STRUCTURE OF GAUSS MAPS

F. L. Zak

UDC 512.763

Introduction

The classical Gauss map γ associates with each point of a nonsingular oriented hypersurface $H \subset R^n$ the unit outer normal vector to H at this point; thus, $\gamma: H \to S^{N-1}$, where S^{N-1} is the unit sphere in R^n. For complex hypersurfaces and manifolds of codimension greater than one, it is natural to call the Gauss map the map which associates a point x of the manifold X with the point in the Grassman manifold corresponding to the tangent space to X at the point x. Many results of classical differential geometry can be interpreted in terms of the Gauss map.

In algebraic geometry it turned out to be more convenient to consider the Gauss map not for affine, but for projective varieties; this is connected, in particular, with the fact that, as we shall see below, the singularities at infinity play a quite important role. Thus, the Gauss map $\gamma: X^n \to G(N, n)$ associates with each point x of the nonsingular projective variety $X^n \subset P^N$ the point in the Grassman manifold $G(N, n)$ of n-dimensional projective subspaces in P^N, corresponding to the (projective) tangent space $T_{X,x}$ to X at the point x. Thus, the fiber of γ over an n-dimensional linear subspace $L^n \subset P^N$ is the set of points (with multiplicities), at which the imbedded tangent space to X coincides with L. Analogously, for any $n \leq m \leq N - 1$ one can define the higher Gauss map γ_m, whose fiber over an m-dimensional linear subspace $L^m \subset P^N$ coincides with the set of points $x \in X$ such that $T_{X,x} \subset L^m$ (i.e., L is tangent to X; more precise definitions which are suitable for the singular case also are given in Sec. 2).

Gauss maps in algebraic geometry have actually been studied since the last centrury and play a very important role (this relates especially to the maps $\gamma = \gamma_n$ and γ_{N-1}). It suffices to say that all the most important invariants of algebraic varieties, including the canonical class, have been defined in terms of them. However, the structure of the Gauss maps was investigated very weakly until recently. In the simplest case, when $m = n$, a basic question is the determination of how much X is "distorted" under the map γ. A classicial conjecture, proved in the present paper, asserts that for complex varieties the map γ is finite (i.e., all fibers of γ are finite) and birational (i.e., γ is an isomorphism almost everywhere), so that the distortion is minimal. Up to now the greatest progress in the direction of proving this conjecture was given by the theorem of Griffiths and Harris [3], asserting that if $X \subsetneq P^N$ is a complete variety, then the field of meromorphic functions on X is a finite extension of the field of meromorphic functions on $\gamma(X)$ (i.e., <u>almost all</u> fibers of γ are made up of a finite number of points). As to the higher Gauss maps γ_m, we give a sharp estimate of the dimension of their fibers and we describe the structure of a generic fiber. In particular, we prove the following <u>tangents theorem</u>:

Let $L^m \subsetneq P^N$ be a linear subspace, tangent to the variety $X^n \subset P^N$ along the subvariety $Y \subsetneq X$. Then $\dim Y \leq m - n$.

For example, for $m = N - 1$ we see that a hyperplane cannot be tangent to a variety $X^n \subset P^N$ along a subvariety of dimension greater than or equal to $N - n$. Another classical conjecture according to which for a nonsingular variety $X^n \subsetneq P^V$, $\dim X^* \geq \dim X$, where $X^* \subset P^{N*}$ is the dual variety, consisting of points of P^{N*}, corresponding to hyperplanes in P^N, tangent to X, follows from this.

The tangents theorem is a consequence of the following result. Let Y be an irreducible r-dimensional subvariety of an irreducible n-dimensional variety $X \subset P^N$, $S(Y, X)$ be the closure of the set of points of P^N, lying on chords, joining points of Y with points of X, and

Central Mathematical Economics Institute, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 21, No. 1, pp. 39-50, January-March, 1987. Original article submitted June 3, 1985.

 0016-2663/87/2101-0032$12.50

$T'(Y, X)$ is the variety swept out by the limits of chords $\langle x, x'\rangle$, when $x, x' \to y \in Y$ (here $\langle x, x'\rangle$ is the line passing through the points x and $x' \neq x$; when X is nonsingular, $T'(X, Y)$ coincides with the union of the imbedded tangent spaces to X at points of Y). Then either $T'(Y, X) = S(Y, X)$, or $\dim T'(Y, X) = r + n$, $\dim S(Y, X) = r + n + 1$. For example, for $Y = X$ we get that the nonsingular variety $X \subset P^N$ can be projected isomorphically to P^M, $M < 2n$, if and only if X can be unramifiedly projected to P^M (the special case of the last assertion for $N \leq 2n$ was proved by Johnson [6]; see also [2]); this result contrasts sharply with the conjecture of Massey on immersions of topological manifolds proved recently by Cohen.

In this paper we consider applications of the results cited above to the geometry of varieties of small codimension, ramification cycles and double points of projections and other questions. It is shown in [12] how one derives the Hartshorn conjecture on linear normality from the tangents theorem. Another application of the results of the paper is given by Faltings, Fujita, Balliko and Chiantini, Ein etc. One should note that in contrast with Faltings and other authors, who apply methods of formal geometry, and also Griffiths and Harris, who make differential-geometric calculations, we use purely geometric considerations, which in this situation are not only simpler, but also lead to considerably sharper results.

A complete description of the structure of Gauss maps is also possible for analytic subvarieties of complex tori. The corresponding results, which also have applications to subvarieties of small codimension and pluricanonical systems, will be published in a separate paper.

1. Relative Secant Varieties and Local Properties of Projections

Let $X^n \subset P^N$ be an irreducible nondegenerate n-dimensional projective variety (i.e., one which is not contained in any hyperplane) over the algebraically closed field K, and let $Y^r \subset X^n$ be a nonempty irreducible r-dimensional subvariety of X. We set $\Delta_Y = (Y \times X) \cap \Delta_X = \{(y, x) \in Y \times X \mid x = y\}$, where Δ_X is the diagonal in $X \times X$, and let $S_{Y,X}{}^0 \subset ((Y \times X) \setminus \Delta_Y) \times P^N$, $S_{Y,X}{}^0 = \{(y, x, z) \mid z \in \langle x, y\rangle\}$, where by $\langle x, y\rangle$ we denote the chord joining points x and y. We denote by $S_{Y,X}$ the closure of $S_{Y,X}{}^0$ in $Y \times X \times P^N$, by $p_i{}^Y$ $(i = 1, 2)$ the projection of $S_{Y,X}$ onto the i-th factor of $Y \times X \times P^N$, and by $\varphi^Y\colon S_{Y,X} \to P^N$ the projection to the third factor, and let $p_{12}{}^Y = p_1{}^Y \times p_2{}^Y\colon S_{Y,X} \to Y \times X$, $S(Y, X) = \varphi^Y(S_{Y,X})$, $T'_{Y,X} = (p_{12}{}^Y)^{-1}(\Delta_Y)$, $\psi^Y = \varphi^Y|_{T'_{Y,X}}$, $T'(Y, X) = \psi^Y(T'_{Y,X})$. For $Y = X$, $S_{Y,X} = S_X$, $S(Y, X) = SX$ is the ordinary secant variety, and $T'(X, X) = T'X$ is the variety of tangent stars (see [12]), and the morphisms $p_i{}^Y$, $p_{12}{}^Y$, φ^Y and ψ^Y are the restrictions of the morphisms p_i, p_{12}, φ and ψ to the subvarieties $S_{Y,X} \subset S_X$ and $T'_{Y,X} \subset T'_X$ (see [12]).

