LINEAR SYSTEMS OF HYPERPLANE SECTIONS ON VARIETIES OF LOW CODIMENSION

F. L. Zak

UDC 512.763

INTRODUCTION

It is well known that every nonsingular n-dimensional projective algebraic variety $X \subset \mathbf{P}$ can be isomorphically projected in the projective space $\mathbf{P}^{2n+1}$. The study of varieties X^n which can be projected isomorphically in a projective space $\mathbf{P}^r$ with $r < 2n + 1$ is particularly interesting. In this case the image of X in $\mathbf{P}^r$ is a "low-codimension" variety, and such varieties, as it is known, enjoy many remarkable properties.

One of the main results of this work is that if the variety $X^n \subset \mathbf{P}$ can be isomorphically projected in a projective space $\mathbf{P}^{2n+1-\delta}$, $\delta > 0$, then the manifold X is itself contained in a linear subspace whose dimension depends only on n and δ. More precisely, let $\mathbf{P}^N = \langle X \rangle$ be the linear envelope of the variety $X \subset \mathbf{P}$, i.e., the minimal linear subspace of $\mathbf{P}$ containing X, and let $M(n, \delta)$ be the maximal value of N for given n and δ. Then

$$2n + 1 - \delta \leqslant N \leqslant M(n, \delta),$$

and

$$M(n, \delta) \leqslant f\left(\left[\frac{n}{\delta}\right]\right),$$

where f is a quadratic function calculated explicitly in Theorem 3, and the square brackets stand for the integer part. Thus, for given δ and $n \geqslant \delta - 1$ the dimension N of the linear envelope of X lies within the limits indicated in Fig. 1. Moreover, our bounds are exact, and one can classify all the varieties X^n admitting an isomorphic projection in $\mathbf{P}^r$, $r < 2n + 1$, for which $N = \dim \langle X \rangle = f([n/(2n + 1 - r)])$ (there exist three series of such extremal varieties and also an isolated 16-dimensional variety).

We remark that in [5] it was shown that if $\delta > n/2$, then $M(n, \delta) = 2n + 1 - \delta = f([n/\delta])$. Moreover, from [6] (see also [4]) it follows that $M(n, n/2) = 3n/2 + 2$; also, in [6] a classification of all varieties X^n, $\delta(X) = n/2$, is given for which $n = 3n/2 + 2$ (there are exactly four such varieties, the so-called Severi varieties; they correspond naturally to the four standard algebras: the ground field, the "complex algebra," the quaternion algebra, and the Cayley algebra). Therefore, one can say that the present work generalizes the results of [5] and [6] to the case $1 \leqslant \delta < n/2$.

The result indicated above may be reformulated in the language of classical algebraic geometry as follows: the dimension of the total linear system of hyperplane sections on the variety $X^n \subset \mathbf{P}^r, r \leqslant 2n$, does not exceed $f([n/(2n + 1 - r)])$. This means that the dimension of the vector space of sections of the restriction to X of the standard linear bundle over $\mathbf{P}^r$, corresponding to a hyperplane, is less than or equal to $f([n/(2n + 1 - r)]) + 1$.

Our methods are no less interesting than the result described above. The main objects we are concerned with are the higher secant varieties. We recall that the k-secant variety S^kX of the variety X is defined as the closure of the union of the k-dimensional linear subspaces spanned by generic collections of k + 1 points of X. In particular, $S^0X = X$ and $S^1X = SX$ is the usual secant (chord) variety of X. We remind the reader that a nonsingular variety $X \subset \mathbf{P}$ can be isomorphically projected in the projective space $\mathbf{P}^r$ if and only if $r > \dim SX$.

We shall write $k_0 = k_0(X)$ to denote the smallest nonnegative integer for which $S^{k_0}X = \mathbf{P}^N$ (here, as above, $\mathbf{P}^N$ denotes the linear envelope of X). It is clear that $k_0 < \infty$, and we obtain an increasing chain of subvarieties

$$X = S^0X \subset SX = S^1X \subset \ldots \subset S^{k_0-1}X \subset S^{k_0}X = \mathbf{P}^N$$

Central Mathematical-Economics Institute, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 3, pp. 1-10, July-September, 1985. Original article submitted October 2, 1984.

0016-2663/85/1903-0165$09.50 © 1986 Plenum Publishing Corporation

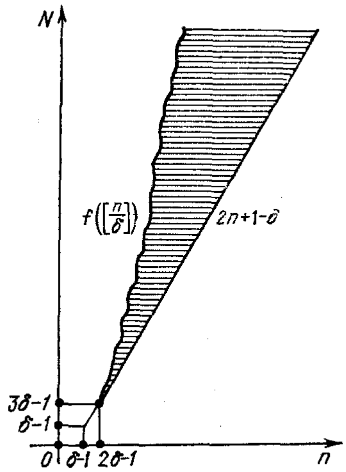


Fig. 1

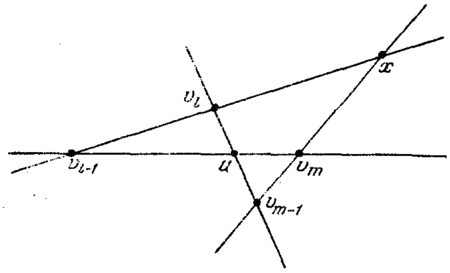


Fig. 2

(it is readily checked that all the inclusions are strict; see Proposition 1).

The invariants of the higher secant varieties yield important information on the projective properties of the variety X. Here we shall be concerned mainly with the dimensions of the higher secant varieties. It turns out that these dimensions for different values of k are connected through specific relations. In particular, we show that for a nonsingular variety X, $k_0(X) \leqslant [n/\delta]$, where $\delta = 2n + 1 - s$, s = dim SX (Theorem 2). In other words, for every nonsingular variety $X^n \subset \mathbf{P}^N S^{[n/\delta]}X = \mathbf{P}^N$. This is precisely the result from which we derive the aforementioned estimate for the dimension of the total linear system of hyperplane sections (Theorem 3).

