ const for sufficiently small x and sufficiently large x, respectively. Denote by [x] the whole part of x.

Proof. The bound of $r_n(t)$ for $n \leq \tau\nu t$ follows from Lemma 1 and the inequality $r_n = O(F_n)$, which is a direct consequence of (9) and the fact that F_k is nondecreasing in k.

If $n \geq \tau\nu t$, then from (9) we obtain the inequality

$$r_n(t) \leq (1-F_n)\sum_{k=1}^{[\tau\nu t]}(\lambda_{k+1}-\lambda_k)F_k + (1-F_n)(\lambda-\lambda_{[\tau\nu t]}) + (\lambda-\lambda_n).$$

Hence, applying Lemma 1, we obtain the sought bound for $r_n(t)$ for $n \geq \tau\nu t$.

Lemma 3. Let $\{F_n(t)\}$ be the solution of the problem (8), (10), (11) given the conditions (i), (a), and the additional condition $\sum_{k=1}^{\infty}(\lambda-\lambda_k) < \infty$. Then the function $s(t) = \sum_{n=1}^{\infty} s_n(t)$, where $s_n(t) = (\lambda-\lambda_n)\varphi_0(F_n)(F_n-F_{n-1})$, is integrable over t, i.e., $\int_0^\infty s(t)dt < \infty$.

Proof. By Lemma 1, we have the inequality

$$\sum_{n=1}^{[\tau\nu t]} s_n(t) \leq (\lambda-\lambda_1)\varphi_0(0)\gamma\,\exp(-\epsilon\nu t)(\nu t)^{-1/2},$$

whence we directly obtain

$$\int_0^\infty\left(\sum_{n=1}^{[\tau\nu t]} s_n(t)\right)dt < \infty. \tag{22}$$

Now, by monotonicity of the sequence $(\lambda-\lambda_n)$ and the inequality

$$\sum_{n=[\tau\nu t]+1}^{\infty}(F_n-F_{n-1}) \leq 1,$$

we obtain

$$\sum_{[n=\tau\nu t]+1}^{\infty} s_n \leq \varphi_0(0)(\lambda-\lambda_{[\tau\nu t]+1}).$$

This combined with the condition of the lemma gives

$$\int_0^\infty \Big(\sum_{[n=\tau\nu t]+1}^{\infty} s_n \Big) dt \le \frac{\varphi_0(0)}{\tau\nu} \sum_{k=1}^{\infty} (\lambda - \lambda_k) < \infty. \tag{23}$$

Lemma 3 now follows from (22) and (23).

Lemma 4. Let $\{F_n(t)\}$ be the solution of problem (8), (11) given conditions (i) and (a), (b). Then the function $A(t) = \sum_{n=1}^{\infty}(1-F_n(t))$ has a bound of the form $A(t) = O(t)$, $t \to \infty$.

Proof. Let $U_n = 1-F_n$. From (8) we obtain the inequality

$$dU_n/dt = \varphi_0(F_n)\lambda_n(U_{n-1} - U_n) + r_n \le \text{const}(U_{n-1} - U_n) + r_n,$$

whence

$$dA(t)/dt \le \text{const} + \sum_{n=1}^{\infty} r^n. \tag{24}$$

Now, from the bounds of Lemma 2 and the condition $\Sigma(\lambda-\lambda_n) < \infty$,

$$\sum_{n=1}^{\infty} r_n \le \text{const} + A(t)\Big[(\lambda - \lambda_{[\tau\nu t]}) + O(t^{-1/2}\exp(-\epsilon t\nu))\Big]. \tag{25}$$

Condition (b) and monotonicity of the sequence $(\lambda-\lambda_n)$ lead to the equality[3]

$$\lim_{n\to\infty} (\lambda-\lambda_n)n^2 = 0.$$

This combined with (24) and (25) produces the bound

$$dA(t)/dt \le \text{const} + A(t)t^{-2}. \tag{26}$$

In order to obtain the bound $A(t) = O(t)$ from (26), we make the substitution $A(t) = v\exp(-1/t)$. Now (26) takes the form $\exp(-1/t)(dv/dt) \le \text{const}$. Hence

$$v \le v(0) + (\text{const})t.$$

Thus, $A(t) = O(t)$. Q.E.D.

