RATIONING AND MARKET: STRUCTURE AND STABILITY OF EQUILIBRIA IN MIXED ECONOMY

F. L. Zak

Central Economics Mathematical Institute
of the Russian Academy of Sciences

Abstract. We consider an equilibrium model of allocation of resources in mixed economy. In the model there are l commodities and m economic agents characterized by their demand functions $f_i(p, w_i)$, $i = 1, ..., m$, where $p \in \mathbb{R}^l_+$ is a price vector and w_i is the income of the i-th participant. The i-th economic agent has initial income α_i, and the state offers him a vector of goods $x_i \in \overline{\mathbb{R}^l_+}$ at a fixed price $q \in \overline{\mathbb{R}^l_+}$. The remaining resources (i. e. those that are not bought at fixed prices) are sold at the free market at a price $p \in \mathbb{R}^l_+$. At this free market the participants can also resell some of the products previously bought from the state at the fixed price q. It is easy to see that equilibrium is defined by the system of l equations $\sum_{i=1}^m f_i(p, \alpha_i + \langle (p-q)^+, x_i\rangle) = y$, where y is the vector of total resources and for $r \in \mathbb{R}^l$ we set $(r^+)^j = \max\{r^j, 0\}$, $j = 1, \ldots, l$.

Considering the quotas x_i^j and prices q^j as control parameters using which the state pursues its economic policy, we study the existence, structure and stability of equilibria in various economies. In particular, there are many economies for which the number of equilibrium prices is odd (which is usual), but there are also many economies for which this number is even (which is quite unusual). Moreover, there is an important class of economies defined by the equations $\sum_{i=1}^m x_i = y$, $\alpha = \sum_{i=1}^m \alpha_i = \langle q, y\rangle$ for which the equilibrium prices form a one-dimensional manifold (ray). These economies can also be described in a different way showing the pertinence of our model for the study of centralized planning. Let y_i be the initial endowment of the i-th economic agent. The state makes him supply a certain amount of goods y_{ik} to the k-th economic agent at a fixed price $q \in \bar{\mathbb{R}}^l_+$. After that, the participants trade both the initial endowments and the newly acquired goods at a free market price p. This model is a special case of the above one with $\alpha = \langle q, y\rangle$, $x_i = y_i + \Delta y_i$, $\Delta y_i = \sum_{k=1}^m (y_{ki} - y_{ik})$, and so the equilibrium prices usually form a one-dimensional manifold. We study the structure of this manifold and show that in certain cases the one-dimensionality of the set of equilibrium prices can be used to explain the phenomenon of "endogenous inflation" in certain price adjustment processes generalizing the Walrasian tâtonnement process.

It is well known that elements of market exist even in the most centralized economies (at least in the hidden form of black market). Coexistence of rationing and market is an important feature of emerging markets undergoing transition from allocation of commodities at fixed prices within prescribed quotas to free trade at competitive prices. At the same time, elements of rationing and two-level price system now play a significant role in traditionally competitive economies. For example, delivery of many commodities, such as raw materials, is largely regulated by long-term contracts specifying quotas and prices. As conditions change, economic agents may resell the commodities at a free market price. Thus it seems important to study essential features of these mixed economies on a model level. A general setting for such a model was proposed by V. Makarov, V. Vasil'ev, A. Kozyrev and V. Marakulin (cf. [1] and [2]). In the present note (based on [6] and [7]) we consider a special case of this base model in which there is no production. Under this assumption, we study the effect of rationing on existence and properties of equilibria and the behavior of price adjustment processes. Our techniques coming from differential topology was first introduced into mathematical economics by Debreu and used by Dierker, Balasko, Mas-Colell et al (cf. [3], [4] and [5] for an approach close to the one we follow in these notes). I am grateful to V. M. Polterovich for useful discussions.

In our model there are m economic agents characterized by their demand functions $f_i(p, w_i)$, $i = 1, ..., m$, where $p \in \mathbb{R}^l_+$ is a price vector and w_i is the total income of the i-th agent. We assume that the demand functions f_i are generated by strictly monotonous strictly concave three times continuously differentiable utility functions u_i (in this case by Debreu's theorem the demand functions are twice continuously differentiable) and that the aggregate demand function $f = \sum_{i=1}^{m} f_i$ satisfies the following desirability condition:

$$\text{(Des)}\qquad \begin{gathered} \textit{if } p_n \to \bar{p},\ p_n \in \mathbb{R}^l_+,\ \bar{p}^j = 0\ \exists j, 1 \le j \le l,\ \underline{\lim}\, w_n > 0, \\ w_n = \sum_{i=0}^{m} (w_i)_n,\ \text{then}\ \|f(p_n; (w_1)_n, ..., (w_m)_n)\| \to \infty \end{gathered}$$

(cf. [3]).