To investigate the connections between the numbers $s_k = \dim S^kX$ we construct an increasing chain of submanifolds $x \in Y_1 \subset Y_2 \subset \ldots \subset Y_{k_0} \subseteq X$, with the property that there exist s_k-dimensional linear subspaces $L_k \subset \mathbf{P}^N$, which are tangent to X along Y_k, $1 \leqslant k \leqslant k_0$, i.e., they contain the (projective) tangent spaces to X at all points $y \in Y_k$. The main technical result of this paper (Theorem 1), from which we deduce Theorem 2, asserts that the function $\delta_k = \dim Y_k$ is subadditive on the segment $[0, k_0]$, i.e., if $k = k_1 + \ldots + k_r$, $k_i \geqslant 0$, $i = 1, \ldots, r$, then $\delta_k \geqslant \delta_{k_1} + \ldots + \delta_{k_r}$ (here $\delta_0 = 0$, $\delta_1 = \delta$).

Throughout this paper we consider nonsingular projective varieties over an algebraically closed field K (for the classification of extremal varieties we must assume, in addition, that the ground field has characteristic zero). The nonsingularity assumption is not particularly essential. For instance, Theorem 4 is valid for singular varieties too, while in the formulation of Theorem 3 it is necessary to introduce an extra term that depends on the dimension of the subvariety of singularities. Here the situation is exactly the same as in Theorem 3a), b) of [5]. Nevertheless, the nonsingularity assumption permits one to shorten considerably the exposition and to simplify formulas.

We shall basically adhere to the notations of [5]. As we have already mentioned, X designates a nonsingular projective variety in the projective space $\mathbf{P}^N$ and, unless otherwise stipulated, we shall assume that X is not contained in a proper projective subspace of $\mathbf{P}^N$. The letter Y (with various indices) designates a closed subvariety of X, and we use S(Y, X) to denote the variety of points lying on the chords that join the points of Y with all the points of X (in particular, S(X,X) = SX is the usual secant (chord) variety of X). For a point $x \in X$ we denote by $T_{X,x}$ the tangent space to X at x, by $TX = \bigcup_{x \in X} T_{X,x}$ the tangent variety, and by $T(Y, X) = \bigcup_{y \in Y} T_{X,y}$ the relative tangent variety. By a generic point we mean a point belonging to an appropriate open everywhere dense subset. Finally, in contrast to [5], $\langle A \rangle$ denotes the linear envelope of the subset $A \subset \mathbf{P}^N$.

1. Higher Secant Varieties

Let $X^n \subset \mathbf{P}^N$ be a projective variety. We set

$$(S_X^k)^0 = \{(x_0, \ldots, x_k; u) \in \underbrace{X \times \ldots \times X}_{k+1} \times \mathbf{P}^N \,|\, \dim \langle x_0, \ldots, x_k \rangle = k, \;\; u \in \langle x_0, \ldots, x_k \rangle\},$$

and denote by S_X^k the closure of $(S_X^k)^0$ in $\underbrace{X \times \ldots \times X}_{k+1} \times \mathbf{P}^N$. We denote by φ^k (or simply by φ the projection mapping from S_X^k into $\mathbf{P}^N$, and by p_i^k (or simply p_i) the projection mapping of S_X^k onto the i-th factor of the product $X \times \ldots \times X$ $(i = 0, \ldots, k)$.

Definition. The variety $S^k X = \varphi^k (S_X^k)$ is called the k-secant variety of the variety X.

Thus, $S^k X$ is the closure of the set of points lying on k-dimensional subspaces passing through k + 1 generic points of X. In particular, $S^0 X = X$, and $S^1 X = SX$ is the usual secant (chord) variety. Clearly,

$$X \subset SX \subset S^2 X \subset \ldots \subset S^k X \subset \ldots \subset S^{k_0 - 1} X \subset S^{k_0} X = \mathbf{P}^N, \quad (1)$$

where $k_0 = \min \{k \mid S^k X = \mathbf{P}^N\}$; moreover, $S^k X$, $0 \leqslant k \leqslant k_0$, are irreducible projective varieties.

There are alternative ways of constructing the variety $S^k X$. For instance, let $a_0 \leqslant \cdots \leqslant a_r$ be nonnegative integers such that $a_0 + \ldots + a_r = k - r$, and let

$$S^0_{S^{a_0}X, \ldots, S^{a_r}X} = \{(v_0, \ldots, v_r;\ u) \in S^{a_0}X \times \ldots \times S^{a_r}X \times \mathbf{P}^N \mid \dim \langle v_0, \ldots, v_r \rangle = r,\ \ u \in \langle v_0, \ldots, v_r \rangle\},$$

and $S_{S^{a_0}X, \ldots, S^{a_r}X}$ the closure of $S^0_{S^{a_0}X, \ldots, S^{a_r}X}$ in $S^{a_0}X \times \ldots \times S^{a_r}X \times \mathbf{P}^N \subset \underbrace{\mathbf{P}^N \times \ldots \times \mathbf{P}^N}_{r+2}$. In this case we still denote by φ (or, in more detail, by $\varphi^{a_0, \ldots, a_r}$) the projection mapping of $S_{S^{a_0}X, \ldots, S^{a_r}X}$ in $\mathbf{P}^N$, and by p_i (or, in more detail, by $p_i^{a_0, \ldots, a_r}$) the projection mapping of $S_{S^{a_0}X, \ldots, S^{a_r}X}$ on $S^{a_i}X$. It is readily seen that

$$S^k X = \varphi^{a_0, \ldots, a_r} (S_{S^{a_0}X, \ldots, S^{a_r}X}). \quad (2)$$

The preceding definition is a particular case of the present one (r = k, $a_0 = \ldots = a_r = 0$), i.e., $S^k X = S_{\underbrace{X, \ldots, X}_{k+1}}$. We shall often use another particular case of representation (2), namely r = 1. In this case it follows from (2) that

$$S^k X = S (S^{a_0}X, S^{a_1}X). \quad (3)$$

In particular, for $a_0 = 0$ we obtain the recursion formula

$$S^k X = S (X, S^{k-1}X). \quad (4)$$

Proposition 1. All the inclusions in (1) are strict. In other words, for $1 \leqslant k \leqslant k_0$, $S^{k-1} X \neq S^k X$.

