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DETERMINANTS OF PROJECTIVE VARIETIES AND THEIR DEGREES

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To Christian Peskine, one of the few who still value elegance in mathematics — and in life

Introduction

Consider the vector space of square matrices of order r and the corresponding projective space $\mathbb{P} = \mathbb{P}^{r^2-1}$. The points of $\mathbb{P}$ are in a one-to-one correspondence with the square matrices modulo multiplication by a nonzero constant. Consider the Segre subvariety $X = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}$ corresponding to the matrices of rank one and a filtration

$$X \subset X^2 \subset \cdots \subset X^{r-1} \subset X^r = \mathbb{P},$$

where, for $1 \leq i \leq r$, X^i denotes the i-th join of X with itself. We recall that by definition X^i (also called the $(i-1)$-st secant variety of X) is the closure of the subvariety of $\mathbb{P}$ swept out by the linear spans of general collections of i points of X. Thus in our case X^i corresponds to the cone of matrices whose rank does not exceed i. In other words, if M is a matrix and $z_M \in \mathbb{P}$ is the corresponding point, then $\operatorname{rk} M = \operatorname{rk}_X z_M$, where for $z \in \mathbb{P}$

$$\operatorname{rk}_X z = \min\{i \mid z \in X^i\}.$$

It is clear that $X^{r-1} \subset \mathbb{P}$ is the hypersurface of degree r defined by vanishing of determinant, which gives a method to define determinant (up to multiplication by a nonzero constant) in purely geometric terms.

Alternatively, one can consider the dual variety $X^* \subset \mathbb{P}^*$ whose points correspond to hyperplanes tangent to X. It is not hard to see that in the case

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when $X = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$ the variety X^* is a hypersurface of degree r in the dual space $\mathbb{P}^*$ which is also defined by vanishing of determinant. This gives another way to define determinant in geometric terms.

It is tempting to study similar notions and interrelations between them for arbitrary projective varieties. The idea is to associate, in a natural way, to any projective variety X a hypersurface (or at least a variety of small codimension) X^{ass} from which one should be able to reconstruct X (or at least some of its essential features). Then geometric and numerical invariants of X^{ass} would yield important information on X itself. On the other hand, varieties of low codimension (and particularly hypersurfaces) are easier to study from algebraic and analytic points of view. A classical way to realize this idea is to consider generic projections, and, indeed, invariants of multiple loci of such projections provide a useful tool for the study of projective varieties. However, generic projections are not canonical, and they preserve the dimension of variety while changing the dimension of the ambient linear space. In contrast to that, the above two constructions of hypersurfaces associated to the Segre variety $X = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$ are well defined and preserve the dimension of the ambient space (we recall that $X^{\mathrm{ass}} = X^{r-1}$ for the first construction and $X^{\mathrm{ass}} = X^*$ for the second one). These constructions generalize to arbitrary projective varieties, and in the present paper we study these generalizations of determinant called respectively *join determinant* and *discriminant*. In particular, we give lower bounds for the degree of associated varieties in these two cases.

As we already observed, in the above examples the degree of associated hypersurface is equal to the order of matrix, and it turns out that, if we define the *order* of an arbitrary nondegenerate variety $X \subset \mathbb{P}^N$ by the formula $\operatorname{ord} X = \operatorname{rk}_X z$, where $z \in \mathbb{P}^N$ is a general point (cf. Definition 1.3), then the degree of associated variety (or determinant) in the above two senses is at least $\operatorname{ord} X$, so that $\operatorname{ord} X$ is the lowest possible value of degree of determinant. Furthermore, for the varieties on the boundary associated variety is a hypersurface, and, in the case of discriminant, we give a complete classification of varieties for which $\deg X^* = \operatorname{ord} X$. It seems that the lower the degree of "determinant" the more the points of the ambient space resemble matrices; in particular, the corresponding varieties tend to be homogeneous, which can be viewed as a generalization of multiplicativity of matrices.

Even though the notion of order of projective variety is quite natural, it is not widely used. We also give another lower bound for the degree of discriminant in terms of dimension and codimension. This bound is also sharp and, rather surprisingly, the varieties on the boundary seem to be the same as for the bound in terms of order. This second approach is based on a study of Hessian matrices of homogeneous polynomials, and, although this topic apparently pertains to pure algebra, it has numerous classical and modern links with fields ranging from differential equations and differential geometry to approximation theory and mathematical physics.

The paper is organized as follows. Section 1 is devoted to join determinants. We obtain a lower bound for their degree in terms of order and consider numerous examples. In section 2 we study dual varieties (discriminants) and obtain a lower bound for their degree (called *codegree*) in terms of order. We also consider various examples and classify varieties on the boundary. To put the problem in a proper perspective, in section 3 (which is of an expository nature) we collect various known results on varieties of small degree and codegree. In section 4 we study Jacobian linear systems and Hessian matrices. As an application, we obtain a lower bound for codegree in terms of dimension and codimension and consider varieties for which this bound is sharp.

To avoid unnecessary complications, throughout the note we deal with algebraic varieties over the field $\mathbb{C}$ of complex numbers.

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1 Join Determinants

Let $X \subset \mathbb{P}^N$, $\dim X = n$ be a nondegenerate projective variety (i.e., X is not contained in a hyperplane).

For an integer $k \geq 1$ we put

$$\widetilde{X^k} = \overline{\{(x_1, \ldots, x_k, u) \in \underbrace{X \times \cdots \times X}_{k} \times \mathbb{P}^N \mid \dim\langle x_1, \ldots, x_k\rangle = k-1, u \in \langle x_1, \ldots, x_k\rangle\}},$$

where, for a subset $A \subset \mathbb{P}^N$, we denote by $\langle A\rangle$ the linear span of A in $\mathbb{P}^N$ and bar denotes projective closure. We denote by φ^k the projection of $\widetilde{X^k}$ to $\mathbb{P}^N$.

Definition 1.1. The variety $X^k = \varphi^k(\widetilde{X^k})$ is called the *k-th join of X with itself.*

It is clear that if $\mathbb{P}^N = \mathbb{P}(V)$, where V is a vector space of dimension $N+1$, and $C_X \subset V$ is the cone corresponding to X, then X^k is the variety corresponding to the cone $\underbrace{C_X + \cdots + C_X}_{k}$. Furthermore, X^k is the join of X with X^{k-1} and $X^{k+1} = X^k$ if and only if $X^k = \mathbb{P}^N$.

It is often necessary to compute tangent spaces to joins.

Proposition 1.2. (Terracini lemma)

(a) *Let $x_1, \dots, x_k \in X$, and let $u \in \langle x_1, \dots, x_k \rangle \subset X^k$. Then*

$$T_{X^k,u} \supset \langle T_{X,x_1}, \dots, T_{X,x_k} \rangle,$$

where $T_{X,x}$ (resp. $T_{X^k,u}$) *is the tangent space to the variety X at the point* x (resp. to the variety X^k at the point u).

(b) *If $x_1, \dots, x_k$ is a general collection of points of X, and u is a general point of the linear subspace $\langle x_1, \dots, x_k \rangle$, then*

$$T_{X^k,u} = \langle T_{X,x_1}, \dots, T_{X,x_k} \rangle.$$

Proof. This is a special case of [Z2, Chapter V, Proposition 1.4].

Definition 1.3. For a point $z \in \mathbb{P}^N$ we put $\operatorname{rk}_X z = \min\{k \mid z \in X^k\}$. The number $\operatorname{rk}_X z$ is called the *rank of z with respect to the variety X*.

The number $\operatorname{ord} X = \min\{k \mid X^k = \mathbb{P}^N\} = \max\limits_{z \in \mathbb{P}^N}\{\operatorname{rk}_X z\}$ is called the *order* of the variety X.

The difference $\operatorname{cork}_X z = \operatorname{ord} X - \operatorname{rk} z$ is called the *corank of z with respect to the variety X*.

Thus $1 \le \operatorname{rk}_X z \le \operatorname{ord} X$, $\operatorname{rk}_X x = 1$ if and only if $x \in X$, $\operatorname{rk}_X z = \operatorname{ord} X$ for a general point $z \in \mathbb{P}^N$ ($z \notin X^{\operatorname{ord} X - 1}$), and we get a strictly ascending filtration

$$X \subset X^2 \cdots \subset X^{\operatorname{ord} X - 1} = X^J \subset X^{\operatorname{ord} X} = \mathbb{P}^N. \tag{1.3.1}$$

Definition 1.4. The filtration (1.3.1) is called the *rank filtration*, and the variety $X^J = X^{\operatorname{ord} X - 1}$ the *join determinant* of X.

The number $\operatorname{codim}_{\mathbb{P}^N} X^J - 1$ is called the *join defect* of X and is denoted by $\operatorname{jodef} X$; it is clear that $0 \le \operatorname{jodef} X \le \dim X = n$.

The degree $\deg X^J$ of the join determinant is called the *jodegree* of X and is denoted by $\operatorname{jodeg} X$.

Examples 1.5. 1) If $X = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}$, then one gets the standard notions of order and rank discussed above. In particular, $\operatorname{ord} X = r$, X^J is the locus of degenerate matrices, $\operatorname{jodeg} X = \operatorname{ord} X$, and X^J is defined by vanishing of determinant.

2) If $X = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \subset \mathbb{P}^{ab-1}$, $a \le b$ and $z_M \in \mathbb{P}^{ab-1}$ is the point corresponding to an $(a \times b)$-matrix M, then $\operatorname{rk} z_M = \operatorname{rk} M$. Furthermore, $\operatorname{ord} X = a$ and $\operatorname{codim}_{\mathbb{P}^{ab-1}} X^J = b - a + 1$. By a formula due to Giambelli, $\operatorname{jodeg} X = \frac{b!}{(a-1)!(b-a+1)!}$ (cf. [Ful, 14.4.14]), which is much larger than $a = \operatorname{ord} X$ if $b \neq a$.

3) Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, so that the ambient $\mathbb{P}^7$ can be interpreted as the space of cubic 2-matrices up to multiplication by a nonzero constant. Then $X^J = X$, $\operatorname{ord} X = 2$ and $\operatorname{jodeg} X = \deg X = 6$.
4) Let $C = v_m(\mathbb{P}^1) \subset \mathbb{P}^m$ be a rational normal curve. The points of the ambient space $\mathbb{P}^m$ can be interpreted as binary forms of degree m modulo multiplication by a nonzero constant. Our definition of rank again coincides with the usual definition for binary forms. Furthermore, $\operatorname{ord} C = \left[\frac{m+2}{2}\right]$, where brackets denote integral part,

$$\operatorname{codim}_{\mathbb{P}^m} C^J = \begin{cases} 1, & m \equiv 0 \pmod 2, \\ 2, & m \equiv 1 \pmod 2 \end{cases}$$

and

$$\operatorname{jodeg} C = \begin{cases} \operatorname{ord} C = \left[\frac{m+2}{2}\right], & m \equiv 0 \pmod 2, \\ \frac{(m+1)(m+3)}{8}, & m \equiv 1 \pmod 2 \end{cases}$$

(this can be easily computed basing on Sylvester's theory of binary forms; cf. [Syl]). Thus the jodegree of C is equal to the order of C for m even and is much larger than the order for m odd.
5) Let $X = v_2(\mathbb{P}^{r-1}) \subset \mathbb{P}^{\frac{r(r+1)}{2}-1}$ be the Veronese variety. The points of the ambient linear space correspond to symmetric matrices of order r, and if M is a matrix and $z_M \in \mathbb{P}^{\frac{r(r+1)}{2}-1}$ is the corresponding point, then $\operatorname{rk} z_M = \operatorname{rk} M$. Here the variety X is a (special) linear section of the Segre variety from 1), and, as in the case 1), $\operatorname{ord} X = r$ and X^J is the hypersurface of degree r defined by vanishing of determinant.
6) Let $X = G(r-1,1) \subset \mathbb{P}^{\binom{r}{2}-1}$ be the Grassmann variety of lines in $\mathbb{P}^{r-1}$. Then $n = \dim X = 2(r-2)$ and the points of $\mathbb{P}^{\binom{r}{2}-1}$ correspond to skew-symmetric matrices of order r. If M is such a matrix and $z_M \in \mathbb{P}^{\binom{r}{2}-1}$ is the corresponding point, then $\operatorname{rk} z_M$ is equal to $\frac{1}{2} \operatorname{rk} M$ (i.e., to the pfaffian rank of M). Furthermore, $\operatorname{ord} X = \left[\frac{r}{2}\right] = \left[\frac{n}{4}\right] + 1$,

$$\operatorname{codim}_{\mathbb{P}^{\binom{r}{2}-1}} X^J = \begin{cases} 1, & r \equiv 0 \pmod 2, \\ 3, & r \equiv 1 \pmod 2 \end{cases}$$

and

$$\operatorname{jodeg} X = \begin{cases} \operatorname{ord} X = \frac{r}{2}, & r \equiv 0 \pmod 2, \\ \frac{1}{4}\binom{r+1}{3}, & r \equiv 1 \pmod 2 \end{cases}$$

(cf. [HT]). Thus the jodegree of X is equal to $\operatorname{ord} X$ for r even and is much larger than $\operatorname{ord} X$ for r odd. In this case X is a (special) linear section of the Segre variety from 1), and X^J corresponds to the skew-symmetric matrices whose rank is less than maximal. In particular, for r even the hypersurface X^J is defined by vanishing of Pfaffian.

Theorem 1.6. *Let $X \subset \mathbb{P}^N$ be a nondegenerate variety. Then* $\operatorname{jodeg} X \geq \operatorname{ord} X + \operatorname{jodef} X \geq \operatorname{ord} X$. *In particular, the jodegree is not less than the order, and equality is possible only if X^J is a hypersurface.*

Sketch of proof. Let $\{x_1, \dots, x_{\operatorname{ord} X}\}$ be a general collection of points of X. By the definition of $\operatorname{ord} X$, the $(\operatorname{ord} X - 1)$-dimensional linear space $\langle x_1, \dots, x_{\operatorname{ord} X}\rangle$ does not lie in X^J, but contains $\operatorname{ord} X$ linear subspaces of the form $\langle x_{i_1}, \dots, x_{i_{\operatorname{ord} X-1}}\rangle$, $\dim \langle x_{i_1}, \dots, x_{i_{\operatorname{ord} X-1}}\rangle = \operatorname{ord} X - 2$, $i_1 < \dots < i_{\operatorname{ord} X-1}$. Thus a general line $l \subset \langle x_1, \dots, x_{\operatorname{ord} X}\rangle$ is not contained in X^J, but meets it in at least $\operatorname{ord} X$ points. Furthermore, adding points, it is easy to construct a linear subspace $L \subset \mathbb{P}^N$, $\dim L = \operatorname{codim}_{\mathbb{P}^N} X^J = \operatorname{jodef} X + 1$ which is not contained in X^J, but meets it in at least $\operatorname{ord} X + \operatorname{jodef} X$ points. Thus $\deg X^J \geq \operatorname{ord} X + \operatorname{jodef} X$, and our claim follows. □

Remarks 1.7. (i) The same argument shows that

$$\deg X^k \geq k + \operatorname{codim}_{\mathbb{P}^N} X^k \tag{1.7.1}$$

usual bound for degree; cf. Theorem 3.4, (i)).

(ii) For $1 \leq k < \operatorname{ord} X$, the variety X^k is not contained in a hypersurface of degree k or less. In fact, suppose that $X^k \subset W$, where $W \subset \mathbb{P}^N$ is a hypersurface. Arguing as in the proof of Theorem 1.6, we see that, for a general collection $\{x_1, \dots, x_{k+1}\}$ of points of X, the intersection $\langle x_1, \dots, x_{k+1}\rangle \cap X^k$ contains $k+1$ hyperplanes of the form $\langle x_{i_1}, \dots, x_{i_k}\rangle$, $i_1 < \dots < i_k$. Then either $\deg W = \deg W \cap \langle x_1, \dots, x_{k+1}\rangle \geq k+1$ or $W \supset \langle x_1, \dots, x_{k+1}\rangle$. In the first case we are done, and in the second case $W \supset X^{k+1}$, so that we arrive at a contradiction by induction.

(iii) Arguing as in the proof of Theorem 1.6, one can show that, for an arbitrary point $z \in X^k$, one has

$$\operatorname{mult}_z X^k \geq k - \operatorname{rk} z + 1, \tag{1.7.2}$$

where $\operatorname{mult}_z X^k$ denotes the multiplicity of X^k at the point z (compare with the bound for multiplicity of points of dual varieties given in Section 2 (cf. (2.11.1)). We observe that, for $z \in X$, (iii) implies (i). For $k = \operatorname{ord} X - 1$ the inequality (1.7.2) assumes the form

$$\operatorname{mult}_z X^J \geq \operatorname{cork}_X z, \qquad z \in X^J. \tag{1.7.3}$$

This bound (which should be compared with the bound for multiplicity in Proposition 2.9) can be used to give a different proof of Theorem 1.6 (compare with the use of Proposition 2.8 in the proof of Theorem 2.7 below).

(iv) Since any variety of degree d and codimension e is contained in a hypersurface of degree $d - e + 1$ (and is in fact a set theoretic intersection of such hypersurfaces; to see this it suffices to project the variety from the linear span of a general collection of $e - 1$ points on it and take the cone over the image), (ii) also yields Theorem 1.6 and, more generally, (i).

(v) Theorem 1.6 and the remarks thereafter were known to the author for many years. Proofs of some special cases can also be found in [Ge, Lecture 7] and [C-J].

Theorem 1.6 gives a nice bound for jodegree in terms of order, but the notion of order might seem a bit unusual. So, it is desirable to give a bound in more usual terms, such as dimension and codimension. One has the following lower bound for order.

Proposition 1.8. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety. Then* $\operatorname{ord} X \geq \dfrac{N - \operatorname{jodef} X}{n+1} + 1 \geq \dfrac{N+1}{n+1}$.

Proof. In fact, for each natural k, $\dim X^k \leq \dim \widetilde{X^k} = kn+k-1$. In particular,

$$N - \operatorname{jodef} X - 1 = \dim X^J \leq (\operatorname{ord} X - 1)(n+1) - 1,$$

which yields the proposition. □

In the case when $C \subset \mathbb{P}^N$ is a curve, one always has $\operatorname{ord} C = \left[\frac{N+2}{2}\right]$, where brackets denote integral part, $\dim C^i = 2i-1$, $1 \leq i \leq \operatorname{ord} C - 1$ and

$$\operatorname{jodef} C = \begin{cases} 0, & N \equiv 0 \pmod 2, \\ 1, & N \equiv 1 \pmod 2 \end{cases}$$

(cf., e.g., [Z2, chapter V, Example 1.6]). Thus the bound for order given in Proposition 1.8 is sharp in this case.

Theorem 1.9. *Let $C \subset \mathbb{P}^N$ be a nondegenerate curve. Then* $\operatorname{ord} C = \left[\frac{N+2}{2}\right]$ *and* $\operatorname{jodeg} C \geq \operatorname{ord} C$. *Furthermore,* $\operatorname{jodeg} C = \operatorname{ord} C$ *if and only if N is even and C is a rational normal curve* (cf. Example 1.5, 4).