The i-th participant has initial income $\alpha_i \in \overline{\mathbb{R}^l_+}$, and the state offers him a vector of goods $x_i \in \overline{\mathbb{R}^l_+}$ at a fixed price $q \in \overline{\mathbb{R}^l_+}$. This means that, if the i-th economic agent has the wish and money, he is permitted to buy any amount of product j not exceeding x_i^j at the price q^j, $j = 1, \ldots, l$. We remark that we do not exclude the cases when $x_i^j = 0$ for certain i and j and $q^j = 0$ for certain j (such "commodities" as housing, health services etc whose cost — up to a certain level — is covered by the state may belong to this last category).

Let $y \in \mathbb{R}^l_+$ be the vector of total resources (output vector) which is distributed between the participants. We emphasize that we do not a priori assume that $x \leqq y$. The reason is that in practice rationing often precedes actual production of goods, and if production plans fail to be fulfilled the state cannot carry out its obligations to deliver certain commodities (which means that $x^j > y^j$ for some $1 \le j \le l$). The remaining resources (i.e. those that have not been bought at fixed prices) are sold at the free market at a price $p \in \mathbb{R}^l_+$. At this free market the participants can also resell (at the same price p) some of the commodities previously bought from the state at price q.

It is clear that the economic agents would buy out their quotas of commodity j if and only if the market price of this commodity is higher than the fixed price, i.e. $p^j > q^j$. It is easy to show that to study the resulting distribution of goods (at an equilibrium) it suffices to assume that the i-th participant enters the market with income $w_i = \alpha_i + \langle (p-q)^+, x_i \rangle$, $i = 1, \ldots, m$ (where, for $r \in \mathbb{R}^l$, $(r^+)^j = \max\{r^j, 0\}$, $j = 1, \ldots, l$) composed of the initial income α_i and the money obtained for reselling the commodities bought at fixed prices provided that these prices are lower than the market ones. The corresponding aggregate demand is

$f(p; w_1, ..., w_m) = \sum_{i=1}^{m} f_i(p, \alpha_i + \langle (p-q)^+, x_i \rangle)$, and p is called an *equilibrium price* if demand is equal to supply, i.e.

$$f(p; w_1, ..., w_m) \sum_{i=1}^{m} f_i(p, \alpha_i + \langle (p-q)^+, x_i \rangle) = y.$$

We remark that here we do not discuss details of short-term loans allowing the participants to acquire their quotas even if they do not have enough money; the only thing that matters is that at equilibrium the agents' liabilities are cleared out.

The above model includes as special cases the pure exchange model (when $x = y$, $\alpha_i = \langle q, x_i \rangle$, $i = 1, \ldots, m$) and the fixed incomes model (e.g. when $x_i = 0$, $i = 1, \ldots, m$ or $x \leqq y$ and the prices q^j are sufficiently high, $j = 1, \ldots, l$) and is obtained by "combining" these two models. In particular, our results allow to establish some new properties of equilibria in the fixed incomes model (e.g. the fact that the number of equilibria is generically odd; cf. [6, Corollary 4]). However it seems more important to find properties of equilibria specific for the above model.

The first problem arising in each equilibrium model is existence of equilibrium. In contrast with the fixed incomes and pure exchange models, in our model equilibrium does not necessarily exist. In fact, from the Walras law it follows that $\alpha = \langle p \wedge q, y \rangle + \langle (p-q)^+, y-x \rangle$, where $\alpha = \sum_{i=1}^{m} \alpha_i$ is the aggregate income and $(p \wedge q)^j = \min\{p^j, q^j\}$, $j = 1, \ldots, l$, so that if e.g. $y \leqq x$, $\alpha > \langle q, x \rangle$, then there is no equilibrium.