Proof. We assume the contrary. From (4) it follows that for each point $x \in X$, $S^{k-1}X$ is the cone with the vertex at x. But this is possible only if $S^{k-1} X = \mathbf{P}^N$, which contradicts the condition $k \leqslant k_0$. This contradiction proves the proposition (see [2], Lemma 7.10).

The following generalization of Terracini's lemma is valid [1, 2, 5].

Proposition 2. Let $v_0 \in S^{a_0}X, \ldots, v_r \in S^{a_r}X$ $\dim \langle v_0, \ldots, v_r \rangle = r$, $u \in \langle v_0, \ldots, v_r \rangle \subset S^k X$, where $k = a_0 + \ldots + a_r + r$, and let $L_u = T_{S^k X, u}$ be the (projective) tangent space to $S^k X$ at the point u. Then:

a) $L_u \supset T_{S^{a_i}X, v_i}$, $i = 0, \ldots, r$, where $T_{S^{a_i}X, v_i}$ is the tangent space to the a_i-secant variety $S^{a_i}X$ at the point v_i;

b) if char K = 0 and u is a generic point of $S^k X$, then

$$L_u = \langle T_{S^{a_0}X, v_0}, \ldots, T_{S^{a_r}X, v_r} \rangle.$$

The proof is carried out by induction on r; the arguments of the proof of the usual Terracini's lemma [1, 2, 5] work with no modifications in the present case — it suffices to use the representation $S^k X = \varphi (S_{S^{a_0}X, S^{a_1 + \ldots a_r + r - 1}X})$ (see (3)), $u \in \langle v_0, v \rangle$, $v \in \langle v_1, \ldots, v_r \rangle$.

Let $u \in S^k X$. Set $Y_u = p_0^k ((\varphi^k)^{-1} (u))$. From Proposition 2 it follows that L_u is tangent to X along the subvariety $Y_u \subset X$. The dimension of Y_u for a generic point $u \in S^k X$ is a projective invariant of the variety X; we set $\delta_k = \dim Y_u$. This invariant is readily expressed through the dimensions of the secant varieties. In fact, let $s_k = \dim S^k X$ (in particular, $s_0 = n$, $s_1 = s = \dim SX$). We use the representation $S^k X = \varphi (S_{X, S^{k-1}X})$ (see (4)).

Then $Y_u = p_0^{0,k-1}((\varphi^{0,k-1})^{-1}(u))$, and if $1 \leqslant k \leqslant k_0$ (which, according to Proposition 1, implies that $S^{k-1}X \neq S^kX$), then

$$\delta_k = \dim Y_u = \dim((\varphi^{0,k-1})^{-1}(u)) = \dim S_{X,S^{k-1}X} - \dim S^kX = s_{k-1} + n + 1 - s_k \tag{5}$$

(in particular, $\delta_1 = 2n + 1 - s$; for brevity we shall write δ instead of δ_1).

From Proposition 2a) it follows that for $k < k_0$ $\delta_k < n$, $\delta_{k_0} \leqslant n$, and formula (5) shows that equality holds if and only if $s_{k_0-1} = N - 1$; finally, for $k \geqslant k_0$, $s_k = N$, and for $k > k_0$, $\delta_k = n$.

Summing equalities (5) for all $1 \leqslant k' \leqslant k$, we get for $k \leqslant k_0$

$$s_k = (k+1)(n+1) - \sum_{i=1}^{k} \delta_i - 1 = \sum_{i=0}^{k} (n - \delta_i + 1) - 1 \tag{6}$$

(we recall that $\delta_0 = \dim Y_X = \dim x = 0$, where $x \in X$ is a generic point).

Example. Suppose that $n = 1$, i.e., X is a curve. As we have already remarked, Proposition 2a) implies that $\delta_k = 0$ for $k < k_0$. From formula (6) it follows that $s_k = \min(2k + 1, N)$. Thus, basically the dimensions of the higher secant varieties of a curve do not depend on the properties of the curve. For higher dimensional varieties this is already far from being true.

Proposition 3. $0 = \delta_0 \leqslant \delta_1 = \delta \leqslant \delta_2 \leqslant \ldots \leqslant \delta_{k_0-1} \leqslant \delta_{k_0} \leqslant n$, $\delta_{k_0-1} \leqslant n - \delta$, and $\delta_k = n$ for $k > k_0$.

Proof. Consider the rational mappings

$$\xi\colon S_{X,X,S^{k-2}X} \to S_{X,S^{k-2}X},$$
$$\xi(x', x, w, u) = (x', w, [x', w] \cap [x, u])$$

and

$$\eta\colon S_{X,X,S^{k-2}X} \to S_{X,S^{k-1}X},$$
$$\eta(x', x, w, u) = (x, [x', w] \cap [x, u], u).$$

It is clear that ξ and η are defined outside $(\varphi^{0,0,k-2})^{-1}(X)$, and

$$p_1^{0,k-2} \circ \xi = p_2^{0,0,k-2}, \qquad \varphi^{0,k-1} \circ \eta = \varphi^{0,0,k-2}.$$

Let x be a generic point of X, v a generic point of $S^{k-1}X$, and $u \in [x, v]$ a generic point of S^kX. Then

$$(\varphi^{0,0,k-2})^{-1}(u) = \eta^{-1}((\varphi^{0,k-1})^{-1}(u)) \supset \eta^{-1}(x, v, u),$$
$$\xi(\eta^{-1}(x, v, u)) = (\varphi^{0,k-2})^{-1}(v),$$

$$Y_u = p_0^{0,0,k-2}((\varphi^{0,0,k-2})^{-1}(u)) \supset p_0^{0,0,k-2}(\eta^{-1}(x,v,u)) = p_0^{0,k-2}(\xi(\eta^{-1}(x,v,u))) = p_0^{0,k-2}((\varphi^{0,k-2})^{-1}(v)) = Y_v.$$

Consequently, $\delta_k = \dim Y_u \geqslant \dim Y_v = \delta_{k-1}$, i.e., the numbers δ_k form a monotonically increasing sequence.