Sketch of proof. The first claim follows from Theorem 1.6 in view of the computation of $\operatorname{codim}_{\mathbb{P}^m} C^J$ made above.

From Proposition 1.2 and the trisecant lemma (cf. [HR, 2.5], [Ha, Chapter IV, §3] or [Mum, §7 B]) it is easy to deduce that the map $\varphi^i : \widetilde{C^i} \to C^i$ is birational for $i < \operatorname{ord} C$. Suppose that $\operatorname{jodeg} C = \operatorname{ord} C$. Then, as we already observed, $N = 2k$, where k is a natural number, and $\operatorname{ord} C = \left[\frac{N+2}{2}\right] = k+1$. From Theorem 1.6 and Remark 1.7 it follows that $\operatorname{mult}_x C^J \geq k$ for each point $x \in C$. Thus, projecting C from the point x to $\mathbb{P}^{2k-1}$, we get a curve $C' \subset \mathbb{P}^{2k-1}$ such that $\deg C' = \deg C - 1$, $\operatorname{ord} C' = k$, and the projection $C^k \dashrightarrow C'^k = \mathbb{P}^{2k-1}$ is birational. Hence the map $\varphi'^k : \widetilde{C'^k} \to C'^k = \mathbb{P}^{2k-1}$ is also birational. Thus to prove our claim it suffices to show that any curve $C' \subset \mathbb{P}^{2k-1}$ with this property is a normal rational curve.

To this end, we take a general point $z \in C'^{k-1}$ and consider the projection $\pi_z : C' \to \mathbb{P}^1$ with center at the linear subspace $T_{C'^{k-1},z}$. From Proposition 1.2 it easily follows that $\deg \pi_z = \deg C' - 2(k-1)$, and so it suffices to verify that the map π_z is an isomorphism. If this were not so, then the map π_z would be ramified and there would exist a point $x \in C$ such that $T_{C,x} \cap T_{C'^{k-1},z} \neq \varnothing$. By the Terracini lemma, the line $\langle x, z \rangle$ lies in the branch locus of the map $\varphi'^k : \widetilde{C'^k} \to \mathbb{P}^{2k-1}$. Varying z in C'^{k-1}, it is easy to see that the branch locus is a hypersurface in $\mathbb{P}^{2k-1}$. The proof is completed by recalling that φ'^k is birational, and so in our case the branch locus has codimension at least two (cf. also [C-J]). □

For higher dimensions there is little chance to obtain a classification of varieties for which the inequality in Theorem 1.6 turns into equality (some of these varieties were listed in Examples 1.5). One of the reasons is that variety X is not uniquely determined by its join determinant X^J. Here follows a typical example of this phenomenon.

Example 1.10. Let $v_2(\mathbb{P}^2) \subset \mathbb{P}^5 \subset \mathbb{P}^6$ be the Veronese surface (cf. Example 1.5, 5) for $r = 3$), let $y \in \mathbb{P}^6$ be a point, let $Y \subset \mathbb{P}^6$ be the cone over $v_2(\mathbb{P}^2)$ with vertex y, and let X be the intersection of Y with a general hypersurface of degree $d \geq 2$. Then $X \subset Y \subset \mathbb{P}^6$ is a nondegenerate surface, $\dim X^2 = 5 = \dim Y^2$, and therefore $X^J = Y^J$ and $\operatorname{jodeg} X = \operatorname{ord} X = \operatorname{ord} Y = \operatorname{jodeg} Y = 3$.

Thus a lot of different surfaces have the same order and the same join determinant.

Remark 1.11. Of course, one can construct similar examples starting from other varieties whose higher self-joins have dimension smaller than expected (cf., e.g., Examples 1.5). Thus, in the case of Segre variety $Y = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$ or Grassmann variety $Y = G(r-1, 1)$ with r even (Examples 1.5, 1), 6)) any general subvariety $X \subset Y$ of sufficiently small codimension has the same order and join determinant as Y. This leads to the notion of *constrained varieties* (cf. [Å]) which deserves to be studied in detail. It should be possible to classify "maximal" (nonconstrained) varieties with jodegree equal to order; then arbitrary varieties with this property should be (constrained) subvarieties of the maximal ones.

It should be noted however that even the lists of nonconstrained varieties of a fixed jodegree are rather big. For example, in view of the above observation, the list of nonconstrained varieties of jodegree three is much longer than the corresponding lists of varieties of degree three (cf. Theorem 3.3, (iii)) or codegree three (cf. Theorem 3.5, (iii)). Still, imposing additional natural conditions, such as inextensibility, can help to make this list shorter (cf. also Remark 2.13, (ii)).

Remark 1.12. A nice series of examples of varieties of small jodegree illustrating Remark 1.11 can be constructed as follows. Let $Y \subset \mathbb{P}^N$ be a nondegenerate variety, and let $X = v_2(Y)$ be the quadratic Veronese reembedding of Y. Then $\operatorname{ord} X \geq N + 1$ (cf. [Z3, Theorem 2.7]), and in [Z3] we construct numerous examples of varieties for which $\operatorname{ord} X = N+1$ and $X^N = Z^N \cap \langle X \rangle$, where $Z = v_2(\mathbb{P}^N)$ and $\langle X\rangle$ is the linear span of X. Since Z^N is a hypersurface of degree $N+1$, for such a variety one has $\operatorname{jodeg} X = \operatorname{ord} X$, so that X is on the boundary of Theorem 1.6.

Example 1.13. A well known example highlighting Remark 1.12 is due to Clebsch (cf. [Cl] and [Z3, Remark 6.2]). Let $X = v_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$. Then $\operatorname{jodeg} X = \operatorname{ord} X = 6$, and the join determinant of X is the locus of quartics representable as a sum of at most five bisquares.

2 Discriminants

Let $X \subset \mathbb{P}^N$ be a projective variety. Consider the subvariety

$$\mathcal{P}_X = \overline{\{(x,\alpha) \mid x \in \operatorname{Sm} X, L_\alpha \supset T_{X,x}\}} \subset X \times \mathbb{P}^{N*},$$

where $\operatorname{Sm} X$ denotes the subset of nonsingular points of X, $L_\alpha \subset \mathbb{P}^N$ is the hyperplane corresponding to a point $\alpha \in \mathbb{P}^{N*}$, $T_{X,x}$ is the (embedded) tangent space to X at x, and bar denotes the Zariski closure. Let $p : \mathcal{P}_X \to X$ and $\pi : \mathcal{P}_X \to \mathbb{P}^{N*}$ be the projections onto the two factors.

Definition 2.1. $\mathcal{P}_X$ is called the *conormal variety* of X, and $X^* = \pi(\mathcal{P}_X) \subset \mathbb{P}^{N*}$ is called the *dual variety* or the *discriminant* of X.

In most cases one can assume that X is nondegenerate because if $X \subset L$, where $L \subset \mathbb{P}^N$ is a linear subspace and X_L^* is the dual of X viewed as a subvariety of L, then X^* is the cone over X_L^* with vertex at the linear subspace ${}^\perp L \subset \mathbb{P}^{N*}$ of all the hyperplanes in $\mathbb{P}^N$ passing through L. Conversely, if $X \subset \mathbb{P}^N$ is a cone with vertex L over a variety $Y \subset \mathbb{P}^M$, $M = N - \dim L - 1$, then $X^* = Y^* \subset \mathbb{P}^{M*} = {}^\perp L \subset \mathbb{P}^{N*}$.

Unlike join determinant, discriminant allows to reconstruct the original variety, viz. one has the following

Theorem 2.2. $X^{**} = X$, *i.e., projective varieties are reflexive with respect to the notion of duality introduced in Definition* 2.1.

We refer to [Tev] for this and other results on dual varieties that are used in this paper.

Typically, for a nonsingular variety X, the dual variety is a hypersurface (in view of Theorem 2.2, one cannot expect this to be true for arbitrary varieties).

Definition 2.3. The number $\operatorname{codim}_{\mathbb{P}^{N*}} X^* - 1$ is called the (dual) *defect* of X and is denoted by $\operatorname{def} X$.

The degree of the dual variety X^* is called the *codegree* or (if $\operatorname{def} X = 0$) *class* of X and is denoted by $\operatorname{codeg} X$ (thus $\operatorname{codeg} X = \deg X^*$).

Proposition 2.4. *Let $X \subset \mathbb{P}^N$ be a projective variety, let $L \subset \mathbb{P}^N$ be a general linear subspace, $L \cap X = \varnothing$, let $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$, $M = N - \dim L - 1$ be the projection with center at L, and let $Y = \pi_L(X) \subset \mathbb{P}^M$. Then $Y^* = X^* \cap {}^\perp L$, where ${}^\perp L \subset \mathbb{P}^{N*}$ parameterizes the hyperplanes in $\mathbb{P}^N$ passing through L and thus is naturally isomorphic to $\mathbb{P}^{M*}$.*

In the case when $\operatorname{def} X > 0$ one can apply Proposition 2.4 (combined with Theorem 2.2) to X^* to show that the dual of a general linear section of X is a general projection of X^*. This yields a reduction to the case of varieties of defect zero (of course, the degree of X^* is stable under general projections).

The Terracini lemma 1.2 yields useful information on the structure of X^*, viz. one gets a descending filtration

$$X^* \supset (X^2)^* \supset \cdots \supset (X^{\operatorname{ord} X - 1})^* \tag{2.4.1}$$

corresponding to the ascending filtration!(1.3.1) (cf. Proposition 4.2 below for a more precise statement).

Definition 2.5. The filtration (2.4.1) is called the *corank filtration*.

Consider the varieties described in Examples 1.5 from the point of view of structure of their duals.