Lemma 5. Under the conditions of Lemma 4, the function $r(t) = \sum_{n=1}^{\infty} r_n(t)$ is integrable, i.e., $\int_0^\infty r(t)dt < \infty$.

Proof. Using Lemmas 2 and 4, we obtain

$$r(t) = \sum_{n=1}^{[\tau\nu t]} r_n + \sum_{[\tau\nu t]+1}^{\infty} r_n = O(t^{1/2}e^{-\epsilon\nu t}) + O(t)(\lambda - \lambda_{[\tau\nu t]}) + \sum_{n=[\tau\nu t]+1}^{\infty} (\lambda - \lambda_n). \tag{27}$$

Integrability over t of the second and third terms in the right-hand side of (27) is a direct consequence of condition (b). Indeed, let

$$\Lambda(t) = \sum_{[\tau\nu t]+1}^{\infty} (\lambda - \lambda_n).$$

Then

$$\int_0^\infty \Lambda(t)dt = \lim_{m\to\infty} (1/\tau\nu) \sum_{n=1}^{m} \sum_{k=1}^{\infty} (\lambda - \lambda_k).$$

Using the Abel transformation, we obtain

$$\sum_{n=1}^{m} \sum_{k=n}^{\infty} (\lambda - \lambda_k) = m\sum_{k=m}^{\infty} (\lambda - \lambda_k) + \sum_{k=1}^{m-1} k(\lambda_{k+1} - \lambda).$$

By (b), the first term tends to zero as $m \to \infty$, and the second tends to a finite limit. Thus, $\int_0^\infty \Lambda(t)dt < \infty$. Q.E.D.

We need another lemma, which is similar to the maximum principle for parabolic equations. Its proof is given in [3] and is not repeated here.

Lemma 6. Let the functions $\Omega_n(t)$ be defined for $n > 0$, $t > 0$, and

$$|\Omega_n| + |d\Omega_n/dt| \le \gamma(t)$$

for all n, t. Also assume that the following conditions hold:

a) $d\Omega_n/dt \le \theta(t)(\Omega_{n-1} - \Omega_n)$, where $0 > a_1 > b_1$;

b) $\Omega_n(0) \le 0$, for $n \ge 0$;

c) $\lim_{n\to\infty} \Omega_n \le 0$, $\Omega_0(t) \le 0$ for all t.

Then $\Omega_n(t) \le 0$ for all n, t.

Let us now proceed directly to the

Proof of Theorem 2. By Lemmas 3 and 5, we can choose $T = T(\epsilon) > 0$ so that

$$(1/\lambda\varphi_0(1)) \int_T^\infty \Big(\sum_{k=1}^\infty |\rho_k(t)| \Big) dt < \epsilon, \ \rho_k = s_k - r_k, \tag{28}$$

where $s_k = (\lambda-\lambda_k)\varphi_0(F_k)(F_k - F_{k-1})$, and r_k is defined by the equality (10).

Consider the system of equations

$$dF_n/dt = \begin{cases} \lambda\varphi_0(F_n)(F_{n-1} - F_n) + \rho_n & \text{if } t \le T(\epsilon), \\ \lambda\varphi_0(F_n)(F_{n-1} - F_n), & \text{if } t \ge T(\epsilon). \end{cases} \tag{29}$$