Definition 1. We call the cone $T'_{Y,X,y} = \psi^Y((p_{12}{}^Y)^{-1}(y \times y))$ the (projective) tangent star to X at y relative to Y. We call the variety $T'(Y, X) = \bigcup_{y \in Y} T'_{Y,X,y}$ the variety of (projective) tangent starts to X relative to Y.

It is clear that $T'_{Y,X,y} \subset T'_{X,y} \subset T_{X,y}$, where $T'_{X,y}$ is the (projective) tangent star to X at y [12], and $T_{X,y}$ is the (projective) tangent space to X at y. If X is nonsingular along Y, i.e., $Y \cap \operatorname{Sing} X = \varnothing$ and $Y \subset \operatorname{Sm} X = X \setminus \operatorname{Sing} X$, then $T'(Y, X) = T(Y, X) = \bigcup_{y \in Y} T_{X,y}$.

The following proposition generalizes Proposition 1 of [12] to the relative case and is proved completely analogously.

Proposition 1. a) Let $y \in Y$, $x \in X$, $x \neq y$, $z \in \langle y, x\rangle$. Then $T_{S(Y,X),z} \supset \langle T_{Y,y}, T_{X,x}\rangle$, where $\langle A\rangle$ denotes the smallest linear subspace of P^N, containing A.

b) Let us assume in addition that $\operatorname{char} K = 0$. Then for generic points $y \in Y$, $x \in X$, $z \in \langle y, x\rangle$ $T_{S(Y,X),z} = \langle T_{Y,y}, T_{X,x}\rangle$.

Let $\mathcal{T}_Y$ be the ideal Δ_Y in $Y \times X$, $\Theta'_{Y,X} = \mathcal{S}\mathrm{pec} \bigoplus_{j=0}^{\infty} \mathcal{T}^j/\mathcal{T}^{j+1}$, $y \in Y$, $\Theta'_{Y,X,y} = \Theta'_{Y,X} \otimes k(y)$.

Definition 2. We call $\Theta'_{Y,X,y}$ the (affine) tangent star to X at y relative to Y.

It is easy to see that $\Theta'_{Y,X,y}$ contains the tangent cone and is contained in the tangent star to X at y (see [6; 12]), which in its own right is contained in the Zariski tangent space $\Theta_{X,y}$. Here $\overline{\Theta'_{Y,X,y}} = T'_{Y,X,y}$ (we assume here that X is imbedded in P^N, and the dash denotes projective closure).

Definition 3. Let $f\colon X \to X'$ be a morphism of algebraic varieties and let $Y \subset X$ be an irreducible subvariety. We say that f is J-unramified relative to Y at the point $y \in Y$, if the morphism $dyf\,|_{\Theta' Y,X,y}$ is quasifinite. If f is J-unramified relative to Y at all points $y \in Y$, we shall say that f is J-unramified relative to Y.

<u>Definition 4</u>. In the notation of Definition 3 we say that f is a J-imbedding relative to Y, if f is J-unramified relative to Y and one-one on $f^{-1}(f(Y))$.

<u>Remark 1</u>. If f is nonsingular along Y, then f is a J-imbedding relative to Y if and only if f is closed imbedding in some neighborhood of Y in X.

The following theorem generalizes Theorem 1 of [12] and is proved completely analogously.

<u>THEOREM 1</u>. Let Y be an irreducible subvariety of X. We consider the following conditions:

a) For a generic linear subspace $L \subset P^N$, $\operatorname{codim} L = m + 1$, a projection $X \to P^m$ with center in L is a J-imbedding relative to Y.

b) There exists an $L^{N-m-1} \subset P^N$ such that a projection $X \to P^m$ with center in L is a J-imbedding relative to Y.

c) $\dim S(Y, X) \leq m$.

d) There exists a Zariski open subset $U \subset Y \times X$ such that for $y \times x \in U$, $\dim \langle T_{Y,y}, T_{X,x} \rangle \leq m$.

e) Let $y \in \operatorname{Sm} Y$, $x \in \operatorname{Sm} X$. Then $\dim \langle T_{Y,y}, T_{X,x} \rangle \leq m$.

Then a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) $\Rightarrow$ d) $\Leftrightarrow$ e). If in addition $\operatorname{char} K = 0$, then all the conditions a)-e) are equivalent.

The following theorem is a generalization to the relative case of Theorem 2 of [12] and is proved by essentially the same method.

<u>THEOREM 2</u>. For an arbitrary irreducible subvariety $Y^r \subset X^n$ exactly one of the following conditions holds:

a) $\dim T'(Y, X) = r + n$, $\dim S(Y, X) = r + n + 1$.

b) $T'(Y, X) = S(Y, X)$.

<u>Proof</u>. Let $t = \dim T'(Y, X)$. It is clear that $t \leq r + n$. If $t = r + n$, the theorem is obvious, since S(Y, X) is an irreducible subvariety $S(Y, X) \supset T'(Y, X)$ and $\dim S(Y, X) \leq r + n + 1$.

Let us assume that $t < r + n$, and let L^{N-t-1} be a linear subspace of P^N such that $L \cap T'(Y, X) = \varnothing$. We denote by $\pi\colon P^N \setminus L \to P^t$ a projection with center in L, and let $X' = \pi(X)$, $Y' = \pi(Y)$. Since $\pi\,|_X$ is a finite morphism, $\dim Y' \times X' = r + n > t$ and it follows from Theorem 3.1 of [2] that $Y_{X'} \times X = (\pi\,|_Y \times \pi\,|_X)^{-1}(\Delta_{P^t})$ is a connected scheme.