Examples 2.6. 1) If $X = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}$, then X^* is the hypersurface defined by vanishing of determinant. Hence $\operatorname{codeg} X = r = \operatorname{jodeg} X = \operatorname{ord} X$. In this case (2.4.1) is just the filtration by the corank of matrix.

2) If $X = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \subset \mathbb{P}^{ab-1}$, $a \le b$, then X^* corresponds to matrices of rank smaller than a in the dual space, $\operatorname{def} X = b - a$, X^* is projectively isomorphic to X^J and, as we saw in Example 1.5, 2), $\operatorname{ord} X = a$ and $\operatorname{codeg} X = \frac{b!}{(a-1)!(b-a+1)!}$ (cf. [Ful, 14.4.14]), which is larger than $a = \operatorname{ord} X$ if $b \ne a$. Here again (2.4.1) is the filtration by corank.

3) Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ (cubic $(2 \times 2 \times 2)$-matrices). Then it is easy to see that $\operatorname{def} X = 0$ and X^* is a hypersurface of degree four (we recall that $\operatorname{jodeg} X = 6$; cf. Example 1.5, 3)). The discriminant (or hyperdeterminant) in this case was first computed by Cayley [Ca]. The variety X is a special case of symmetric Legendrean varieties all of which have codegree four (cf. [Mu], [LM] and Remark 3.6 below). More generally, if $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \subset \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1}$, $n_1 \le n_2 \le \cdots \le n_k$, then it is easy to see that $\operatorname{def} X = \max\{n_k - \sum_{i=1}^{k-1} n_i, 0\}$, and one can use tools from combinatorics to compute the codegree (cf. [GKZ, Chapter 14B] for the case of $\operatorname{def} X = 0$), which, in our opinion, does not make much sense; anyhow, the codegree is very large.

4) Let $C = v_m(\mathbb{P}^1) \subset \mathbb{P}^m$ be a rational normal curve. Then C^* is the hypersurface in $\mathbb{P}^{m*}$ swept out by the osculating $(m-2)$-spaces to the curve in $\mathbb{P}^{m*}$ parameterizing the osculating hyperplanes to C and defined by vanishing of discriminant of binary form. Thus, for $m > 2$, $\operatorname{codeg} C = 2m-2$ is

larger than $\operatorname{ord} C = \left[\frac{m+2}{2}\right]$, but (2.4.1) is again the filtration by the corank of binary form. It should be noted that in this case $\operatorname{codeg} C > \operatorname{jodeg} C$ for $m > 2$ even while $\operatorname{codeg} C < \operatorname{jodeg} C$ for $m > 3$ odd (cf. Example 1.5, 4)).

5) If $X = v_2(\mathbb{P}^{r-1}) \subset \mathbb{P}^{\frac{r(r+1)}{2}-1}$ is a Veronese variety, then X^* is the hypersurface of degenerate matrices in the dual space of symmetric matrices of order r defined by vanishing of determinant; in particular, (2.4.1) is the filtration by corank and $\operatorname{codeg} X = \operatorname{ord} X$. Furthermore, the dual variety X^* is projectively isomorphic to X^J and, in particular, $\operatorname{codeg} X = \operatorname{jodeg} X$.

6) Let $X = G(r-1,1) \subset \mathbb{P}^{\binom{r}{2}-1}$ be the Grassmann variety of lines in $\mathbb{P}^{r-1}$. Then

$$\operatorname{def} X = \begin{cases} 0, & r \equiv 0 \pmod 2, \\ 2, & r \equiv 1 \pmod 2, \end{cases}$$

and (2.4.1) is the filtration by pfaffian corank. Furthermore,

$$\operatorname{codeg} X = \begin{cases} \operatorname{ord} X = \frac{r}{2}, & r \equiv 0 \pmod 2, \\ \frac{1}{4}\binom{r+1}{3}, & r \equiv 1 \pmod 2 \end{cases}$$

(cf. [HT]). The dual variety corresponds to the skew-symmetric matrices whose rank is less than maximal. In particular, if r is even, then the hypersurface X^* is defined by vanishing of Pfaffian. Thus the codegree is equal to $\operatorname{ord} X$ if r is even and is much larger than $\operatorname{ord} X$ if r is odd. We observe that also in this case X^* is projectively isomorphic to X^J and, in particular, $\operatorname{codeg} X = \operatorname{jodeg} X$.

The following result is an analogue of Theorem 1.6 for codegree.

Theorem 2.7. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety. Then $d^* = \operatorname{codeg} X \geq \operatorname{ord} X$. Furthermore, if X is not a cone, then $d^* \geq \operatorname{ord} X + \operatorname{def} X \geq \operatorname{ord} X$. In particular, the codegree is not less than the order, and equality is possible only if X^* is a hypersurface in its linear span $\langle X^* \rangle$.*

Sketch of proof. The idea is to produce points of high multiplicity in the dual variety X^*. For a point $\alpha \in X^*$ we denote by $\operatorname{mult}_\alpha X^*$ the multiplicity of X^* at α. If $\varLambda$ is a general $(\operatorname{def} X + 1)$-dimensional linear subspace of $\mathbb{P}^{N*}$ passing through the point α, then $\varLambda$ meets X^* at α and $d^* - \operatorname{mult}_\alpha X^*$ other points. Furthermore, if U is a small neighborhood of α in X^* and $\varLambda'$ is a general $(\operatorname{def} X + 1)$-dimensional subspace of $\mathbb{P}^{N*}$ sufficiently close to $\varLambda$, then $\varLambda'$ meets U in $\operatorname{mult}_\alpha X^*$ nonsingular points.

The multiplicity defines a stratification of the dual variety. To wit, put

$$X^*_k = \{\alpha \in X^* \mid \operatorname{mult}_\alpha X^* \geq k\},$$

and let

$$k_m = \max\{k \mid X^*_k \neq \varnothing\}.$$

Then

$$X^* = X_1^* \supset X_2^* \supset \cdots \supset X_{k_m}^*.$$

To give a lower bound for d^* it suffices to bound the number k_m from below. In fact, it is easy to prove the following

Proposition 2.8. *If X is not a cone, then $d^* \geq k_m + \operatorname{def} X + 1$.*

Proof. If $\operatorname{def} X = 0$, then there is nothing to prove. Suppose that $\operatorname{def} X > 0$, and let $\alpha \in X_{k_m}^*$ be a general point. Let $\varpi : \mathbb{P}^{N*} \dashrightarrow \mathbb{P}^{N-1*}$ be the projection with center at α, and let $X^{*\prime} = \varpi(X^*)$. Since X is nondegenerate, X^* is not a cone, and so $\dim X^{*\prime} = \dim X^*$ and $\deg X^{*\prime} = \deg X^* - \operatorname{mult}_\alpha X^* = d^* - k_m$. Moreover, $\operatorname{codim}_{\mathbb{P}^{N-1*}} X^{*\prime} = \operatorname{codim}_{\mathbb{P}^{N*}} X^* - 1 = \operatorname{def} X$. Since, by our assumption, X^* is nondegenerate, $X^{*\prime}$ has the same property, and so $\deg X^{*\prime} \geq \operatorname{def} X + 1$ (cf. Theorem 3.4, (i) below). Thus $d^* - k_m \geq \operatorname{def} X + 1$ and we are done. □

To give a bound for k_m, we consider the corank filtration (2.4.1) introduced in Definition 2.5.

Proposition 2.9. *For each natural number k, $k < \operatorname{ord} X$ one has $(X^k)^* \subset X_k^*$.*

Sketch of proof. To prove the claim one can argue by induction. Let $(x_1, \dots, x_k) \in \underbrace{X \times \cdots \times X}_{k}$ be a general collection of k points of X, let $\mathcal{P}_{x_i} = \{\alpha \mid L_\alpha \supset T_{X,x_i}\} \subset X^*$, $i = 1, \dots, k$ (we recall that L_α denotes the hyperplane corresponding to α), and let $\alpha_k \in \mathcal{P}_{x_1} \cap \cdots \cap \mathcal{P}_{x_k}$. By the Terracini lemma, it suffices to show that $\operatorname{mult}_{\alpha_k} X^* \geq k$. Suppose that, for a general point $\alpha_{k-1} \in \mathcal{P}_{x_1} \cap \cdots \cap \mathcal{P}_{x_{k-1}}$, in a neighborhood of α_k we already know that $\operatorname{mult}_{\alpha_{k-1}} X^* \geq k-1$. Then, joining α_{k-1} with a general point $\alpha \in \mathcal{P}_{x_k}$ close to α_k, adding $\operatorname{def} X$ general points of $\mathbb{P}^{N*}$, taking the linear span of these $\operatorname{def} X + 2$ points (which, since X is nondegenerate, meets X^* in finitely many points), and passing to a limit, one can show that $\operatorname{mult}_{\alpha_k} X^* \geq \operatorname{mult}_{\alpha_{k-1}} X^* + \operatorname{mult}_\alpha X^* \geq k$. □

Corollary 2.10. $k_m \geq \operatorname{ord} X - 1$.

Theorem 2.7 immediately follows from Proposition 2.8 and Corollary 2.10. □

Remarks 2.11. (i) Arguing as in the proof of Proposition 2.9, one can show that, for $1 \leq i \leq k < \operatorname{ord} X$ and an arbitrary point $\alpha \in (X^k)^*$, one has

$$\operatorname{mult}_\alpha (X^i)^* \geq k - i + 1. \tag{2.11.1}$$

This is an analogue of (1.7.2).