It remains to show that if $S^kX \neq \mathbf{P}^N$, then $\delta_k \leqslant n - \delta$. Let u be a generic point of S^kX and $L = T_{S^kX,u}$. Proposition 2a) implies that $T(Y_u, X) \subset L$. Hence, using Proposition 2.5 of [4] we conclude that $s = \dim SX \geqslant \dim S(Y_u, X) = \delta_k + n + 1$, i.e., $\delta_k \leqslant s - n - 1 = n - \delta$, as asserted.

Proposition 3 is proved.

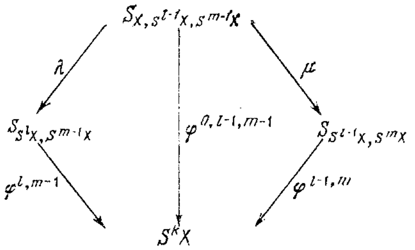
The next result sharpens the assertion of Proposition 3 concerning the monotonicity of the sequence δ_k.

THEOREM 1. Let $0 \leqslant k \leqslant k_0$ be an integer and let $k = k_1 + \ldots + k_r$ be a decomposition of k into a sum of r integers $k_1, \ldots, k_r \geqslant 0$. Then $\delta_k \geqslant \delta_{k_1} + \ldots + \delta_{k_r}$. That is to say, function δ is subadditive on the segment $[0, k_0]$.

Proof. First of all, notice that it suffices to consider the case $r = 2$. In fact, the assertion of the theorem for arbitrary $r \geqslant 2$ follows from the case $r = 2$ in view of the chain of inequalities

$$\delta_k \geqslant \delta_{k_1+\ldots+k_{r-1}} + \delta_{k_r},\ \delta_{k_1+\ldots+k_{r-1}} \geqslant \delta_{k_1+\ldots+k_{r-2}} + \delta_{k_{r-1}}, \ldots, \delta_{k_1+k_2} \geqslant \delta_{k_1} + \delta_{k_2}.$$

Thus, let $k = l + m$, $1 \leqslant l \leqslant m \leqslant k - 1$ (recall that, by definition, $\delta_0 = 0$). Consider the commutative diagram of rational mappings



(7)

in which for generic points $x \in X$, $v_{l-1} \in S^{l-1}X$, $v_{m-1} \in S^{m-1}X$, and $u \in \langle x, v_{l-1}, v_{m-1}\rangle \subset S^kX$ we put $\lambda(x, v_{l-1}, v_{m-1}, u) = (v_l, v_{m-1}, u)$, where $v_l = \langle x, v_{l-1}\rangle \cap \langle v_{m-1}, u\rangle$, and $\mu(x, v_{l-1}, v_{m-1}, u) = (v_{l-1}, v_m, u)$, where $v_m = \langle x, v_{m-1}\rangle \cap \langle v_{l-1}, u\rangle$ (see Fig. 2). It is clear that $\varphi^{l,m-1}[\lambda(\mu^{-1}(v_{l-1}, v_m, u))] = u$. Hence,

$$\lambda^{-1}[\lambda(\mu^{-1}(v_{l-1}, v_m, u))] \subset (\varphi^{0,l-1,m-1})^{-1}(u),$$
$$p_0^{0,\,l-1,\,m-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{l-1}, v_m, u)))] \subset p_0^{0,\,l-1,\,m-1}((\varphi_0^{0,\,l-1,\,m-1})^{-1}(u)) = Y_u$$

and

$$\delta_k = \dim Y_u \geqslant \dim p_0^{0,l-1,m-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{l-1}, v_m, u)))]. \tag{8}$$

Obviously,

$$\dim \lambda(\mu^{-1}(v_{l-1}, v_m, u)) = \dim \mu^{-1}(v_{l-1}, v_m, u) = \dim((\varphi^{0,m-1})^{-1}(v_m)) = \delta_m. \tag{9}$$

On the other hand, for a generic point $(v_l, v_{m-1}, u) \in S_{S^lX, S^{m-1}X}$

$$\dim \lambda^{-1}(v_l, v_{m-1}, u) = \dim((\varphi^{0,l-1})^{-1}(v_l)) = \delta_l. \tag{10}$$

From (9) and (10), it follows that

$$\dim \lambda^{-1}[\lambda(\mu^{-1}(v_{l-1}, v_m, u))] \geqslant \delta_l + \delta_m. \tag{11}$$

Formulas (8) and (11) show that in order to prove Theorem 1 it suffices to verify that the mapping $p_0^{0,l-1,m-1}$ is finite at a generic point of the variety $\lambda^{-1}[\lambda(\mu^{-1}(v_{l-1}, v_m, u))]$. Let y be a generic point of $p_0^{0,l-1,m-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{l-1}, v_m, u)))]$. The preimage of y in $\lambda^{-1}[\lambda(\mu^{-1}(v_{l-1}, v_m, u))]$ consists of the triples (y, v_{l-1}, v_{m-1}), where $\langle y, v_{l-1}\rangle \ni v_l$, $v_l = \langle x, v_{l-1}\rangle \cap \langle v_{m-1}, u\rangle$, $\langle x, v_{m-1}\rangle \ni v_m$ (see Fig. 3). It therefore suffices to show that the subvarieties $Y^{m-1}_{v_{k-1}} = p_1^{l-1,m-1}[(\varphi^{l-1,m-1})^{-1}(v_{k-1})]$, where $u \in \langle y, v_{k-1}\rangle$ and $Y^{m-1}_{v_m} = p_1^{0,\,m-1}[(\varphi^{0,m-1})^{-1}(v_m)]$, intersect each other at a finite number of points.