We show $\operatorname{Supp}(Y_{X'} \times X) = \Delta_Y$. Let us assume the contrary. Then by definition, for all $y \times x \in \overline{(Y_{X'} \times X)} \setminus \Delta_Y$ $\varphi^Y((p_{12}{}^Y)^{-1}(y \times x)) \cap L \neq \varnothing$ and consequently for each point $y \times y \in \Delta_Y \cap (\overline{(Y_{X'} \times X) \setminus \Delta_Y})$ $T'(Y, X) \cap L \supset T_{Y,X,y} \cap L = \varphi^Y((p_{12}^Y)^{-1}(y \times y)) \cap L \neq \varnothing$ contrary to the choice of L.

Thus, $\operatorname{Supp} Y_{X'} \times X = \Delta_Y$ and consequently $L \cap S(Y, X) = \varnothing$, since $t \leq \dim S(Y, X) \leq N - \dim L - 1 = t$ and $S(Y, X) = T'(Y, X)$, i.e., condition b) holds. Theorem 2 is proved.

Just as in [12], we get the following corollaries.

<u>COROLLARY 1</u>. $\operatorname{codim}_{S(Y,X)} T'(Y, X) \leq 1$.

<u>COROLLARY 2</u>. Let $\pi\colon X^n \to P^m$ be a projection with center in the subspace $L^{N-m-1} \subset P^N$. Let us assume that π is J-unramified relative to the irreducible subvariety $Y^r \subset X^n$ and that $m < r + n$ (i.e., $\dim L \geq N - n - r$). Then π is a J-imbedding relative to Y.

<u>Proof</u>. It is easy to see that it follows from our assumptions that $L \cap T'(Y, X) = \varnothing$. Consequently, $\dim L + \dim T'(Y, X) < N$ and $\dim T'(Y, X) < N - \dim L = m + 1 \leq n + r$. Hence Theorem 2 shows that $T'(Y, X) = S(Y, X)$. Thus, $L \cap S(Y, X) = \varnothing$, which is what was required.

<u>COROLLARY 2'</u>. Under the conditions of Corollary 2 let us assume that π is unramified at all points of Y. Then in a neighborhood of Y, π is an isomorphism.

Remark 2. Completely analogously, Corollaries 2, 2', and 2" can be proved for arbitrary morphisms $X \to P^m$ (not just for projections). The corresponding results generalize the results of Sec. 5 of [2].

THEOREM 3. Let $Y^r \subset X^n$ be a subvariety of the projective variety $X^n \subset P^N$. Let us assume that there exists a point $u \in P^N \setminus X$ such that projection $X \to P^{N-1}$ with center at u is a J-imbedding relative to Y. Then $\operatorname{codim}_{P^N} X = N - n \geq \frac{r-b}{2} + 1$, where $b = \dim(Y \cap \operatorname{Sing} X)$.

Proof. It is clear that it suffices to consider the case when Y is irreducible. Let $s = \dim(Y, X)$ and let z be a generic point of $S(Y, X)$. We set $L = T_{S(Y,X),z}$, $Q_z = p_1{}^Y((\psi^Y)^{-1}(z))$. It follows from Theorem 2 that either $T'(Y, X) = S(Y, X)$, or $s = n + r + 1$. In the latter case $N \geq s + 1 = n + r + 2$ and $\operatorname{codim}_{P^N} X = N - n \geq r + 2 > \frac{r-b}{2} + 1$. Consequent- we assume that $T'(Y, X) = S(Y, X)$, $Q_z \neq \varnothing$ and $\dim Q_z = r + n - s$. Obviously, $L \supset T_{X,x}$ for all points $x \in Q_z \setminus \operatorname{Sing} X$.

Let $M \subset P^N$ be a generic linear subspace of codimension $b + 1$, $X' = X \cap M$, $Y' = Y \cap M$, $Q'_z = Q_z \cap M$, $L' = L \cap M$. Then the variety X' is nonsingular along Y' and $T(Q'_z, X') \subset L'$. On the other hand, $X' \not\subset L'$. Hence $S(Q'_z, X') \neq T'(Q'_z, X')$ and it follows from Theorem 2 that $\dim S(Q'_z, X') = \dim Q'_z + \dim X' + 1 = (4 + n - s - b - 1) + (n - b - 1) + 1 = 2n + r - s - 2b - 1$.

On the other hand

$$\dim S(Q'_z, X') \leqslant \dim (S(Y, X) \cap M) = s - b - 1.$$

Consequently, $2n + r - s - 2b - 1 \leq s - b - 1$, $2s \geq 2n + r - b$ and $2N \geq 2s + 2 \geq 2n + r - b + 2$, i.e., $\operatorname{codim}_{P^N} X = N - n \geq \frac{r-b}{2} + 1$. Theorem 3 is proved.

COROLLARY 3. Let Y^r be a subvariety of the variety $X^n \subset P^N$, where X is nonsingular in a neighborhood of Y. Let us assume that there exists a point $u \in P^N \setminus X$ such that the projection π: $X \to P^{N-1}$ with center at u is an isomorphism in a neighborhood of Y (according to Corollary 2" of Theorem 2 for $N \leq n + r$ this is equivalent to π being unramified at all points of Y). Then $N \geq n + \frac{r+3}{2}$.

Remark 3. For $Y = X$ Theorem 3 gives $n \leq \frac{2N + b - 2}{3}$, which is somewhat weaker than the estimate $n \leq \frac{2N + b}{3} - 1$, proved in Theorem 3a) of [12]. This is explained by the fact that for $Y = X$ the subvariety Q_z can be replaced by the subvariety $Y_z = p_1(\varphi^{-1}(z))$ of dimension one larger. However, for $Y \neq X$ the estimate in Theorem 3 is sharp. We demonstrate this with the following examples.

Example 1 (to simplify the arguments we assume that $\operatorname{char} K = 0$). a) Let $X \subset P^4$ be a rational surface F_1 of degree 3. Then X is the image of P^2 under the rational map defined by a linear system of quadrics passing through a fixed point of P^2, i.e., by projection of the Veronese surface $v_2(P^2) \subset P^5$ from some point of it. We denote by Y a minimal section of F_1 (so that Y is an exceptional curve of the first kind on F_1). Then Y is a line and the imbedding $X \hookrightarrow P^4$ is given by the complete linear system $|Y + 2F|$, where F is the fiber of F_1. Since the tangent plane at an arbitrary point of X contains the fiber passing through this point, and consequently intersects Y, it follows from Proposition 1b) that $\dim S(Y, X) = r + n = n + \frac{r+1}{2} = 3$, so that $S(Y, X) = T(Y, X) \neq P^4$ and there exists a projection π: $X \to P^3$, which is an isomorphism in a neighborhood of Y (in a suitable coordinate system $\pi(X)$ is given by the equation $u_0 u_3{}^2 = u_1 u_2{}^2$). Here $N = 4 = n + \frac{r+3}{2}$.

b) Let $X^6 = G(4, 1) \subset P^9$ and let $Y = P^3$ be the linear subspace of lines passing through a fixed point of P^4. Then for generic points $y \in Y$, $x \in X$, $T_{Y,y} \cap T_{X,x}$ is the line corresponding to the line in P^4 passing through the fixed point and intersecting a fixed line. It follows from Proposition 1b) that $\dim S(Y, X) = \dim T(Y, X) = 8 = r + n - 1 = n + \frac{r+1}{2} = N - 1$.