(ii) In view of Proposition 1.8, from Theorem 2.7 it follows that

$$d^* \geq \frac{N - \operatorname{jodef} X}{n+1} + 1 \geq \frac{N+1}{n+1}. \tag{2.11.2}$$

Furthermore, if $N' = N - \operatorname{def} X$, $\mathbb{P}^{N'} \subset \mathbb{P}^N$ is a general linear subspace and $X' = \mathbb{P}^{N'} \cap X$, so that $n' = \dim X' = n - \operatorname{def} X$, X'^* is a general projection of X^* to $\mathbb{P}^{N'*}$ (cf. Proposition 2.4) and $\operatorname{codeg} X' = \operatorname{codeg} X = d^*$, then the same argument shows that

$$\operatorname{ord} X' \geq \frac{N'+1}{n'+1}. \tag{2.11.3}$$

Thus from Theorem 2.7 it follows that

$$d^* = \operatorname{codeg} X \geq \frac{N'+1}{n'+1} \geq \frac{N+1}{n+1}. \tag{2.11.4}$$

Suppose further that

$$(X_l'^*)^2 \not\subset X'^* \tag{2.11.5}$$

for some l such that

$$X_l'^* \neq \varnothing. \tag{2.11.6}$$

Then one can choose points $\alpha, \beta \in X_l'^*$ so that the line $\langle \alpha, \beta \rangle$ is not contained in X'^*, from which it follows that

$$d^* \geq 2l. \tag{2.11.7}$$

By Proposition 1.8, $\frac{N'-n'}{n'+1} \leq \operatorname{ord} X' - 1$, and thus there exists an $l \geq \frac{N'-n'}{n'+1}$ satisfying (2.11.6) If (2.11.5) holds for such an l, then (2.11.7) yields

$$d^* \geq 2l \geq 2 \cdot \frac{N'-n'}{n'+1} \geq 2 \cdot \frac{N-n}{n+1}, \tag{2.11.8}$$

which is almost twice better than (2.11.3).
Inequalities (2.11.4) and (2.11.8) give lower bounds for the codegree in terms of dimension and codimension. We will obtain a still better (universal and sharp) bound of this type in Section 4 (cf. Theorem 4.7).

(iii) Our proof of Theorem 2.7 is based on producing points of high multiplicity in the dual variety. More precisely, in Proposition 2.9 we showed that the points in X^* corresponding to k-tangent hyperplanes have multiplicity at least k. There is another natural way to produce highly singular points in X^*. To wit, if the hyperplane section of X corresponding to a point $\alpha \in X^*$ has only isolated singular points $x_1, \dots, x_l \in \operatorname{Sm} X$, then it is easy to show that $\alpha \in X^*_\mu \setminus X^*_{\mu+1}$, where $\mu = \sum_{i=1}^l \mu_i$ and μ_i is the Milnor number of x_i. In fact, since the singularities are isolated, in this case X^* is a hypersurface, and if Λ is a general line passing through α and Λ' is a line sufficiently close to Λ, then, in a small neighborhood of α, Λ' meets X^* in $\operatorname{mult}_{\alpha X^*}$ nonsingular points corresponding to hyperplane sections having a unique nondegenerate quadratic singularity. Now our claim follows from the definition of Milnor number as the number of ordinary quadratic points for a small deformation of function (cf. [Dim] for a less direct proof).

This method allows to show that d^* is quite high (in particular, higher then all the bounds that we discuss here) if all (or at least sufficiently many, in a certain sense) hyperplane sections of X have only isolated singularities (which, for example, is the case when X is a nonsingular complete intersection). However, we cannot assume this, and indeed in our examples "on the boundary" (cf. Examples 2.6, 1), 5), 6) (for r even) and Theorem 3.5 (iii), II.4) all singularities of hyperplane sections are either nondegenerate quadratic or nonisolated (cf. Condition $(*)$ in [Z1]). So, we will not further discuss this method here.

Conjectured Theorem 2.12. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety. Then the following conditions are equivalent:*

(i) $d^* = \operatorname{codeg} X = \operatorname{ord} X$*;*

(ii) X *is* (a cone over) *one of the following varieties:*

I. X *is a quadric,* $N = n+1$, $d^* = 2$*;*

II. X *is a Scorza variety* (cf. [Z2, Chapter VI])*. More precisely, in this case there are the following possibilities*:

II.1. $X = v_2(\mathbb{P}^n)$ *is a Veronese variety,* $N = \frac{n(n+3)}{2}$, $d^* = n+1$*;*

II.2. $X = \mathbb{P}^a \times \mathbb{P}^a$, $a \geq 2$ *is a Segre variety,* $n = 2a$, $N = a(a+2) = \frac{n(n+4)}{4}$, $d^* = a+1$*;*

II.3. $X = G(2m+1, 1)$ *is the Grassmann variety of lines in* $\mathbb{P}^{2m+1}$, $m \geq 2$, $n = 4m$, $N = m(2m+3) = \frac{n(n+6)}{8}$, $d^* = m+1$*;*

II.4. $X = E$ *is the variety corresponding to the orbit of highest weight vector in the lowest dimensional nontrivial representation of the group of type* E_6, $n = 16$, $N = 26$, $d^* = 3$.

Idea of proof. The implication (ii)$\Rightarrow$(i) is immediate (cf., e.g., [Z2, Chapter VI]). Thus we only need to show that (i)$\Rightarrow$(ii).

The case $\operatorname{ord} X = 2$ being obvious, one can argue by induction on $\operatorname{ord} X$. Given a variety $X \subset \mathbb{P}^N$ with $d^* = \operatorname{ord} X$, we consider its projection $X' \subset \mathbb{P}^{N-n-1}$ from the (embedded) tangent space $T_{X,x}$ at a general point $x \in X$. Then $\operatorname{ord} X' = \operatorname{ord} X - 1$, and it can be shown that $\operatorname{codeg} X' = \operatorname{codeg} X - 1$. Thus X' also satisfies the conditions of the theorem, and, reducing the order, we finally arrive at the case $\operatorname{ord} X = 3$, which can be dealt with directly (for smooth varieties, cf. Theorem 3.5, (iii)). Then a close analysis allows to reconstruct X. $\square$

Remarks 2.13. (i) Of course, the above proof is not complete, particularly in the nonsmooth case. Moreover, some details have not been verified yet. A complete proof will hopefully be given elsewhere.

(ii) Even without giving a complete classification of varieties satisfying (i), one can show that for such a variety $k_m = \operatorname{ord} X - 1$, $X^*_{k_m} = (X^{\operatorname{ord} X - 1})^*$ and $(X^*_{k_m})^{k_m} = X^*$. Thus $\operatorname{ord} X^*_{k_m} = \operatorname{ord} X$, and so $X^*_{k_m}$ is on the boundary of Theorem 1.6 for jodegree. In other words, in the extremal case the join determinant of $X^*_{k_m}$ coincides with the discriminant of X, and so classification of varieties of minimal codegree is “contained” in that of varieties of minimal jodegree. A posteriori, $X^*_{k_m}$ is projectively isomorphic to X, and so X also has minimal jodegree. On the other hand, there exist varieties for which jodegree is minimal while codegree is far from the boundary; cf., e.g., Examples 1.5, 4) and 2.6, 4).

(iii) It is worthwhile to observe that all the varieties in Theorem 2.12 are homogeneous. More precisely, varieties II.1–II.4 are the so called *Scorza varieties* (cf. [Z2, Chapter VI]). Thus in the case II the points of the ambient linear space correspond to (Hermitean) matrices over composition algebras (cf. [Z2, Chapter VI, Remark 5.10 and Theorem 5.11] and also [Ch]), and the points of the variety correspond to matrices of rank one. Furthermore, homogeniety is induced by multiplication of matrices. Thus the true meaning of Theorem 2.12 is that, in a certain sense, the varieties of Hermitean matrices of rank one over composition algebras are characterized by the property that their discriminant has minimal possible degree.

(iv) In Theorem 2.12 we give classification of varieties of minimal codegree, i.e., varieties X for which $\operatorname{codeg} X = \operatorname{ord} X$. This classification “corresponds” to classification of varieties of minimal degree (cf. Theorem 3.4 below). The next step in classification of varieties of small degree is to describe (linearly normal) varieties whose degree is close to minimal. The difference $\Delta_X = \deg X - \operatorname{codim} X - 1$ is called the Δ-*genus* of X. Thus, variety has minimal degree if and only if its Δ-genus vanishes. Classification of varieties of Δ-genus one (Del Pezzo varieties) is well known; much is also known about varieties whose Δ-genus is small (cf., e.g., [Fu]). Similarly, one can define ∇-*genus* of X by putting $\nabla_X = \operatorname{codeg} X - \operatorname{ord} X$ (or $\nabla_X = \operatorname{codeg} X - \operatorname{ord} X - \operatorname{def} X$). An interesting problem is then to give classification of projective varieties of small ∇-genus.

3 Varieties of Small Degree and Codegree

Theorems 2.7 and 2.12 give a nice idea of what it means for a variety to have small codegree. However, the answer is given in terms of the notion of order of variety which is not very common. A similar question about varieties of small degree has been studied for a century and a half. To put the problem in a proper perspective, in this section (which is of an expository nature) we collect various known results on varieties of small degree and codegree.

We start with giving sharp bounds for the codegree of nonsingular curves and surfaces in terms of standard invariants, such as codimension and degree.

In the case of curves one gets the following counterpart of Theorem 1.9.

Proposition 3.1. (i) *Let $C \subset \mathbb{P}^N$ be a nondegenerate nonsingular curve of degree d and codegree d^*. Then $d^* \geq 2d - 2$ with equality holding if and only if C is rational.*

(ii) *Let $C \subset \mathbb{P}^N$ be an arbitrary nondegenerate curve. Then $d^* = \operatorname{codeg} C \geq 2 \operatorname{codim} C = 2N - 2$ with equality holding if and only if C is a normal rational curve.*

Proof. First of all, it is clear that $\operatorname{def} C = 0$ for an arbitrary curve C. For any nonsingular curve C of genus g and degree d one has

$$\operatorname{codeg} C = 2(g + d - 1) \tag{3.1.1}$$

(this is essentially the Riemann–Hurwitz formula; its generalization to singular curves is immediate, but does not yield the inequality in (i)).