In view of the genericity property, Theorem 1 now follows from the next lemma.

<u>LEMMA.</u> Suppose that $S^{k-1}X \neq \mathbf{P}^N$. Let v_{k-1} be a generic point of $S^{k-1}X$ and $Y^{m-1}_{v_{k-1}} = p_1^{l-1,m-1}[(\varphi^{l-1,m-1})^{-1}(v_{k-1})]$. Then the dimension of the variety $S(Y^{m-1}_{v_{k-1}}, X)$, consisting of the chords joining the points of $Y^{m-1}_{v_{k-1}}$ with those of X, is equal to $\dim Y^{m-1}_{v_{k-1}} + n + 1$.

<u>Proof of the Lemma.</u> According to Proposition 2a), $T'(Y^{m-1}_{v_{k-1}}, S^{m-1}X) \subset L$, where $L = T_{S^{k-1}X, v_{k-1}}$, and $T'(Y^{m-1}_{v_{k-1}}, S^{m-1}X)$ designates the relative variety of tangent stars. From Proposition 2.5 of [4] it follows that

$$\dim S(Y^{m-1}_{v_{k-1}}, S^{m-1}X) = \dim Y^{m-1}_{v_{k-1}} + s_{m-1} + 1. \tag{12}$$

But, with obvious notations,

$$S(Y^{m-1}_{v_{k-1}}, S^{m-1}X) = S(S(Y^{m-1}_{v_{k-1}}, X), S^{m-2}X).$$

Therefore, the dimension of the preimage of a generic point of $S(Y^{m-1}_{v_{k-1}}, S^{m-1}X)$ under the morphism

$$S_{Y^{m-1}_{v_{k-1}}, S^{m-1}X} \to S(Y^{m-1}_{v_{k-1}}, S^{m-1}X)$$

is not less than the dimension of the preimage of a generic point of $S(Y^{m-1}_{v_{k-1}}, X)$ under the morphism

$$S_{Y^{m-1}_{v_{k-1}}, X} \to S(Y^{m-1}_{v_{k-1}}, X).$$

Since, according to (12), the first of these dimensions is equal to zero, the second is also equal to zero, and hence

$$\dim S(Y^{m-1}_{v_{k-1}}, X) = \dim S_{Y^{m-1}_{v_{k-1}}, X} = \dim Y^{m-1}_{v_{k-1}} + n + 1.$$

This completes the proof of the lemma and hence that of Theorem 1 too.

COROLLARY. For $0 \leqslant k \leqslant k_0, \delta_k \geqslant k\delta$.

To prove this it suffices to apply Theorem 1 with r = k, $k_1 = \ldots = k_r = 1$.

THEOREM 2. $k_0 \leqslant [n/\delta]$, i.e., $S^{[n/\delta]}X = \mathbf{P}^N$ (here the square brackets stand for the integer part; for δ = 0 the assertion of the theorem is void).

Proof. From the definition of k_0 it follows that for $a \geqslant k_0$, $S^a X = \mathbf{P}^N$, so that in order to prove the theorem it suffices to check that $n/\delta \geqslant k_0$. From the Corollary to Theorem 1 we deduce that $\delta_{k_0} \geqslant k_0\delta$; on the other hand, $\delta_{k_0} \leqslant \dim X = n$. Consequently, $k_0\delta \leqslant n$ and $k_0 \leqslant n/\delta$ (alternatively, one can use the corollary to Theorem 1 for k = $k_0 - 1$ and the inequality $\delta_{k_0-1} \leqslant n - \delta$, proved in Proposition 3).

Theorem 2 is proved.

COROLLARY. (Hartshorne's conjecture on linear normality; see [1, 2, 5]). If $SX \neq \mathbf{P}^N$, then $\delta \leqslant n/2$ and $n \leqslant 2/3\,(N-2)$.

Proof of the Corollary. For $\delta > n/2, [n/\delta] = 1$ and hence, according to Theorem 2, $SX = \mathbf{P}^N$. Further, for $SX \neq \mathbf{P}^N$, $N \geqslant s + 1 = 2n + 2 - \delta \geqslant 3n/2 + 2$, i.e., $n \leqslant 2/3\,(N-2)$, as claimed.

2. Maximal Imbeddings of Varieties of Low Codimension

We denote by M(n, δ) (m(n, δ)) the maximal (respectively, minimal) integer N for which there exists a nonsingular projective variety $X \subset \mathbf{P}^N$ such that dim X = n, δ(X) = δ (as usual, we assume that $X \neq \mathbf{P}^N$ and is not contained in a hyperplane). It is not hard to see that functions m and M are defined on the set of all pairs $(n, \delta) \in \mathbf{Z}^2$, for which $0 \leqslant \delta \leqslant n$. Moreover, it is natural to put M(n, 0) = ∞.

Proposition 4. (i) $m(n, \delta) = 2n + 1 - \delta$;

(ii) $M(n, \delta - 1) \geqslant M(n, \delta) + 1$;

(iii) $M(n-1, \delta-1) \geqslant M(n, \delta) - 1$.