Let $X^n \subset P^N$ be a nonsingular variety and let D_m (respectively R_m) be the set of double points (respectively the ramification set) with respect to a generic projection $P^N \to P^m$, $m \geq n$. In other words, if L^{N-m-1} is a generic linear subspace of P^N, then

$$D_m = \overline{\{x \in X \mid \exists x' \in x,\ x' \neq x,\ \langle x, x' \rangle \cap L \neq \varnothing\}},$$
$$R_m = \{x \in X \mid T_{X,x} \cap L \neq \varnothing\}.$$

COROLLARY 4. Let $2(m - n) \leq r \leq n - 1$. Then for any subvariety $Y^r \subset X^n$ $Y^r \cap D_m \neq \varnothing$. If in addition $r > 0$, then for any subvariety $Y^r \subset X^n$ $Y^r \cap R_m \neq \varnothing$.

Proof. Since $R_m \subset D_m$, it suffices to prove the assertion with respect to R_m. Let us assume that $Y^r \cap R_m = \varnothing$. Then for a generic linear subspace $L^{N-m-1} \subset P^N$ $T(Y, X) \cap L = \varnothing$ and consequently $\dim T(Y, X) \leq m$. On the other hand, it follows from Theorem 3 that for $r > 0$, $\dim T(Y, X) \geq n + \frac{r + 1}{2}$. Thus, under our assumptions $m \geq n + \frac{r + 1}{2}$, i.e., $r \leq 2(m - n) - 1$. Consequently for $r \geq \max\{1, 2(m - n)\}$, $Y^r \cap R_m \neq \varnothing$. Corollary 4 is proved.

Remark 4. The assertion of Corollary 4 is only meaningful for $m \leq \frac{3n - 1}{2}$. It is clear that for the validity of Corollary 4 it suffices to require that X be nonsingular in a neighborhood of Y (we restrict ourselves to the nonsingular case in order not to introduce the definitions of ramification cycles and multiple points in the general situation). The preceding examples show that the estimate in Corollary 4 is sharp.

Remark 5. For $m = n$ it is easy to deduce from Corollary 4 that the divisor R_n is ample on X (see Proposition 3 of Sec. 2).

2. Tangents Theorem. Structure of Gauss Mappings

Definition 5. Let $L \subset P^N$ be a linear subspace. We say that L is tangent to the variety $X \subset P^N$ along the subvariety $Y \subset X$ (respectively L is J-tangent to X along Y, respectively L is J-tangent to X relative to Y), if $L \supset T_{X,y}$ (respectively $L \supset T'_{X,y}$, respectively $L \supset T'_{Y,X,y}$) for all points $y \in Y$.

It is clear that if L is tangent to X along Y, then L is J-tangent to X along Y, and if L is J-tangent to X along Y, then L is J-tangent to X relative to Y.

THEOREM 4. Let $X^n \subset P^N$ be a nondegenerate variety, $Y^r \subset X^n$ and $Z^b \subset Y^r$ be closed subvarieties, and let $L^m \subset P^N$, $n \leq m \leq N - 1$ be a linear subspace, J-tangent to X relative to Y along $Y \setminus Z$ (i.e., $L \supset T'_{Y,X,y}$ for all points $y \in Y \setminus Z$). Then $r \leq m - n + b + 1$.

Proof. Let M be a generic linear subspace of P^N of codimension $b + 1$. We set $X' = X \cap M$, $Y' = Y \cap M$, $L' = L \cap M$. It is clear that $n' = \dim X' = n - b - 1$, $r' = \dim Y' = r - b - 1$, $m' = \dim L' = m - b - 1$, and L' is J-tangent to X' relative to Y' along Y'. In other words, $T'(Y', X') \subset L'$. On the other hand, it follows from the nondegeneracy of X that the variety $S(Y', X')$, containing X', does not lie in the subspace L'. Consequently, $S(Y', X') \neq T'(Y', X')$, and it follows from Theorem 2 that $\dim T'(Y', X') = r' + n' = r + n - 2(b + 1)$. Since $L' \supset T'(Y', X')$, $m' \geq r' + n'$, i.e., $r' \leq m' - n' = m - n$ and $r \leq m - n + b - 1$. Theorem 4 is proved.

COROLLARY 5. If the linear subspace $L^m \subset P^N$ is tangent to the nondegenerate variety $X^n \subset P^N$ along the closed subvariety $Y^r \subset X^n$, then $r \leq m - n$.

Corollary 5 is called the tangents theorem (see [2]).

Remark 6. It is clear that if Z does not contain a component of Y, then one can assume that $Z \subset Y \cap \operatorname{Sing} X$.

We give an example showing that the estimate in Theorem 4 is sharp.

Example 2. Let $X \subset P^N$, $N = 2n - b - 2$ be the cone with vertex P^b over the nonsingular projective variety $X' = P^1 \times P^{n-b-2} \subset P^{2n-2b-3}$. Then $X^* = (X')^* = P^1 \times P^{n-b-2} \subset (P^b)^* = P^{2n-2b-3}$ (here and in what follows the asterisk denotes the dual variety) and the subspace $L^m \subset P^N$, $n \leq m \leq N - 1$, is tangent to X at the point $x \in \operatorname{Sm} X$ (and all points of $\langle x, P^b\rangle \setminus P^b$, where $\langle x, P^b\rangle$ is $(b + 1)$-dimensional linear subspace generated by x and P^b), if and only if the $(N - m - 1)$-dimensional subspace L^* lies in the $(N - n - 1)$-dimensional subspace $T^*_{X,x} \subset X^*$. It is easy to see that an arbitrary $(n - b - 3)$-dimensional linear subspace lying in X^* coincides with $(T_{X,x})^*$ for some $x \in X$. Let $P^{n-b-2} \subset X^*$ and let L^* be an arbitrary $(N-m-1)$-dimensional subspace of P^{n-b-2}. Then the m-dimensional subspace $L = (L^*)^*$ is tangent to X at all points of $Y \setminus P^b$, where $Y = \mathbf{P}^{m-n+b+1} \supset \mathbf{P}^b$, $Y = \{x \in X \mid L^* \subset (T_{X,x})^* \subset \mathbf{P}^{n-b-2}\}$. Thus, for the subspace L, the variety $Y = P^{m-n+b+1} \subset P^{n-1} \subset X$, and the subvariety $Z = \operatorname{Sing} X = P^b$ in Theorem 4, equality holds.