Assertion (i) immediately follows from (3.1.1) ((i) is trivially false without the assumption of smoothness; to see this, it suffices to consider the case $N = 2$ and use Theorem 2.2).

In the case of nonsingular curves, assertion (ii) follows from (i) since $d \geq N$ for an arbitrary nondegenerate curve C with equality holding if and only if C is a normal rational curve (this is a very special case of Theorem 3.4 below).

If $C \subset \mathbb{P}^N$ is a nondegenerate, but possibly singular curve and (x, y) is a general pair of points of C, then it is easy to see that the points α and β on the hypersurface C^* corresponding to the osculating hyperplanes at x and y respectively have multiplicity at least $N - 1$ (cf. also Remark 2.11, (iii)). From the Bertini theorem it follows that the line $\langle \alpha, \beta \rangle$ joining the points α and β in $\mathbb{P}^{N*}$ is not contained in C^* (in other words, the self-join of the dual (osculating) curve of C is not contained in the dual hypersurface). This proves the inequality $d^* \geq 2N - 2$. From our argument it also follows that equality $d^* = 2N - 2$ is only possible for curves without hyperosculation points, i.e., for rational normal curves. □

Lower bounds for the class can be also obtained for smooth surfaces (in this case the defect is always equal to zero, cf. [Z1]).

Proposition 3.2. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular surface of degree d and codegree d^*.*

(i) *$d^* \geq d - 1$. Furthermore, $d^* = d - 1$ if and only if X is the Veronese surface $v_2(\mathbb{P}^2)$ or its isomorphic projection to $\mathbb{P}^4$, and $d^* = \operatorname{codeg} X = \deg X = d$ if and only if X is a scroll over a curve.*

(ii) *$d^* \geq N - 2$. Furthermore, $d^* = \operatorname{codeg} X = \operatorname{codim} X = N - 2$ if and only if X is the Veronese surface $v_2(\mathbb{P}^2)$, and $d^* = N - 1$ if and only if X is either an isomorphic projection of the Veronese surface $v_2(\mathbb{P}^2)$ to $\mathbb{P}^4$ or a rational normal scroll.*

Proof. Assertion (i) is proved in [Z1, Proposition 3] (cf. also [M], [G1], [G2]).

Assertion (ii) follows from (i) since $d \geq N - 1$ for an arbitrary nondegenerate smooth surface X with equality holding if and only if X is either the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or a normal rational scroll (this is a special case of Theorem 3.4 below). □

Some work has been done in the direction of classification of smooth surfaces and, to a smaller degree, threefolds of small class (cf. [L], [LT], [LT2], [TV], but the corresponding results are based on computations involving Betti numbers or Chern classes and do not extend to higher dimensions.

Furthermore, it is not even clear which values of class should be considered "small". In most cases, in the existing literature class is compared to degree, which only seems reasonable for low-dimensional varieties (anyhow, the bounds obtained in [Bal] are very rough). On the other hand, in [TV] the authors give a classification of smooth surfaces whose class does not exceed twenty five, which does not seem to be a "small" number. It is clear that results of such type rely on classification of curves and surfaces and can hardly be generalized to higher dimensions.

To understand the situation better and pose the problem in a reasonable way, it makes sense to consider a similar question for varieties of low *degree*. There are two types of results concerning such varieties, viz. classification theorems for varieties whose degree is absolutely low and bounds and classification theorems for varieties whose degree is lowest possible (in terms of codimension).

An example of the first type is given by the following

Theorem 3.3. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety of degree d. Then*

(i) $d = 1$ *if and only if* $X = \mathbb{P}^N$;
(ii) $d = 2$ *if and only if* X *is a quadric hypersurface in* $\mathbb{P}^N$;
(iii) $d = 3$ *if and only if* X *is one of the following varieties*:
 I. X *is a cubic hypersurface in* $\mathbb{P}^N$;
 II. $X = \mathbb{P}^1 \times \mathbb{P}^2$ *is a Segre variety,* $n = \dim X = 3$*,* $N = 5$;
 II'_1. $X = \mathbb{F}_1$ *is a nonsingular hyperplane section of the variety* II*, i.e.,* $n = \dim X = 2$*,* $N = 4$*, and* X *is the scroll to which* $\mathbb{P}^2$ *is mapped by the linear system of conics passing through a point*;
 II'_2. $X = v_3(\mathbb{P}^1)$ *is a nonsingular hyperplane section of the variety* II'_1*, i.e.,* $n = \dim X = 1$*,* $N = 3$*, and* X *is a twisted cubic curve*;
 II''. X *is a cone over one of the varieties* II, II'_1, II'_2 (in this case $n = \dim X$ and $N = n + 2$ can be arbitrarily large).

It should be noted that, apart from hypersurfaces and cones, there exists only a finite number of varieties of degree three.

Proof. (i) and (ii) are obvious, and (iii) is proved in [X.X.X]. □

Swinnerton–Dyer [S-D] obtained a similar classification for varieties of degree four (apart from complete intersections, all such varieties are obtained from the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ by projecting, taking linear sections, and forming cones). Singularities create additional difficulties for higher degrees, but Ionescu [Io1], [Io2] gave a classification of all *nonsingular* varieties up to degree eight.

Clearly, one cannot hope to proceed much further in this way, and it is necessary to specify which degrees should be considered "low". To this end, there is a well known classical theorem dating back to Del Pezzo and Bertini.

Theorem 3.4. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety of degree d. Then*

(i) $d \geq N - n + 1$;
(ii) *If $d = N - n + 1$, then there are the following possibilities*:
 I. $X = \mathbb{P}^n$, $N = n$, $d = 1$;
 II. X *is a quadric hypersurface,* $N = n + 1$, $d = 2$;
 III. $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ *is a Veronese surface,* $n = 2$, $N = 5$, $d = 4$;
 IV. $X = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ *is a Segre variety,* $N = 2n - 1$, $d = n$;
 IV′ X *is a linear section of* IV (the hyperplane sections of I and II have the same type as the original varieties, and the irreducible hyperplane sections of III are also linear sections of $\mathbb{P}^1 \times \mathbb{P}^3$)*. It should be noted that varieties of this type are scrolls* (cf. [E-H]; in particular, if $n = 1$, then X is a rational normal curve);
 III″–IV″ X *is a cone over* III *or over one of the varieties described in* IV *and* IV′ (cones over I and II are of the same type as the original varieties).

Proof. Cf. [E-H]. □

A next step is to classify nondegenerate varieties $X^n \subset \mathbb{P}^N$ for which the difference $\Delta_X = d - (N - n + 1)$ is small (if X is linearly normal, then Δ_X is called the Δ-*genus* of X; cf. Remark 2.13, (iv)). Much is known about classification of varieties with small Δ-genus and a related problem of classification of varieties of small sectional genus (cf., e.g., [Fu]); we will not go into details here.

There is an analogue of Theorem 3.3 for codegree.

Theorem 3.5. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety of codegree d^*. Then*

(i) $d^* = 1$ *if and only if* $X = \mathbb{P}^N$;
(ii) $d^* = 2$ *if and only if* X *is a quadric hypersurface in* $\mathbb{P}^N$ (furthermore, $\operatorname{def} X$ is the dimension of singular locus (vertex) of X);
(iii) *If X is smooth, then $d^* = 3$ if and only if X is one of the following varieties*:
 I. $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ *is a Segre variety. In this case* $n = 3$, $\operatorname{def} X = 1$, *and X^* is isomorphic to X*;
 I′ $X = \mathbb{F}_1 \subset \mathbb{P}^4$ *is a nonsingular hyperplane section of the Segre variety from* I, *and X^* is the projection of the dual Segre variety from the point corresponding to the hyperplane*;
 II. X *is a Severi variety* (cf. [Z2, Chapter IV])*. More precisely, in this case there are the following possibilities*:
 II.1. $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ *is a Veronese surface, and X^* is projectively isomorphic to X^2* (cf. Examples 1.5, 5) and 2.6, 5));

II.2. $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ *is a Segre fourfold, and* X^* *is projectively isomorphic to* X^2 (cf. Examples 1.5, 1) and 2.6, 1));

II.3. $X = G(5,1) \subset \mathbb{P}^{14}$ *is an eight-dimensional Grassmann variety of lines in* $\mathbb{P}^5$*, and* X^* *is projectively isomorphic to* X^2 (cf. Examples 1.5, 6) and 2.6, 6));

II.4. $X = E \subset \mathbb{P}^{26}$ *corresponds to the orbit of highest weight vector of the lowest dimensional nontrivial representation of the group of type* E_6*,* $\dim X = 16$*, and* X^* *is projectively isomorphic to* X^2;

II$'$. X *is an isomorphic projection of one of the Severi varieties* $Y^n \subset \mathbb{P}^{\frac{3n}{2}+2}$ *from* II *to* $\mathbb{P}^{\frac{3n}{2}+1}$*,* $n = 2^i$*,* $1 \le i \le 4$*, and* X^* *is the intersection of the corresponding* Y^* *with the hyperplane corresponding to the center of projection* (as in II, here we obtain four cases II$'$.1–II$'$.4).