Let $\{F_n^{(T)}\}$ be the solution of equation (29), and $\{F_n\}$ the solution of (7) for the same initial conditions (11). Let

$$\Delta_n = \sum_{k=1}^m (\Phi(F_k) - \Phi(F_k^{(T)})),$$

where

$$\Phi(F) = (1/\lambda) \int_F^1 df/(\varphi_0(f)).$$

By (29) for $t \ge T(\epsilon)$ we have the equality

$$d\Delta_n/dt = \sum_{k=1}^n \Big((d\Phi(F_k)/dt - (d\Phi(F_k^{(T)})/dt \Big)$$

$$= \sum_{k=1}^n \Big[(F_k - F_{k-1}) - (F_k^{(T)} - F_{k-1}^{(T)}) - (\rho_k/\lambda\varphi_0) \Big] \tag{30}$$

$$= F_n - F_n^{(T)} - \sum_{k=1}^n (\rho_k/\lambda\varphi_0).$$

Since $\Delta_n - \Delta_{n-1} = \Phi(F_n) - \Phi(F_n^{(T)})$, we also have the relationship

$$F_n = \Phi^{(-1)}(\Delta_n - \Delta_{n-1} + \Phi(F_n^{(T)})), \tag{31}$$

where Φ^{-1} is the inverse of the function Φ,.

From (30) and (31) we obtain the equality

$$d\Delta_n/dt = \Phi^{(-1)}(\Delta_n, \Delta_{n-1} + \Phi(F_n^{(T)})) - F_n^{(T)} - \sum_{k=1}^n \rho_k/(\lambda\varphi_0(F_k)). \tag{32}$$

The last relationship can be rewritten in the form

$$d\Delta_n/dt = \theta(\Delta_n, \Delta_{n-1}, F_n^{(T)})(\Delta_{n-1} - \Delta_n) - \sum_{k=1}^n \rho_k/(\lambda\varphi_0(F_k)), \tag{33}$$

where $-\theta$ is the derivative of the function $\Phi^{(-1)}$, at some point that depends on the values of $\Delta_n, \Delta_{n-1}, F_n^{(T)}$, $0 < a_1 \le \theta \le b_1$. Let

$$\Omega_n(t) = \Delta_n(t) - (1/a_1) \int_T^t \Big(\sum_{k=1}^\infty |\rho_k(t)| \Big) dt,$$

$$a_1 = \varphi_0(1)\lambda, \quad b_1 = \lambda\varphi_0(0).$$

From (33) we obtain for $t \ge T = T(\epsilon)$

$$d\Omega_n/dt = \theta(\Omega_{n-1} - \Omega_n) - \omega_n, \tag{34}$$

where

$$\omega_n(t) = (1/a_1) \sum_{k=1}^\infty |\rho_k(t)| - \sum_{k=1}^n \rho_k/(\lambda\varphi_0(F_k)) \ge 0. \tag{35}$$

Apply Lemma 6 to Eq. (34), taking $T(\epsilon)$ as the initial time instead of zero. Since $F_0 = F_0^{(T)} \equiv 0$, we have $\Omega_0 \le 0$.

From (8) we have

$$d\Phi(F_n)/dt = F_n - F_{n-1} - \rho_n/\lambda\varphi_0(F_k),$$

whence

$$\sum_{n-1}^m \Phi(F_n(t)) = \sum_{n=1}^m \Phi(F_n(T)) + \int_T^t F_m(\tau) d\tau \tag{36}$$

$$- \int_T^t \Big(\sum_{n=1}^m \rho_n/\lambda\varphi_0(F_n) \Big) dt,$$

where $T = T(\epsilon)$.