Proposition 2. Let $X^n \subset P^N$ be a nondegenerate variety which has the property R_k (see [4, Chap. IV_2, (5.8.2)], i.e., X is regular in codimensions $\leq k$ (in other words, $b = \dim(\operatorname{Sing} X) < n-k$), and let L be an m-dimensional linear subspace of P^N. We set $X' = X \cdot L$, and let $b' = \dim(\operatorname{Sing} X')$. Then $b' \leq 2N - m - n + b - 1 = c + \varepsilon + b - 1$, i.e., X' has the property $R_{k-c-2\varepsilon+1}$, where $c = \operatorname{codim}_{P^N} X$, $\varepsilon = \operatorname{codim}_{P^N} L$.

Proof. For arbitrary point λ of $(\varepsilon - 1)$-dimensional linear subspace $L^* \subset P^{N*}$ we set $X_\lambda = X \cdot \lambda^*$, where λ^* is the hyperplane corresponding to λ. It is clear that $X' = \bigcap_{\lambda \in L^*} X_\lambda$. Let $Y = \operatorname{Sing} X'$, $Y_\lambda = \operatorname{Sing} X_\lambda$, $\lambda \in L^*$. It is easy to see that $Y \subset \bigcup_{\lambda \in L^*} Y_\lambda$, so that $b' = \dim Y \leq \max_{\lambda \in L^*} b_\lambda + \varepsilon - 1$, where $b_\lambda = \dim Y_\lambda$. Obviously, the hyperplane λ^* is tangent to X at all points of $Y_\lambda \setminus \operatorname{Sing} X$. Hence it follows from Theorem 4 that $b_\lambda \leq c + b$. Consequently, $b' \leq c + \varepsilon + b - 1$. Proposition 2 is proved.

The following simple example shows that the estimate in Proposition 2 is sharp.

Example 3. Let $X^{N-1} \subset P^N$ be a quadratic cone with vertex P^b and let $\left[\frac{N+b}{2}\right] + 1 \leq m \leq N - 1$ (the square brackets denote the greatest integer in the number). Then X^* is a singular quadric in the $(N-b-1)$-dimensional space $P^{b*} \subset P^{N*}$. As is known, X^* contains a projective subspace of dimension $\left[\frac{N-b-2}{2}\right]$. Let L^* be an arbitrary $(N-m-1)$-dimensional linear subspace of it. We set $L = (L^*)^*$, $X' = X \cdot L$. Then $\dim L = m$, and it is easy to see that $Y = \operatorname{Sing} X'$ is an $(N-m+b)$-dimensional linear subspace.

The following two corollaries follow quickly from Proposition 2, Proposition 5.8.5 and Theorem 5.8.6 of [4, Chap. IV_2].

COROLLARY 6. If the variety $X^n \subset P^N$ has the properties $S_{\varepsilon+1} = S_{N-m+1}$ and $R_{c+2\varepsilon-1} = R_{3N-2m-n-1}$, and $L^m \subset P^N$ is a linear subspace, for which $\dim(X^n \cdot L^m) = m + n - N$, then the scheme $X \cdot L$ is reduced. In particular, if X is nonsingular, $N < 2/3(m+n+1)$ and $\dim(X \cdot L) = m + n - N$, then $X \cdot L$ is a reduced scheme.

COROLLARY 7. If the variety $X^n \subset P^N$ has the properties $S_{\varepsilon+2} = S_{N-m+2}$ and $R_{c+2\varepsilon} = R_{3N-2m-n}$, then for an arbitrary linear subspace $L^m \subset P^N$ such that $\dim(X^n \cdot L^m) = n + m - N$, the scheme $X \cdot L$ is normal (and consequently irreducible and reduced). In particular, if X is nonsingular, $N \leq 2/3(m+n)$ and $\dim(X \cdot L) = m + n - N$, then $X \cdot L$ is a normal scheme.

Particularly important for applications is the case when L is a hyperplane. We formulate our results for this case specially.

COROLLARY 8. a) If the variety $X^n \subset P^N$ is normal and $N < 2n - b - 1$, where $b = \dim(\operatorname{Sing} X)$, then all hyperplane sections of X are reduced. In particular, for $N < 2n$ all hyperplane sections of a nonsingular variety are reduced.

b) If the variety $X^n \subset P^N$ has properties S_3 and R_{N-n+2} (the latter means that $N < 2n - b - 2$), then all hyperplane sections of X are normal (and consequently irreducible and reduced). In particular, if X is nonsingular and $N \leq 2n - 1$, then all hyperplane sections of X are normal.

Remark 7. We note that Corollary 7 gives considerably more precise information than theorems of Bertini type in which one speaks of generic hyperplane sections; however, as Examples 4 and 5 below show, in order that it be valid it is necessary to impose some conditions on the codimension of X in P^N.

Remark 8. If $K = C$, and $b = -1$, then the irreducibility of hyperplane sections of X follows from the Barth–Larsen theorem, according to which for $N < 2n - 1$, $\operatorname{Pic} X \cong Z$ is generated by the class of a hyperplane section of X (see [9]).

We give examples showing that the estimates in Corollary 8 cannot be improved.

Example 4. Let $X_0 = P^1 \times P^{n-b-1} \subset P^{2n-2b-1}$ and let $Y_0 = x \times P^{n-b-2} \subset X_0$ be a linear subspace. We denote by $X_1 \subset P^{2(n-b-1)}$ the section of X_0 by a generic hyperplane passing through Y_0. It is easy to see that X_1 is a nonsingular projectively normal variety. Let

$X^n \subset P^N$, $N = 2n - b - 1$ be a projective cone with vertex P^b and base X_1. It is clear that X is a normal variety where $\dim(\operatorname{Sing} X) = b$, so that X has properties S_2 and R_{N-n}. However, X has a nonreduced hyperplane section corresponding to a hyperplane in $P^{2n-2b-1}$, tangent to X_0 along Y_0 (see Example 2).

Example 5. Let $X_0 = P^1 \times P^{n-b-2} \subset P^{2n-2b-3}$ and let X be a projective cone with vertex P^b and base X_0. Then $X^n \subset P^N$, $N = 2n - b - 2$ is a Cohen–Macauley variety, where $\dim(\operatorname{Sing} X) = b$, so that X has properties S_3 and R_{N-n+1}. However for any hyperplane L such that $L^* \in X^* = X_0^*$, $L \cdot X = H_1 \cup H_2$ is reducible and consequently a nonnormal variety (here $\operatorname{Sing}(L \cdot X) = H_1 \cap H_2 = P^{n-2}$; see Example 2).