To solve the existence problem, one needs to consider the hypersurface Ω_0 in the parameter space $\Omega = \{(x^j; y^j; \alpha_i; q^j)\}$, $i = 1, \ldots, m$; $j = 1, \ldots, l$ swept out by the economies for which $y = \sum_{i=1}^{m} f_i(p, \langle p, x_i \rangle)$ when p runs through the unit simplex. The economic meaning of Ω_0 is that this hypersurface parametrizes the possible aggregate demand when the agents are endowed with their quotas (the incomes α_i and the prices q^j are not involved in the definition of Ω_0; we observe that from the Arrow-Debreu theorem it follows that Ω_0 contains all points for which $x = y$). The hypersurface Ω_0 divides Ω into several connected domains one of which denoted by Ω_- contains the orthant $x \leq y$ and the other one denoted by Ω_+ contains the orthant $x \geq y$. Moreover, in a neighborhood of an economy ω for which $x = y$, $\alpha_i = \langle q, x_i \rangle$, $i = 1, \ldots, m$ (i. e. all production is rationed and the quotas are set with regard to the incomes) and the distribution x_i is Pareto optimal (such points are particularly important from the economic viewpoint) Ω_0 divides Ω into exactly two domains Ω_- and Ω_+. We show that in each of these domains the number of equilibria is almost everywhere constant modulo two. In particular, for a general economy $\omega \in \Omega_-$ the number of equilibria is odd, and for a general economy $\omega \in \Omega_+$ the number of equilibria is even. From this it follows that for all $\omega \in \Omega_-$ (and thus for all ω for which $x \leq y$)) there exists an equilibrium. By the above, this implies existence of equilibrium in fixed income economies; furthermore, for a general fixed income economy the number of equilibria is odd (a result for which I was unable to find reference elsewhere).

In the other domains the situation is more complicated, and one can only give some sufficient conditions for the existence of equilibrium (such conditions for some economies from Ω_+ are given in [6, Theorem 7]). It is particularly interesting to study the existence problem for $\omega \in \Omega_0$; here the answer depends on the prices

q^j and incomes α_i. One could argue that the hypersurface Ω_0 has measure zero, and one may assume that the case $\omega \in \Omega_0$ is not actually observable. However, if we assume that all the resources y are offered to the participants at fixed prices (within the limits of individual quotas), then we see that the case when $y = x$ (full rationing) is worth special consideration. In this case we show that equilibrium exists if and only if $\alpha \leq \langle q, x\rangle$. Moreover, if $\alpha < \langle q, x\rangle$, then almost always there exists a finite (odd) number of equilibria, and if $\alpha = \langle q, x\rangle$ (i.e. the prices q^j are set so that all money is spent to buy out the quotas), then the set of equilibrium prices is isomorphic to a union of an odd number of rays (along which the prices of all commodities tend to infinity) and a finite number of arcs. As we already mentioned, a special case of these economies is given by pure exchange economies. By homogeneity, in this case our results imply the existence of equilibrium in pure exchange economies and show that for a general economy ω of this type the set of equilibrium prices has the form $\mathbb{R}_+ \cdot \mathrm{P}^n_\omega$, where P^n_ω is the set of normalized prices consisting of an odd number of elements (this is a classical result of Debreu and Dierker).

In the study of the above model it is natural to consider the quotas x_i^j and the prices q^j as control parameters using which the state pursues its economic policy (of course, in more general models one should also take into consideration the influence of the state on the vector of resources y and the incomes α_i). But what are the state's goals and what are the advantages of controlling the quotas x_i^j in comparison with controlling the financial parameters α_i and q^j? To answer this question, one should first decide which equilibrium states are "good". The most common unpleasant phenomena in equilibrium theory are multiple and unstable equilibria (in our model all equilibrium states of an economy can be stable only if there exists a unique equilibrium). Thus it is natural to require that, varying control parameters, one should be able to transfer the economy into a state for which there exists a unique equilibrium price vector stable with respect to the tâtonnement process. Since our model is obtained by "combining" the fixed incomes and pure exchange models and since for general demand functions there is no reason to expect uniqueness in the fixed incomes situation while in the pure exchange setting there always exists a unique stable equilibrium in a neighborhood of the Pareto manifold (cf. [3]), the following result cf. [6, Theorems 4 and 5] seems quite natural.