Proof. (i) In fact, for every variety X with dim X = n, δ(X) = δ, there holds the inequality $m(n, \delta) \geqslant s_X = 2n + 1 - \delta$. Taking for X the intersection of n + 1 − δ generic hypersurfaces $H_i \subset \mathbf{P}^{2n+1-\delta}$, $\deg H_i > 1$, $i = 1, \ldots, n + 1 - \delta$, we see that $m(n, \delta) \leqslant 2n + 1 - \delta$, as asserted.

(ii) Suppose that $X \subset \mathbf{P}^{M(n,\delta)}$ is a variety such that dim X = n, δ(X) = δ, and let $C_X \subset \mathbf{P}^{M(n,\delta)+1}$ be the projective cone over X with the vertex at a generic point $u \in \mathbf{P}^{M(n,\delta)+1}$. Let $X' \subset \mathbf{P}^{M(n,\delta)+1}$ denote the intersection of C_X with a generic hypersurface $H \subset \mathbf{P}^{M(n,\delta)+1}$, $\deg H > 1$. Then it is readily checked that X' is a nonsingular variety contained in no hyperplane, dim X' = dim X = n, SX' = SC_X = C_{SX}, where C_{SX} is the cone over SX with vertex u, $s_{X'}$ = dim SX' = dim C_{SX} = s_X + 1, and $\delta(X') = 2n + 1 - s_{X'} = \delta - 1$. This proves (ii).

(iii) Let $X \subset \mathbf{P}^{M(n,\delta)}$ be a variety such that dim X = n, δ(X) = δ, and denote by $X' \subset \mathbf{P}^{M(n,\delta)-1}$ the intersection of X with a generic hyperplane $H \subset \mathbf{P}^{M(n,\delta)}$. Then it is readily checked that X' is a nonsingular variety contained in no hyperplane and such that dim X' = n − 1, $SX' = SX \cap H$, $s_{X'} = s_X - 1$ and δ(X') = δ − 1. This proves (iii) and completes the proof of Proposition 4.

Definition. We call an extremal variety any nonsingular variety $X \subset \mathbf{P}^N$, such that dim X = n, δ(X) = δ, and N = M(n, δ).

THEOREM 3.

$$M(n, \delta) \leqslant f([n/\delta]) = \frac{n(n+\delta+2)}{2\delta} + \frac{1}{2}\left\{\frac{n}{\delta}\right\}\left(\delta - \delta\left\{\frac{n}{\delta}\right\} - 2\right) = \frac{n(n+\delta+2) + \varepsilon(\delta - \varepsilon - 2)}{2\delta},$$

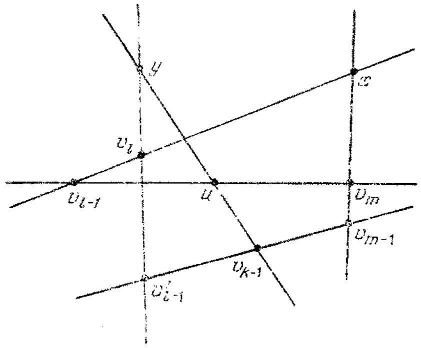


Fig. 3

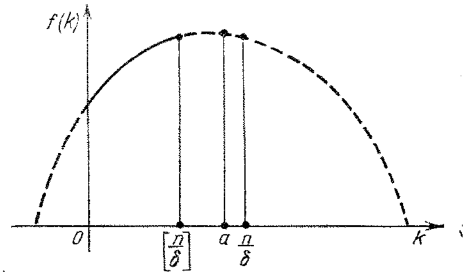


Fig. 4

where

$$f(k)=(k+1)(n+1)-\frac{k(k+1)}{2}\delta-1,$$

and $\varepsilon=\delta\left\{\frac{n}{\delta}\right\}=n \pmod{\delta}$ (here the square brackets and the braces denote respectively the integer and fractional part of a number).

Proof. Let $X \subset \mathbf{P}^{M(n,\delta)}$ be a nonsingular variety with dim X = n and δ(X) = δ. From Theorem 1 and formula (6) of Sec. 1 it follows that for $k \leqslant k_0$

$$s_k=(k+1)(n+1)-\sum_{i=1}^{k}\delta_i-1 \leqslant (k+1)(n+1)-\left(\sum_{i=1}^{k} i\right)\delta-1=f(k). \tag{13}$$

The graph of the function f(k) is a parabola (see Fig. 4); f(k) attains its maximum (equal to $\frac{(2n+\delta+2)^2}{8\delta}-1$) at $k=\frac{2n-\delta+2}{2\delta}=a$, and f(k) is monotonically increasing for $0 \leqslant k \leqslant a$. By the definition of k_0, $M(n,\delta)=s_{k_0} \leqslant f(k_0)$, while Theorem 2 implies that $k_0 \leqslant [n/\delta]$. If $k_0 < [n/\delta]$, then $k_0 \leqslant [n/\delta]-1 \leqslant n/\delta-1 < a$ and $f(k_0) \leqslant f(n/\delta-1)=f(n/\delta)-1<f([n/\delta])$ (we remark that from (13) it follows that f([n/δ]) is an integer). Therefore, one has always $M(n,\delta) \leqslant f([n/\delta])$, and to prove Theorem 3 it suffices to calculate f([n/δ]) explicitly.

Remarks. 1. For δ > n/2, [n/δ] = 1 and Theorem 3 gives $N \leqslant 2n+1-\delta=s$. Since $s \leqslant N$, $s=N$, $SX=\mathbf{P}^N$, and we obtain again a proof of Hartshorne's conjecture on linear normality (see Corollary to Theorem 2). In this case $N = s = 2n + 1 - \delta \leqslant (3n+1)/2$.

2. For δ = n/2 (n even), Theorem 3 gives $N \leqslant 3n/2+2$. Here we must distinguish two cases:

i) $SX=\mathbf{P}^N$, $N=s=3n/2+1$;

ii) $SX \neq \mathbf{P}^N$, $N=s+1=3n/2+2$.