Let $X^n \subset P^N$ be a nondegenerate variety. For $n \leq m \leq N - 1$ we set

$$\mathcal{P}_m = \overline{\{(x,\ \alpha) \in \operatorname{Sm} X \times G(N,\ m) \mid L_\alpha \supset T_{X,x}\}},$$

where G(N, m) is the Grassmann manifold of m-dimensional linear subspaces of P^N, L_α is the linear subspace corresponding to the point $\alpha \in G(N, m)$, and the dash denotes the closure in $X \times G(N, m)$, and we denote by $p_m\colon \mathcal{P}_m \to X$ (respectively $\gamma_m\colon \mathcal{P}_m \to G(N, m)$) the map of projection to the first (respectively second) factor.

Definition 6. The map γ_m is called the m-th Gauss map, and its image $X_m^* = \gamma_m(\mathcal{P}_m)$ is called the variety of m-dimensional tangent subspaces to the variety X.

Remark 9. The two extremal cases deserve special attention: $m = n$ and $m = N - 1$. For $m = n$ we get essentially the usual Gauss map $\gamma\colon X \dashrightarrow G(N, n)$, and for $m = N - 1$, $X_{N-1}^* \subset P^{N*}$ is the dual variety.

Let $X^n \subset P^N$ be a nondegenerate variety, $\dim(\operatorname{Sing} X) = b \geq -1$.

THEOREM 5. a) For any point $\alpha \in \gamma_m(p_m^{-1}(\operatorname{Sm} X))$, $\dim \gamma_m^{-1}(\alpha) \leq m - n + b + 1$.

a') $\dim X_m^* \geq (m - n)(N - m - 2) + (m - b - 1)$.

b) For a generic point $\alpha \in X_m^*$, $\dim \gamma_m^{-1}(\alpha) \leq \max\{b + 1, m + n - N - 1\}$.

b') $\dim X_m^* \geq \min\{(m - n)(N - m) + (n - b - 1), (m - n + 1)(N - m) + 1\}$.

c) Let $\operatorname{char} K = 0$ and let $\gamma_m = \nu_m \circ \tilde{\gamma}_m$ be the Stein factorization. Then $\partial\alpha\nu_m$ is a birational isomorphism, and the generic fiber of the morphism γ_m (and $\tilde{\gamma}_m$) is a linear subspace of P^N of dimension $\dim \mathcal{P}_m - \dim X_m^*$.

Proof. Assertion a) follows quickly from Theorem 4, and assertion a') from a) (since $\dim \mathcal{P}_m = \dim X + \dim G(N - n - 1, m - n - 1) = n + (m - n)(N - m)$).

b) Let us first assume that $m = N - 1$. It is clear that $\dim \gamma_{N-1}^{-1}(\alpha) \leq n - 1$, and it suffices to verify that if $n - 1 \geq b + 2$, i.e., $n \geq b + 3$, then $\dim \gamma_{N-1}^{-1}(\alpha) \neq n - 1$. Let us assume the contrary, and let x be a generic point of X. Since $n - 1 > b + 1$, it follows from Theorem 4 that the system of divisors $Y_\alpha = p_{N-1}(\gamma_{N-1}^{-1}(\alpha))$, $\alpha \in (T_{X,x})^*$, is mobile and consequently $X = \overline{\bigcup_\alpha Y_\alpha}$, where α runs through the set of generic points of $(T_{X,x})^*$. Consequently, for generic points $y \in X$ there exists a hyperplane $\Lambda_y \subset (T_{X,x})^*$ such that for a generic point $\beta \in \Lambda_y$, $L_\beta \supset T_{X,y}$. But then $\langle T_{X,x}, T_{X,y}\rangle \subset (\Lambda_y)^* = P^{n+1}$, i.e., for a generic pair of points $x, y \in X$, $\dim(T_{X,x} \cap T_{X,y}) = n - 1$. It follows from this that either all n-dimensional linear spaces from $\gamma_n(X)$ are contained in some linear subspace $P^{n+1} \subset P^N$, or they all pass through a generic $(n - 1)$-dimensional linear subspace $P^{n-1} \subset P^N$. But in the first case X is a hyperplane and $\dim Y_\alpha = n - 1 \leq b + 1$ contrary to our assumption, and in the second case $X^* = P^{N-n}$, and it is easy to see that the intersection of X with a generic linear subspace $P^{N-n+1} \subset P^N$ is a nonsingular strange curve, so that $\operatorname{char} K = 2$, and X is a quadric, and again we arrive at a contradiction. Thus, assertion b) is valid in the case $m = N - 1$ (if $\operatorname{char} K = 0$, then the proof is noticeably simpler).

Now let us assume that assertion b) is proved for $m = k + 1$, and we prove it for $m = k$. It is clear that for generic points $\alpha_k \in X_k^*$, $\alpha_{k+1} \in X_{k+1}^*$, $\dim Y_{\alpha_k} \leq \dim Y_{\alpha_{k+1}}$. Hence, if $b + 1 \geq k + n - N$, then $\dim Y_{\alpha_k} \leq \dim Y_{\alpha_{k+1}} \leq b + 1$. Let us assume that $\dim Y_{\alpha_{k+1}} \leq k + n - N$, $k + n - N > b + 1$. If $\dim Y_{\alpha_k} < Y_{\alpha_{k+1}}$, then assertion b) is obviously valid. Let us assume the contrary. Then for a generic point $x \in X$ and a generic point $\alpha_{k+1} \in X_{k+1}^*$ such that $Y_{\alpha_{k+1}} \ni x$, each hyperplane in $L_{\alpha_{k+1}}$, containing $T_{X,x}$, is tangent to X at all nonsingular points on X of some $\dim Y_{\alpha_{k+1}}$-dimensional component of $Y_{\alpha_{k+1}}$, and it follows from Theorem

4 that $\dim Y_{\alpha_{k+1}} \le b + 1$. But then $\dim Y_{\alpha_k} = \dim Y_{\alpha_{k+1}} \le b + 1$, so that again $\dim Y_{\alpha_k} \le \max\{b + 1, k + n - N - 1\}$. Assertion b) is proved.