Proof. (i) and (ii) are almost obvious (they also follow from Theorem 2.2, Proposition 2.4 and Theorem 3.3 (i), (ii)), and (iii) is proved in [Z2, Chapter IV, §5, Theorem 5.2]. □

Remark 3.6. Comparing Theorems 3.3 (iii) and 3.5 (iii), one observes that classification of varieties of codegree three is already much harder than that of varieties of degree three and that there exist varieties of codegree three having large codimension (while the codimension of varieties of degree three is at most two). It is likely that one can classify all smooth varieties of codegree four. It is interesting that, unlike varieties of low degree, smooth varieties of low codegree tend to be homogeneous. They also have nice geometric properties. Thus, the "main series" II in Theorem 3.5 (iii) is formed by Severi varieties which are defined as varieties of lowest codimension that can be isomorphically projected (i.e., they do not have apparent double points) and the "main series" of varieties of codegree four is conjecturally formed by the homogeneous Legendrean varieties which, incidentally, have one apparent double point (i.e., their projection from a general point acquires only one singularity; cf. [Hw], [CMR], [LM], [Mu]).

In what follows we address the problem of finding a sharp lower bound for the codegree in terms of dimension and codimension, i.e., proving an analogue of Theorem 3.4 (i) and describing the varieties on the boundary, i.e., proving an analogue of Theorem 3.4 (ii).

4 Jacobian Systems and Hessian Matrices

Let $X \subset \mathbb{P}^N$, $\dim X = n$ be a nondegenerate variety, and let $d^* = \operatorname{codeg} X$. Applying Proposition 2.4 and Theorem 2.2, one can replace X by its generic linear section $X' = \mathbb{P}^{N'} \cap X$ such that $N' = N - \operatorname{def} X$, $n' = \dim X' = n - \operatorname{def} X$, $\operatorname{def} X' = 0$, and X'^* is the projection of X^* from the linear subspace ${}^{\perp}\mathbb{P}^{N'} \subset \mathbb{P}^{N*}$ of hyperplanes passing through $\mathbb{P}^{N'}$. It is clear that $\operatorname{codeg} X' = \operatorname{codeg} X = d^*$ and the dual variety X'^* is a hypersurface of degree d^* in $\mathbb{P}^{N'*}$

defined by vanishing of a form F of degree d^* in $N' + 1 = N - \operatorname{def} X + 1$ variables $x_0, \dots, x_{N'}$.

Let $\mathcal{J} = \mathcal{J}_F$ be the Jacobian (or polar) linear system on $\mathbb{P}^{N'*}$ spanned by the partial derivatives $\frac{\partial F}{\partial x_i}$, $i = 0, \dots, N'$, and let $\phi = \phi_{\mathcal{J}} : \mathbb{P}^{N'*} \dashrightarrow \mathbb{P}^{N'}$ be the corresponding rational map. By Theorem 2.2, the Gauß map $\gamma = \phi\big|_{X'^*}$ associating to a smooth point $\alpha \in X'^*$ the point in $\mathbb{P}^{N'}$ corresponding to the tangent hyperplane $T_{X'^*,\alpha}$ maps X'^* to X'. Thus one gets a rational map $\gamma = \phi\big|_{X'^*} : X'^* \dashrightarrow X'$, and $n' = n - \operatorname{def} X$ is equal to the rank of the differential $\mathrm{d}_\alpha \gamma$ at a general point $\alpha \in X'^*$. Since γ is defined by the partial derivatives of F, it is not surprising that, at a smooth point $\alpha \in X'^*$, the rank n' of the differential $\mathrm{d}_\alpha \gamma$ can be expressed in terms of the Hessian matrix $H_\alpha = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}(\alpha) \right)$, $0 \le i, j \le N'$.

We use the following general

Definition 4.1. Let M be a matrix over an integral ring A, and let $\mathfrak{a} \subset A$ be an ideal. We say that M *has rank* r *modulo* $\mathfrak{a}$ and write $\operatorname{rk}_{\mathfrak{a}} M = r$ if all minors of M of order larger than r are contained in $\mathfrak{a}$, but there exists a minor of order r not contained in $\mathfrak{a}$. If $\mathfrak{a} = (a)$ is a principal ideal, then we simply say that M *has rank* r *modulo* a and write $\operatorname{rk}_a M = r$.

Thus $\operatorname{rk}_e M = 0$ for every invertible element $e \in A$, $\operatorname{rk}_0 M = \operatorname{rk} M$ is the usual rank, and $0 \le \operatorname{rk}_{\mathfrak{a}} M \le \operatorname{rk} M$ for an arbitrary ideal $\mathfrak{a} \in A$.

It turns out that the rank of Hessian matrix modulo F determines the rank of the differential $\mathrm{d}_\alpha \gamma$ at a general point $\alpha \in X'^*$, which is equal to n'.

Proposition 4.2. *In the above notations,* $\operatorname{rk}_F H = n' + 2$.

Proof. This can be verified by an easy albeit tedious computation (cf. [S, Theorem 2]).

For a more direct geometric proof, we observe that, for a point $\xi \in \mathbb{P}^{N'*}$, the Hessian matrix $H_\xi = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}(\xi) \right)$ is the matrix of the (projective) differential $\mathrm{d}_\xi \phi : \mathbb{P}^{N'*} \dashrightarrow \mathbb{P}^{N'}$. Since, for a general point $\alpha \in X'^*$,

$$\operatorname{rk}_F H = (N' + 1) - \dim \operatorname{Ker} \mathrm{d}_\alpha \phi \tag{4.2.1}$$

and

$$\dim \operatorname{Ker} \mathrm{d}_\alpha \gamma = \operatorname{def} X'^* = N' - n' - 1 = N - n - 1, \tag{4.2.2}$$

to prove the proposition it suffices to verify that for $\alpha \in \operatorname{Sm} X'^*$ one has

$$\operatorname{Ker} \mathrm{d}_\alpha \phi = \operatorname{Ker} \mathrm{d}_\alpha \gamma. \tag{4.2.3}$$

Suppose to the contrary that

$$\operatorname{Ker} \mathrm{d}_\alpha \phi \supsetneq \operatorname{Ker} \mathrm{d}_\alpha \gamma. \tag{4.2.4}$$

Then

$$\operatorname{Im} \mathrm{d}_\alpha \phi = \operatorname{Im} \mathrm{d}_\alpha \gamma. \tag{4.2.5}$$

In particular, if $\nu = \left(\frac{\partial F}{\partial x_0}(\alpha), \ldots, \frac{\partial F}{\partial x_{N'}}(\alpha)\right)$ and $\bar{\nu}$ is the complex conjugate vector, then there exists a vector v such that

$$\nu^* \cdot v = 0, \qquad H_\alpha \cdot \bar{\nu} = H_\alpha \cdot v \tag{4.2.6}$$

(here and in what follows $\nu = \nu^{(N+1)\times 1}$ is a column vector and $\nu^* = \nu^{*\,1\times(N+1)}$ is a row vector). Denoting a vector corresponding to the point α by the same letter and recalling that H_α is a symmetric matrix, one sees that (4.2.6) yields the following chain of equalities:

$$\begin{aligned}(H_\alpha \cdot \alpha)^* \cdot v &= (\alpha^* \cdot H_\alpha) \cdot v = \alpha^* \cdot (H_\alpha \cdot v) \\ &= \alpha^* \cdot (H_\alpha \cdot \bar{\nu}) = (\alpha^* \cdot H_\alpha) \cdot \bar{\nu} = (H_\alpha \cdot \alpha)^* \cdot \bar{\nu}.\end{aligned} \tag{4.2.7}$$

To complete the proof, we recall that, by Euler's formula,

$$H_\alpha \cdot \alpha = (d^* - 1)\nu, \tag{4.2.8}$$

so that, in view of (4.2.6) and (4.2.7),

$$0 = (d^* - 1)\nu^* \cdot v = (H_\alpha \cdot \alpha)^* \cdot v = (H_\alpha \cdot \alpha)^* \cdot \bar{\nu} = (d^* - 1)\nu^* \cdot \bar{\nu}. \tag{4.2.9}$$

But from (4.2.9) it follows that $\nu = 0$, which is only possible if $\alpha \in \operatorname{Sing} X'^*$, contrary to the assumption that $\alpha \in X'^*$ is smooth. □

We now use the following result which generalizes the obvious fact that if all elements of a square matrix M of order m are multiples of p, then $\det M$ is a multiple of p^m.

Proposition 4.3. *Let M be a matrix over an integral ring A, let $\mathfrak{p} \subset A$ be a prime ideal, and let k be a natural number. Then* $\operatorname{rk}_{\mathfrak{p}^k} M \leq \operatorname{rk}_{\mathfrak{p}} M + k - 1$.