Let $\omega \in \Omega_-$ be an economy close to an economy ω_0 for which $\{x_i(\omega_0)\}$ is a Pareto optimal distribution, $y(\omega_0) = x(\omega_0)$, $\alpha_i(\omega_0) = \langle q(\omega_0), x_i(\omega_0)\rangle$, $i = 1, \ldots, m$. Then, under sufficiently general assumptions, in ω there is a unique equilibrium, and this equilibrium is stable with respect to the tâtonnement process.

This result is similar to Balasko's result cited above, but while in Balasko's model the quotas x_i are simply given to the agents, we offer the economic agents to buy out their quotas at fixed prices q (if they wish to do so). We remark that the above result is not obvious since in the economy ω_0 itself and in some neighboring economies from Ω_0 there are infinitely many equilibrium prices (the set of such prices is diffeomorphic to a ray; cf. [6, Theorem 1]). Moreover, the equilibrium prices in ω are not bounded: for suitable incomes the prices tend to infinity when $\omega \to \omega_0$ (if $\|y - x\| = o(\alpha - \langle q, x\rangle)$, then the unboundedness of prices is clear; this is a case of inflation caused by high incomes).

We remark that the requirements for the amount of information about the economic agents in the above result seem natural — we assume that the state has a rough idea of the preferences and incomes of the economic agents. If the state does not have such information, then it is unrealistic to expect equilibrium to be unique; on the other hand, it is also unrealistic to assume that the state has complete knowledge.

We also consider economies ω from Ω_+ which are close to ω_0 and show that if the incomes of the participants are not too high, then in ω there are exactly *two* equilibrium price vectors; one of them is "small" and stable and the other one is "large" and unstable (cf. [6, Theorem 7]).

In [6] we also study some other problems. For example, in Theorem 6 we consider the problem of uniqueness and stability of equilibrium in the case of gross substitutability.

Up to now we concentrated mainly on more or less general economies $\omega \in \Omega$. However the special case when $y = x$, $\alpha = \langle q, x\rangle$ deserves special attention. To see that, we give a different interpretation of this case which shows that it reflects some important features of the system of mandatory reciprocal commodity deliveries at fixed prices organized by the state.

Let y_i be the initial endowment of the i-th economic agent. The state makes him to supply a certain amount of goods y_{ik} to the k-th economic agent, $k = 1, ..., m$ at fixed prices $q = (q^1, \dots, q^l)$ (provided that the k-th agent is willing to buy those goods at that price). After that, the participants trade both the initial endowments and the newly acquired goods at free market prices $p = (p^1, ..., p^l)$. It is easy to see that this model is basically equivalent to the previous one with $\alpha_i = \langle q, y_i\rangle$, $x_i = y_i + \Delta y_i$, $\Delta y_i = \sum\limits_{k=1}^{m} (y_{ki} - y_{ik})$.

As we have already mentioned, the set of equilibrium prices $p \geqq q$ for a general economy is a union of a finite number of rays along which the prices of all commodities tend to infinity and a finite number of arcs. If the distribution $\{x_i\}$ is close to a Pareto optimal distribution, then such a ray is unique, and if in addition the costs of net supplies at fixed prices $|\langle q, \Delta y_i\rangle|$, $i = 1, \dots, m$ are small, then there are no arcs. We remark that while it seems natural for the state to be willing to place an economy near a Pareto optimum, the actual equilibrium distribution (which is clearly Pareto optimal) varies along the ray whose existence is granted by the above theorem and the resulting distribution may deviate from the "planned" distribution $\{x_i\}$. However when the prices grow and tend to infinity the corresponding equilibrium tends to the pure exchange equilibrium with initial endowments $\{x_i\}$. In this situation some of the participants are interested in raising prices while the others are interested in lowering them.