Assertion b') follows quickly from b).

c) Let α_m be a generic point of X_m^*. The linear subspace $L_{\alpha_m}^m \subset P^N$ is tangent to X at all points of the submanifold $Y_{\alpha_m} \cap \operatorname{Sm} X$, where $Y_{\alpha_m} = p_m(\gamma_m^{-1}(\alpha_m))$, while it is not hard to see that $Y_{\alpha_m} \cap \operatorname{Sm} X = \bigcap (Y_\alpha \cap \operatorname{Sm} X)$, where α runs through the set of points of X^* for which $L_\alpha \supset L_{\alpha_m}$. It follows from the classical reflexivity theorem of Segre (see, e.g., [8]) that if $\operatorname{char} K = 0$, then for a generic point $\alpha \in X^*$ the subvariety $Y_\alpha = p_{N-1}(\gamma_{N-1}^{-1}(\alpha))$ coincides with the linear subspace $(T_{X^*,\alpha})^*$. Consequently, $Y_{\alpha_m} = \overline{Y_{\alpha_m} \cap \operatorname{Sm} X} = \bigcap\limits_{L_\alpha \supset L_{\alpha_m}} (T_{X^*,\alpha})^*$ is also linear subspace of P^N. Since $\operatorname{char} K = 0$, the morphism γ_m is separable and consequently smooth at a generic point. Hence the field of functions $K(X_m^*)$ is algebraically closed in $K(\mathcal{P}_m)$ and ν_m is a birational isomorphism.

Theorem 5 is proved.

<u>COROLLARY 9</u>. If $\operatorname{char} K = 0$, $X^n \subset P^N$ is a nonsingular variety, $N - n + 1 \le m \le N - 1$, then a generic n-dimensional tangent subspace is tangent to X along a no more than $(m + n - N - 1)$-dimensional linear subspace (for $N \ge 2n$ this estimate is better than the estimate given in Theorem 4). For $n \le m \le N - n + 1$ a generic m-dimensional tangent subspace is tangent to X at a single unique point.

<u>COROLLARY 10</u>. Let $X^n \subset P^N$, $X^n \ne P^n$, $n^* = \dim X^*$, $b = \dim(\operatorname{Sing} X)$. Then $n^* \ge n - b - 1$. In particular, for a nonsingular manifold X, $n^* \ge n$. If $n \ge b + 3$, then $n^* \ge N - n + 1$ (this estimate is better than the preceding one for $N \ge 2n - b - 1$).

Both estimates in Corollary 10 are sharp: for example, for a Segre variety $X^n = P^1 \times P^{n-1} \subset P^{2n-1} = P^N$, $X^* \simeq X$, and $n^* = n = N - n + 1$.

<u>Remark 10</u>. If $\operatorname{char} K = 0$, $b = -1$, the inequality $n^* \ge N - n + 1$ was proved independently by Landman (see [7]). In this case another proof was given previously by the author (see [11], where the case $n = 2$ is considered; the general case is completely analogous).

<u>COROLLARY 11</u>. Let $X^n \subset P^N$, $X^n \ne P^n$, $b = \dim(\operatorname{Sing} X)$. Then $\dim \gamma_n(X) \ge n - b - 1$. In particular, for a nonsingular variety X^n, $\dim \gamma_n(X) = \dim X$ and γ_n is a finite morphism. If in addition $\operatorname{char} K = 0$, then γ_n is a birational isomorphism (i.e., γ_n is a normalization morphism).

<u>Remark 11</u>. When $K = C$, $b = -1$, Griffiths and Harris [3] proved that $\dim \gamma_n(X) = \dim X$. After the appearance of the announcement of the author's results [2], Ein [1] and Ran [10] gave different proofs of the finiteness of γ_n in this case. Our original proof of Corollary 11 (and also of the general Theorem 5) used methods of formal geometry. We give a sketch of this proof of Corollary 11.

It is clear that we can assume that $b = -1$. Let us assume that contrary to the assertion of Corollary 11, the n-dimensional subspace L, corresponding to the point $\alpha_L \in G(N, n)$, is tangent to X along an irreducible subvariety Y, $\dim Y > 0$, i.e., $Y \subset \gamma_n^{-1}(\alpha_L)$. Let $\mathfrak{X} = X_{/Y}$ be the completion of X along Y and let $\mathfrak{G} = \gamma_n(X)_{/\alpha_L}$ be a formal neighborhood of the point α_L in the projective variety $\gamma_n(X) \subset G(N, n)$. Since $\dim \gamma_n(X) > 0$ (because $X \ne P^n$) and $H^0(\mathfrak{G}, \mathcal{O}_{\mathfrak{G}}) \hookrightarrow H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an infinite-dimensional vector space over K. On the other hand, let $M \subset P^N$ be a linear subspace such that $\dim M = N - n - 1$, $L \cap M = \varnothing$, and let $\pi: X \to P^n$ be a projection with center in M. Then $\pi_{/Y}: \mathfrak{X} \to \mathbf{P}^n_{/\pi(Y)}$ is an isomorphism of formal spaces and consequently $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq H^0(\mathfrak{L}, \mathcal{O}_{\mathfrak{L}})$, where $\mathfrak{L} = L_{/Y} \simeq \mathbf{P}^n_{/\pi(Y)}$ is the completion of the subspace L along Y. But by a familiar theorem on formal functions [5] for $\dim Y > 0$ $H^0(\mathfrak{L}, \mathcal{O}_{\mathfrak{L}}) = K$. The contradiction found proves Corollary 11.

Besides the interpretation given in Definition 6, the Gauss map $\gamma_n: X \to G(N, n)$, where $X^n \subset P^N$, $X^n \ne P^n$, is a nonsingular variety, admits another interpretation. First of all, γ_n is the map corresponding to the vector bundle $\mathcal{N}(-1)$ (where $\mathcal{N}$ is the normal bundle to X in P^N) with distinguished $(N + 1)$-dimensional vector space of sections corresponding to points of K^{N+1} (where $P^N = (K^{N+1} \setminus 0)/K^*$).

Further, let $L \subset P^N$, $\lim L = N - n - 1$ be a generic linear subspace and let $\pi_L: X \to P^n$ be a projection with center in L. We denote by R_L the ramification divisor of the finite covering π_L, $R_L = \{x \in X \mid L \cap T_{X,x} \ne \varnothing\}$. The linear system $|R_L|$ generated by divisors R_L, $L \in G(N, N - n - 1)$ gives the Gauss map γ_n. Here $|R_L|$ has no basis points, but the ramification divisors R_L corresponding to all linear subspaces $L^{N-n-1} \subset P^N$ are the preimages of Schubert divisors on $G(N, n)$.

Proposition 3. The linear system $|R_L|$ is ample.

Proof. Proposition 3 follows quickly from Corollary 11 in view of [4, Chap. II, Cor. (6.6.3)].

Remark 12. In the case $\operatorname{char} K = 0$ Ein [1] proved that the complete linear system generated by ramification divisors is ample for an arbitrary nonsingular finite covering of P^n.