Therefore, by Proposition 1,

$$\sum_{n=1}^{\infty} \Phi(F_n(t)) = \sum_{n=1}^{\infty} \Phi(F_n(T)) + (t-T) - \int_T^t \Big(\sum_{n=1}^{\infty} \rho_n/(\lambda\varphi_0(F_n)\Big) dt.$$

Moreover, by (29) and conservation of the first integral, we obtain

$$B_F(t) = \sum_{n=1}^{\infty} \Phi(F_n^{(T)}(t)) - t = \sum_{n=1}^{\infty} \Phi(F_n(T)) - T = B_F(T). \qquad (37)$$

From the last two equalities it follows that

$$\lim_{n\to\infty} \Omega_n(t) = -\int_T^t \Big(\sum_{k=1}^{\infty} \rho_k/(\lambda\varphi_0(F_k)\Big) dt - (1/a_1)\int_T^t \Big(\sum_{k=1}^{\infty} |\rho_k(t)|\Big) dt \leq 0.$$

Thus, condition c) of Lemma 6 is satisfied. Condition b) is obvious, because $F_n(T) = F_n^{(T)}(T)$. Condition a) follows from (34) and (35). From (28), (36), and (37) it follows that $|\Omega_n(t)|$ can be bounded by a linear function of time. The absolute value of the right-hand side of (34) is also bounded by a linear function of Ω_n, Ω_{n-1}. From Lemma 6 we obtain that $\Omega_n(t) \leq 0$ for all $t > T$, i.e.,

$$\Delta_n(t) \leq (1/a_1) \int_T^t \sum_{k=1}^{\infty} |\rho_k(t)| \leq \epsilon.$$

Similarly, applying Lemma 6 to

$$\Omega_n(T) = -\Delta_n(t) - (1/a_1) \int_T^t (\sum_{k=1}^{\infty} |\rho_k(t)|)dt,$$

we can show that $\Delta_n(t) \geq -\epsilon$. Combining these bounds with (31), we obtain

$$|F_n(t) - F_n^{(T)}(t)| \leq 2b_1\epsilon_1, \quad t \geq T(\epsilon), \quad b_1 = \varphi_0(0)\lambda. \qquad (38)$$

By Theorem 1, there exist constants $a(\epsilon)$ and $T_1(\epsilon)$ such that for all n

$$|F_n^{(T)}(t) - F^*(n - ct - a(\epsilon))| \leq \epsilon, \quad t \geq T_1(\epsilon), \qquad (39)$$

where F^* is the solution of the equation

$$\Big(\int_0^1 dy/\lambda\varphi_0(y)\Big)^{-1} dF/dx = \lambda\varphi_0(F(x))(F(x) - F(x-1)).$$

Hence it follows, in particular, that for $\epsilon_1 > \epsilon_2 > 0$,

$$|F^*(n - ct - a(\epsilon_1)) - F^*(n - ct - a(\epsilon_2))| \leq \epsilon_1 + \epsilon_2.$$

for $t \geq T_1(\epsilon_2)$ and all n.

Let x^0 be an arbitrary point where $(dF^*(x^0))/dx \neq 0$. Take n and t so that $n - ct = a(\epsilon_1) + x_0$. Then

$$|F^*(x^0) - F^*(x^0 + a\epsilon_1) - a(\epsilon_2))| \leq \epsilon_1 + \epsilon_2.$$

Hence for small $\epsilon_1 + \epsilon_2$ we have

$$|a(\epsilon_1) - a(\epsilon_2)| \leq 2(\epsilon_1 + e_2)/dF^*(x^0)/dx,$$

and so $a(\epsilon) \to a(0)$ as $\epsilon \to 0$. Combining this result with (38), (39), we obtain that

$$\sup_n |F_n(t) - F^*(n - ct - a(0))| \longrightarrow 0, \quad t \to \infty.$$

This completes the proof of Theorem 2. Q.E.D.

Notes

1. Since F_n is large, we may ignore the fact that it is discrete; a similar assumption is made in [1-3].
2. This is done in order to simplify the presentation. If γ_n is the retention norm on level n, then all the results of this paper remain valid under appropriate conditions for the sequence $\lambda_n\gamma_n$. In particular, we may take $\gamma_n = \gamma > 0$.
3. It suffices to note that if $\xi_r \geq 1/r^2$ for some r for nonincreasing positive ξ_k, then $\sum_{k=r}^{2r} k\xi_k \geq 1/4$.

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