The variety for which s = 3n/2 + 1 and N > s are called Severi varieties. Hence, ii) means that every n-dimensional Severi variety lies in the (3n/2 + 2)-dimensional projective space (and projectively generates this space). Incidentally, this result follows from the classification of Severi varieties [4, 6].

3. From Remark 1 it follows that if n is odd and $SX \neq \mathbf{P}^N$, then $\delta \leqslant (n-1)/2$, i.e., $s \geqslant (3n+3)/2$. Let us consider the case where δ = (n − 1)/2. According to Theorem 3, for n = 3 (δ = 1) $N \leqslant n(n+3)/2=9=s+3$, whereas for n > 3

$$N \leqslant \frac{3n(n+1)}{2(n-1)}+\frac{1}{2}\cdot\frac{2}{n-1}\left(\frac{n-1}{2}-\frac{n-1}{2}\cdot\frac{2}{n-1}-2\right)=\frac{3n+7}{2}=s+2.$$

Thus, for $n \equiv 1 \pmod 2$, $\delta=(n-1)/2$, the following cases are possible:

i) $SX=\mathbf{P}^N$, $N=s=\frac{3n+3}{2}$; ii) $N=\frac{3n+5}{2}$; iii) $N=\frac{3n+7}{2}$;

iv) $n=3$, $N=9$.

As an example of realization of ii) one can take the nonsingular hyperplane section of any of the Severi varieties; an example of iii) is provided by the five-dimensional Segre variety $\mathbf{P}^2 \times \mathbf{P}^3 \subset \mathbf{P}^{11}$, and an example of iv) is the Veronese variety $v_2(\mathbf{P}^3) \subset \mathbf{P}^9$ (see Remark 4)

4. The estimates in Theorem 3 are exact (though, of course, not for all the pairs (n, δ)). Examples of extremal varieties $X^n \subset \mathbf{P}^N$ for which $N = M(n, \delta) = f([n/\delta])$ are provided by the Severi varieties (δ = n/2; see Remark 2) and also by the Veronese varieties $v_2(\mathbf{P}^n) \subset \mathbf{P}^{n(n+3)/2}$ $(\delta = 1)$, the Segre varieties $\mathbf{P}^a \times \mathbf{P}^b \subset \mathbf{P}^{(a+1)(b+1)-1}$ with $|a - b| \leqslant 1$ $(\delta = 2)$ and the Grassman varieties $G(m, 1) \subset \mathbf{P}^{\binom{m+1}{2}-1}$ $(\delta = 4)$. One can show that every variety $X^n \subset \mathbf{P}^N$, for which $N = M(n, \delta) = f([n/\delta])$ (i.e., equality is attained in Theorem 3) coincides with one of the varieties listed in this remark.

THEOREM 4. Let $X^n \subset \mathbf{P}^r$ be a nonsingular projective variety (here we do not require that there be no proper linear subspace of $\mathbf{P}^r$ containing X). Then the dimension of a complete linear system of hyperplane sections of X does not exceed f([n/(2n + 1 − r)]), where f is the function given by formula (13). In other words,

$$h^0(X, \mathcal{O}_X(1)) \leqslant f\left(\left[\frac{n}{2n+1-r}\right]\right) + 1 \leqslant \left[\frac{(4n-r+3)^2}{8(2n-r+1)}\right].$$

Proof. Let $X' \subset \mathbf{P}^N$ be the imbedding of X given by the complete linear system $|\mathcal{O}_X(1)|$, $N = h^0(X, \mathcal{O}_X(1)) - 1$. Then $\delta(X') = 2n + 1 - s_{X'} = 2n + 1 - s_X \geqslant 2n + 1 - r$. According to Proposition 4 (ii), $N \leqslant M(n, \delta(X')) \leqslant M(n, 2n + 1 - r)$. Hence, Theorem 3 implies that $h^0(X, \mathcal{O}_X(1)) \leqslant f([n/(2n + 1 - r)]) + 1$. On the other hand, it is readily verified that the second term in the expression $f\left(\left[\frac{n}{2n+1-r}\right]\right) = \frac{n(3n-r+3)}{2(2n-r+1)} + \frac{2n-r-\varepsilon-1}{2(2n-r+1)}$ attains (for fixed n and r) its maximum, equal to $(2n - r - 1)^2/8(2n - r + 1)$, for ε = (2n − r − 1)/2. Therefore,

$$f\left[\left(\frac{n}{2n+1-r}\right]\right) + 1 \leqslant \frac{n(3n-r+3)}{2(2n-r+1)} + \frac{(2n-r-1)^2}{8(2n-r+1)} + 1 = \frac{(4n-r+3)^2}{8(2n-r+1)}.$$

Theorem 4 is proved.

COROLLARY. Let $X^n \subset \mathbf{P}^r$ be a nonsingular variety, $r \leqslant 2n$. Then $h^0(X, \mathcal{O}_X(1)) \leqslant \binom{n+2}{2}$.

Proof. Since Theorem 4 is valid even if X is contained in a linear subspace of $\mathbf{P}^r$, we may assume, without loss of generality, that r = 2n. In this case the corollary is a straightforward consequence of Theorem 4, because $f(n) = n(n + 3)/2 = \binom{n+2}{2} - 1$.

Remarks. 1. The particular case of the Corollary to Theorem 4, in which X is a complex variety and $r = s_X = \dim SX = \dim TX = 2n$, was proved in [3, 6] where other (analytic) methods were used.

2. It turns out that if for the variety $X^n \subset \mathbf{P}^{2n}$, $h^0(X, \mathcal{O}_X(1)) = \binom{n+2}{2}$, then $X \simeq \mathbf{P}^n$, and the imbedding $\mathbf{P}^n \subsetneq \mathbf{P}^{2n}$ is given by a generic collection of 2n + 1 quadratic forms.