In [7, Theorem 2] we show that the tâtonnement process in our model is *locally stable* provided that the distribution $\{x_i\}$ is close to a Pareto optimum and either all $|\langle q, \Delta y_i\rangle|$ are small or the initial price vector $p(0)$ is large, and is *globally stable* if the excess demand satisfies the condition of gross substitutability. However while the standard tâtonnement process guarantees against inflation since $\dfrac{d\langle p(t), p(t)\rangle}{dt} = 2\left\langle \dfrac{dp(t)}{dt}, p\right\rangle = 0$ by the Walras law, the above results show that in many cases one may hope for a good behavior of prices only if the prices are high enough. And while in the pure exchange model it is natural to assume that the state price-setting agency is impartial, in our situation the state pursues an active distribution policy and is unlikely to be objective. It is more natural to assume that the state counts the individual excess demands with certain weights, which leads to a system of differential equations

$$\frac{dp}{dt} = E_\lambda(p) = \sum_{i=1}^{m} \lambda_i E_i(p),$$

where $\lambda_i > 0$ and $E_i(p) = f_i(p, \langle p, x_i\rangle - \langle q, \Delta y_i\rangle) - x_i(p)$, $p \geqq q$ is the excess demand function of the i-th participant upon implementation of the planned deliveries, $i = 1, \dots, m$. It seems reasonable to set weights depending on the balance of mutual deliveries. For example, the weight of a "net-consumer" ($\langle q, \Delta y_i\rangle > 0$) should probably be lower than that of a "net-supplier" ($\langle q, \Delta y_i\rangle < 0$) since the the second one contributes more to realization of the state's goal s. More generally, it is natural to assume that weight is a decreasing function of $\langle q, \Delta y\rangle$ so that if there exists a participant for whom $\langle q, \Delta y_i\rangle \neq 0$ and

$$\langle q, \Delta y_{i_1}\rangle \le \langle q, \Delta y_{i_2}\rangle \le \dots \le \langle q, \Delta y_{i_k}\rangle \le 0 < \langle q, \Delta y_{i_{k+1}}\rangle \le \dots \le \langle q, \Delta y_{i_m}\rangle,$$

then $\lambda_{i_1} \ge \lambda_{i_2} \ge \dots \ge \lambda_{i_m} > 0$ and $\lambda_{i_a} \neq \lambda_{i_b}$ if $\langle q, \Delta y_{i_a}\rangle \neq \langle q, \Delta y_{i_b}\rangle$.

We observe that under the above conditions the function $E_\lambda(p)$ does not satisfy the Walras law and the trajectories of our price adjustment process satisfy the inequality $\dfrac{d\langle p(t), p(t)\rangle}{dt} > 0$, so that the prices grow along the trajectories. In general the linear system $\dfrac{dp}{dt} = E_\lambda(p)$ does not have stable solutions in the set of equilibria of our model, but if $\{x_i\}$ is close to a Pareto optimum and p is a sufficiently large equilibrium price, then $\|E_\lambda(p)\|$ is small. For example, in [7, Theorem 3] we show that if all participants have normal demands satisfying the condition of gross substitutability, then the above price adjustment process is globally stable, the length of the vector $p(t)$ grows indefinitely, and the normalized price vector $\dfrac{p(t)}{\|p(t)\|}$ converges to a solution of the system of equations $\sum_{i=1}^{m} \lambda_i[f_i(p, \langle p, x_i\rangle) - x_i] = 0$. If $\{x_i\}$ is close to a Pareto optimum or all weights are close to 1, then the limit normalized price vector is close to the equilibrium price vector in the pure exchange model with initial endowments $\{x_i\}$ which in its turn is the limit of normalized price vectors in our model.

We remark that, since the demand functions are homogeneous, to formulate our results it is more natural to use the notion of *projective space* of prices whose hyperplane at the infinity parametrizes the equilibrium prices in pure exchange models.

The above theorem can be generalized in various directions, but the heart of the problem is not in concrete statements. We motivated the introduction of weights by partiality of the state with respect to participants in the process of distribution of commodities. However weights are relevant even in the study of price adjustment processes in pure exchange economies. In fact, both deliberate and accidental errors arise in the process of evaluating and summing up the individual excess demand functions, and a simple way to take them into account is to ascribe higher weights to more reliable data. In the pure exchange model such errors do not lead to inflation since the individual excess demand functions satisfy the Walras law and $d\dfrac{\|p(t)\|^2}{dt} = 2\langle p, E_\lambda(p)\rangle = 0$. However in our model, if the weights are chosen reasonably, one has "endogenous inflation" related to the price adjustment process whose rate depends on the weights and the costs of net deliveries (in rigid prices). The above result shows that, under suitable conditions, this type of inflation may play a positive role by helping to balance the market (at least approximately).

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CEMI RAS, Nakhimovskiĭ av. 47, Moscow 117418, Russia
E-mail address: zak@cemi.rssi.ru