The exact sequences

$$0 \to \mathscr{T}_X \to \mathcal{O}^{N+1} \to \mathscr{N}(-1) \to 0,$$
$$0 \to \mathcal{O}_X(-1) \to \mathscr{T}_X \to \Theta_X(-1) \to 0,$$

where Θ_X is the tangent bundle to X, and $\mathscr{T}_X$ is the vector bundle of rank $n + 1$, the projectivizations of the fibers of which correspond naturally to the projective tangent spaces to X, show that $\gamma_n^*(\mathcal{O}_{G(N,n)}(1)) \simeq \det \mathscr{T}_X^* \simeq K_X(n+1) = K_X \otimes \mathcal{O}_X(n+1)$, where K_X is the canonical bundle. We note that precisely the fact that a section of the line bundle $K_X(n + 1)$ vanishes along R_L lies at the base of the classical definition of the canonical class. The next corollary follows quickly from Proposition 3.

COROLLARY 12. Let $X^n \subset P^N$, $X^n \neq P^n$ be a nonsingular variety. Then $K_X(n + 1)$ is an ample line bundle.

Remark 13. In fact, under the hypotheses of Corollary 12, the bundle $K_X(n + 1)$ is even very ample, at least if $\operatorname{char} K = 0$ (see [1]). This is easy to prove by induction on n, using the fact that for X there exist sufficiently many nonsingular hyperplane sections, while by Kodaira's vanishing theorem for such a section $H^{n-1} \subset X^n$ the complete linear system $|K_H + nH^2| = |K_X + (n + 1)H \cdot H|$ is cut out by the linear system $|K_X + (n + 1)H|$ (here K_H is the canonical class on H).

Proposition 4. Let $X^n \subset P^N$ be a nondegenerate variety, $b = \dim(\operatorname{Sing} X)$, $c = \operatorname{codim}_{P^N} X$ and let $Y^r \subset X$ be a subvariety of X for which $m - r = \operatorname{codim}_L Y < c = N - n$, where $L^m = \langle Y \rangle$ is the linear span of Y. Then $r \leqslant \min\left\{n-1, \left[\frac{N+b}{2}\right]\right\}$, where the square brackets denote the greatest integer in a number.

Proof. Without loss of generality we can assume that $Y \not\subset \operatorname{Sing} X$. It follows from the hypotheses that for an arbitrary point $y \in Y$, $\dim(T_{X,y} \cap L) \geq \dim T_{Y,y} \geq r$. Consequently,

$$\gamma_X(Y) = \overline{\gamma_X(Y \cap \operatorname{Sm} X)} \subset \{\alpha \in G(N, n) \mid \dim L_\alpha \cap L \geqslant r\} = S(L, r) \subset G(N, n),$$

where $S(L, r)$ is a Schubert cell. Since by hypothesis $m - r < N - n$, $n + m - r < N$, so that for any point $y \in Y \cap \operatorname{Sm} X$ there exists a hyperplane M, containing L and tangent to X at y. We set

$$S(M, L, r) = \{\alpha \in G(N, n) \mid L_\alpha \subset M, \ \dim L_\alpha \cap L \geqslant r\}.$$

Then $S(M, L, r) \subset S(L, r)$, $\dim S(M, L, r) = (r + 1)(m - 1) + (n - r)(N - n - 1)$, $\dim S(L, r) = (r + 1)(m - r) + (n - r)(N - n)$ and $\operatorname{codim}_{S(L,r)} S(M, L, r) = n - r = \operatorname{codim}_X Y$. Replacing r by $\min_{y \in Y}(\dim T_{X,y} \cap L)$ if necessary, we can assume that $\gamma_X(Y) \cap S(M, L, r) \cap \operatorname{Sm}(S(L, r)) \neq \varnothing$. Then

$$\dim(\gamma_X(Y) \cap S(M, L, r)) \geqslant \dim \gamma_X(Y) - \operatorname{codim}_{S(L,r)} S(M, L, r) = (r - f) - (n - r) = 2r - n - f,$$

where f is the dimension of a generic fiber of $\gamma_X|_Y$. On the other hand,

$$\gamma_X(Y) \cap S(M, L, r) = \overline{\gamma_X(\{y \in Y \cap \operatorname{Sm} X \mid T_{X,y} \subset M\})},$$

and it follows from Theorem 4 that

$$\dim(\gamma_X(Y) \cap S(M, L, r)) \leqslant N - n + b - f.$$

Combining the last two formulas, we see that $2r - n - f \leq N - n + b - f$, i.e, $r \leqslant \left[\frac{N+b}{2}\right]$, which is what was required.

Remark 14. For K = C, b = −1 Proposition 4 can also be derived from the Barth–Larsen theorem [9].

Remark 15. It is easy to construct examples showing that the estimate in Proposition 4 is sharp (singular varieties for which equality holds in Theorem 6 can be constructed as cones with vertex P^b over nonsingular ones).

Remark 16. It is interesting to compare Proposition 4 with the famous classical result (first proved rigorously by Lewis apparently), in which conditions are not imposed on r, but instead it is assumed that L is a generic linear subspace.

COROLLARY 13. If $X \neq P^n$, then X does not contain linear subspaces of dimension greater than $\left[\frac{N+b}{2}\right]$. If X is not a hypersurface, then X does not contain projective hypersurfaces of dimension greater than $\left[\frac{N+b}{2}\right]$.

LITERATURE CITED

1. L. Ein, "The ramification divisors for branched coverings of P_k^n," Math. Ann., 261, 483-485 (1982).
2. W. Fulton and R. Lazarsfeld, "Connectivity and its applications in algebraic geometry," Lect. Notes Math., 862, 26-92 (1981).
3. P. Griffiths and J. Harris, "Algebraic geometry and local differential geometry," Ann. Sci. Ec. Norm. Sup., 12, 355-432 (1979).
4. A. Grothendieck, Elements de Geometrie Algebrique, Chap. II, Publ. Math. IHES, No. 8, Chap. IV_2 (1961); Elements de Geometrie Algebrique, Chap. II, Publ. Math. IHES, No. 24 (1965).
5. H. Hironaka and H. Matsumura, "Formal functions and formal embeddings," J. Math. Soc. Jpn., 20, 52-82 (1968).
6. K. Johnson, "Immersion and embedding of projective varieties," Acta Math., 140, 49-74 (1978).
7. S. L. Kleiman, "Numerical theory of singularities," Usp. Mat. Nauk, 35, No. 6, 69-148 (1980).
8. S. L. Kleiman, "Concerning the dual variety," in: Proc. 18th Scand. Congress Math. Aarhus, Birkhäuser Verlag, Basel–Boston–Stuttgart (1981), pp. 386-396.
9. M. E. Larsen, "On the topology of complex projective manifolds," Invent. Math., 19, 251-260 (1972).
10. Z. Ran, "The structure of Gausslike maps," Compos. Math., 52, 171-177 (1984).
11. F. L. Zak, "Surfaces with null Lefschetz cycles," Mat. Zametki, 13, 869-880 (1973).
12. F. L. Zak, "Projections of algebraic varieties," Mat. Sb., 116, 593-602 (1981).