3. Needless to say, if r > 2n, then $h^0(X, \mathcal{O}_X(1))$ can assume arbitrarily large values: it suffices to take a nonsingular linearly normal variety $X' \subset \mathbf{P}^N$ ($\dim X' = n$, $h^0(X', \mathcal{O}_X(1)) = N + 1$) and project it isomorphically in $\mathbf{P}^r$; the image of X' under this projection is a nonsingular variety $X \subset \mathbf{P}^r$, for which dim X = n and $h^0(X', \mathcal{O}_X(1)) = N + 1$. This assertion is only a reformulation of the fact that M(n, 0) = ∞.

Theorem 4 is conveniently interpreted using the notion of index of nonnormality.

Definition. Let $X^n \subset \mathbf{P}^r$ be a nonsingular variety. Then the number $\lambda(X) = h^0(X, \mathcal{O}_X(1)) - r - 1$ is called the index of (linear) nonnormality of the variety X^n.

In other words, the index of nonnormality measures how large in comparison with r the number N can be, for which there is a variety $X' \subset \mathbf{P}^N$ which projects onto X.

In particular, the linearly normal varieties are the varieties with index of nonnormality equal to zero, the index of nonnormality of the projection of a Severi variety is equal to one (see Remark 2 following Theorem 3); for δ > 0 the index of nonnormality for curves is equal to zero, for surfaces it does not exceed one, and for three-dimensional varieties it may attain the value 3.

Theorem 4 can now be reformulated as follows:

THEOREM 4'. For every n-dimensional nonsingular projective variety X with $s_X \leqslant s$, we have

$$\lambda(X) \leqslant \frac{(2s-3n)(s-n-1)}{2(2n-s+1)} + \frac{\varepsilon(2n-s-\varepsilon-1)}{2(2n-s+1)} = \frac{(2s-3n)(s-n-1)}{2(2n-s+1)} +$$

$$+\left(n-\frac{s+1}{2}\right)\left\{\frac{n}{2n-s+1}\right\}-\left(n-\frac{s-1}{2}\right)\left\{\frac{n}{2n-s+1}\right\}^2,$$

where $\varepsilon = n(\mathrm{mod}(2n-s+1))$. In particular, if $s_X \leqslant 2n$ (or, equivalently, SX = TX, see [5], Theorem 2), then $\lambda(X) \leqslant n(n-1)/2$.

LITERATURE CITED

1. T. Fujita and J. Roberts, "Varieties with small secant varieties: the extremal case," Am. J. Math., 103, No. 5, 953-976 (1981).
2. W. Fulton and R. Lazarsfeld, "Connectivity and its applications in algebraic geometry," in: Proc. Midwest Conf. on Algebraic Geometry, Univ. of Illinois, 1980, Lecture Notes in Math., Vol. 862, Springer-Verlag, Berlin—Heidelberg—New York (1981), pp. 26-92.
3. Ph. Griffiths and J. Harris, "Algebraic geometry and local differential geometry," Ann. Sci. Ecole Norm. Sup., 12, No. 3, 355-452 (1979).
4. R. Lazarsfeld and A. Van der Ven, "Topics in the geometry of projective space: Recent work of F. L. Zak," Birkhäuser Verlag, Basel—Boston—Stuttgart (1984).
5. F. L. Zak, "Projections of algebraic varieties," Mat. Sb., 116, No. 4, 593-602 (1981).
6. F. L. Zak, "Severi varieties," Mat. Sb., 126, No. 1, 115-132 (1985).

MONODROMY AND VANISHING CYCLES OF BOUNDARY SINGULARITIES

G. G. Il'yuta

UDC 513.836+517.919

INTRODUCTION

The Weyl groups of the classical Lie algebras have symplectic analogs, which are certain infinite groups of automorphisms of a lattice with symplectic structure. The theory of singularities gives a natural definition of these groups. In this paper we describe symplectic versions of the Weyl groups B_μ, C_μ, F_4, G_2.

In the theory of singularities, with each critical point of a function (on a manifold with boundary or without boundary) there are connected two (monodromy) groups, which are subgroups of the group of automorphisms of an integral lattice [2]. One group is generated by reflections in the basis vectors with respect to some symmetric bilinear form, the other by "reflections" (transvections) in the basis vectors with respect to some skew-symmetric bilinear form. At the present time there is a description of the monodromy groups of nonboundary critical points: symmetric ones in [4, 22], and skew-symmetric ones in [14-18] (see also [5-12] and [19-21]). Little is known about monodromy groups of boundary critical points: it is proved in [1] that the symmetric monodromy groups of simple boundary critical points B_μ, C_μ, F_4 coincide with the Weyl groups of the corresponding types. The present paper is devoted to the description of the skew-symmetric monodromy groups of boundary critical points of functions. As an example of the results found we describe the skew-symmetric monodromy groups of simple singularities B_μ, C_μ, F_4 (i.e., the symplectic analogs of the Weyl groups of the same name): a linear automorphism of the corresponding lattice belongs to the skew-symmetric monodromy group of a singularity B_μ, C_μ, or F_4, if and only if it is fixed on the kernel of the skew-symmetric form, preserves the skew-symmetric form and the orbits of the basis vectors under the action of the monodromy group. In the paper we explicitly give the orbits of the basis vectors (or, using the geometric terminology, the set of vanishing cycles).

Arnol'd [1] gave a construction imbedding the monodromy group of a boundary singularity in the monodromy group of a nonboundary singularity (in particular, to the singularities B_μ, C_μ, F_4 there correspond, respectively, $A_{2\mu-1}$, $D_{\mu+1}$, E_6). Namely, the lattice of a non-

M. V. Lomonosov Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 3, pp. 11-21, July-September, 1985. Original article submitted March 28, 1984.

0016-2663/85/1903-0173$09.50
 © 1986 Plenum Publishing Corporation