In particular, if M is a square matrix of order m, then, for each prime ideal $\mathfrak{p} \subset A$, $\det M \in \mathfrak{p}^{\operatorname{cork}_{\mathfrak{p}} M}$, *where* $\operatorname{cork}_{\mathfrak{p}} M = m - \operatorname{rk}_{\mathfrak{p}} M$ *is the corank of M modulo $\mathfrak{p}$.*

Proof. By an easy induction argument, it suffices to consider the case when M is a square matrix of order m and to show that

$$\nu_{\mathfrak{p}}(\det M) \geq \min_{1 \leq i,j \leq m} \nu_{\mathfrak{p}}(\det M_{ij}) + 1, \tag{4.3.1}$$

where, for an element $a \in A$, $\nu_{\mathfrak{p}}(a) = \max\{k \mid a \in \mathfrak{p}^k\}$ and M_{ij} is the $(m-1)\times(m-1)$-matrix obtained from M by removing the i-th row and the j-th column. To this end, consider the matrix $\operatorname{adj} M$ for which $(\operatorname{adj} M)_{ij} = (-1)^{i+j} \det M_{ji}$. It is well known and easy to check that

$$M \cdot \operatorname{adj} M = \det M \cdot I_m, \tag{4.3.2}$$

where I_m is the identity matrix of order m (in standard courses of linear algebra it is shown that if A is a field and $\det M \neq 0$, then M is invertible and $M^{-1} = (\det M)^{-1} \cdot \operatorname{adj} M$). If $\det M = 0$, then (4.3.1) is clear; otherwise, taking determinants of both sides of (4.3.2), we see that

$$\det(\operatorname{adj} M) = (\det M)^{m-1} \tag{4.3.3}$$

and so

$$m \cdot \min_{1 \le i,j \le m} \nu_{\mathfrak{p}}(\det M_{ij}) \le \nu_{\mathfrak{p}}\big(\det\big(\operatorname{adj} M\big)\big) = (m-1) \cdot \nu_{\mathfrak{p}}(\det M). \tag{4.3.4}$$

Thus

$$\nu_{\mathfrak{p}}(\det M) \ge \frac{m}{m-1} \cdot \min_{1 \le i,j \le m} \nu_{\mathfrak{p}}(\det M_{ij}) > \min_{1 \le i,j \le m} \nu_{\mathfrak{p}}(\det M_{ij}), \tag{4.3.5}$$

which implies (4.3.1). □

If the ring A is nötherian, then, by Krull's theorem, $\bigcap_k \mathfrak{p}^k = 0$ and the numbers $\nu_{\mathfrak{p}}(a)$ are finite for each element $a \in A$, $a \neq 0$. However, we do not need this fact here.

Applying Proposition 4.3 to the Hessian matrix $H = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$, $0 \le i, j \le N'$ and using Proposition 4.2, one gets the following

Corollary 4.4. *$F^{N'-n'-1} | \det H$.*

Since all the entries of H have degree $d^* - 2$, one has $\deg \det H = (N' + 1)(d^* - 2)$. On the other hand, $\deg F^{N'-n'-1} = d^*(N' - n' - 1)$, and so Corollary 4.4 yields the following

Corollary 4.5. *Suppose that $h = \det H \neq 0$. Then*

$$\operatorname{codeg} X = d^* \ge 2 \cdot \frac{N'+1}{n'+2} = 2 \cdot \frac{N - \operatorname{def} X + 1}{n - \operatorname{def} X + 2} \ge 2 \cdot \frac{N+1}{n+2}.$$

Corollary 4.5 gives the desired lower bound for codegree in terms of dimension and codimension. Clearly, unless X is a quadratic cone, this bound can be sharp only if $\operatorname{def} X = 0$, i.e., X^* is a hypersurface. Furthermore, Examples 2.6, 1), 5) and 6) (for r even) and Theorem 3.5 (iii), II.4 show that the bound given in Corollary 4.5 is indeed sharp.

The catch is that we do not know how to tell whether or not the determinant $h = h_F$ of the matrix H (called the *hessian* of F) vanishes identically for a given form F. The simplest example of this phenomenon is when, after a linear change of coordinates, the form F depends on fewer than $N' + 1$ variables (cf. [H1, Lehrsatz 2]). Geometrically, this means that X'^* is a cone, and therefore the variety X' is degenerate. But then X is also degenerate, contrary to our assumptions. Thus in our setup F always depends on all the variables.

Hesse claimed that the converse is also true, i.e., in our language, $h_F = 0$ if and only if X' is degenerate (cf. [H1, Lehrsatz 3]). Moreover, his paper [H2] written eight years later is, in his own words, devoted to "giving a stronger foundation to this result". However, as Gauß put it, "unlike with lawyers for whom two half proofs equal a whole one, with mathematicians half proof equals zero, and real proof should eliminate even a shadow of doubt". This was not the case here, and, twenty five years after the publication of [H1], Gordan and Nöther [GN] found that Hesse's claim was wrong. To wit, they did not produce an explicit example of form with vanishing hessian, but rather verified the existence of solutions of certain systems of partial differential equations yielding such forms. In particular, they checked that, while Hesse's claim is true for $N' = 2$ and $N' = 3$, it already fails for $N' = 4$ (i.e., for quinary forms).

Even though constructing examples of forms with vanishing hessian might not be evident from the point of view of differential equations, it is easy from geometric viewpoint. From the above it is clear that $h = h_F \equiv 0$ if and only if the Jacobian map $\phi = \phi_{\mathcal{J}} : \mathbb{P}^{N'*} \dashrightarrow \mathbb{P}^{N'}$ fails to be surjective.

Example 4.6. Let $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be the Segre variety considered in Theorem 3.5 (iii), I, so that $X^* \simeq X$ and $\deg X = \deg X^* = 3$, and let $X' = \mathbb{F}_1 \subset \mathbb{P}^4$ be a nonsingular hyperplane section of X. Then X'^* is the projection of X^* from the point corresponding to the hyperplane (the center of projection is not contained in X^*; cf. Theorem 3.5 (iii), I$'$). I claim that the hessian $h = h_F$ of the cubic form F defining X'^* is identically equal to zero (this was first discovered by Perazzo [P] who used the theory of polars).

In fact, it is clear that the entry locus of the center of projection in $\mathbb{P}^{5*}$ is a nonsingular quadric surface. The restriction of the projection on this quadric is a double covering of the image, a plane $\Pi \subset \mathbb{P}^{4*}$, ramified along a nonsingular conic $C \subset \Pi$. Thus $\operatorname{Sing} X'^* = \Pi$ and X'^* is swept out by a family of planes $\mathbb{P}^2_\alpha$, $\alpha \in C$ such that $\mathbb{P}^2_\alpha \cap \Pi = T_{C,\alpha}$, i.e., the planes of the family meet Π along the tangent lines to the conic. Furthermore, the pencil of hyperplanes passing through a fixed plane $\mathbb{P}^2_\alpha$ is parameterized by a fibre f_α of the scroll $X' = \mathbb{F}_1$, the corresponding hyperplane sections of X'^* are a union of $\mathbb{P}^2_\alpha$ and a nonsingular quadric, and these quadrics meet $\mathbb{P}^2_\alpha$ along a union of $T_{C,\alpha}$ and lines from the pencil defined by the point $\alpha \in \mathbb{P}^2_\alpha \cap C$. Finally, the hyperplane sections of X'^* corresponding to the points of the exceptional section s of the scroll $X' = \mathbb{F}_1$, i.e., those cut out by the hyperplanes in $\mathbb{P}^{4*}$ passing through Π, have the form $\mathbb{P}^2_\alpha + 2\Pi$.

In this example $\mathcal{J}$ is a linear system of quadrics with base locus Π. We need to check that $\dim \phi(\mathbb{P}^{4*}) < 4$, i.e., the fibres of ϕ are positive-dimensional. To this end, we observe that, by the above, a general point $\xi \in \mathbb{P}^{4*}$ is contained in a unique hyperplane of the form $\mathbb{P}^3_\alpha = \langle \mathbb{P}^2_\alpha, \Pi \rangle$. Restricted to $\mathbb{P}^3_\alpha$, the linear system $\mathcal{J}$ has fixed component Π. Thus $\phi\big|_{\mathbb{P}^3_\alpha}$ is a linear projection, and to prove our claim it suffices to recall that $\phi\big|_{X'^*} = \gamma$ and $\phi(X'^*) = X'$. In particular, ϕ blows down the plane $\mathbb{P}^2_\alpha$ to a line, hence it maps $\mathbb{P}^3_\alpha$ to a plane, and the fibres of ϕ are lines. Furthermore, $Z = \phi(\mathbb{P}^{4*}) \subset \mathbb{P}^4$ is a quadratic cone with vertex at the minimal section s of the scroll $X' = \mathbb{F}_1$, and $Z = T(s, X') = \bigcup\limits_{x\in s} T_{X',x} = l \cdot X'$ is the join of s with X'. $\square$

Some work has been done towards better understanding the condition of vanishing of the hessian (cf. [P], [Fr], [C], [Il], [PW]), but it does not seem to be helpful in dealing with our problem.

In what follows we show that Corollary 4.5 is true for an arbitrary nondegenerate variety X, even though the corresponding hessian may vanish.

Theorem 4.7. *In the above notations,*

$$\operatorname{codeg} X = d^* \geq 2 \cdot \frac{N'+1}{n'+2} = 2 \cdot \frac{N - \operatorname{def} X + 1}{n - \operatorname{def} X + 2} \geq 2 \cdot \frac{N+1}{n+2}.$$

In view of Proposition 4.2, Theorem 4.7 is equivalent to the following elementary statement not involving any algebraic geometry.

Theorem 4.8. *Let F be an irreducible form of degree d in m variables which cannot be transformed into a form of fewer variables by a linear change of coordinates, and let $H = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$, $1 \leq i, j \leq m$ be its Hessian matrix. Then $\operatorname{rk}_F H \geq \frac{2m}{d}$.*

Before proceeding with the proof of Theorem 4.7, we need to study the rational map $\phi = \phi_{\mathcal{J}} : \mathbb{P}^{N'*} \dashrightarrow \mathbb{P}^{N'}$ in more detail. From the Euler formula it follows that ϕ is regular outside of $\Sigma = \operatorname{Sing} X'^*$, and it is clear that $Z = \phi(\mathbb{P}^{N'*}) \subset \mathbb{P}^{N'}$ is an irreducible variety.

Proposition 4.9. (i) *$Z \supset X'$, $r = \dim Z = \operatorname{rk} H_\xi - 1$, where $\xi \in \mathbb{P}^{N'*}$ is a general point, and $n' + 1 \leq r \leq N'$.*

(ii) *Let $z \in Z$ be a general point, and let $\mathcal{F}_z = \phi^{-1}(z)$ be the corresponding fibre. Then $\mathcal{F}_z$ is an $(N' - r)$-dimensional linear subspace passing through the $(N' - r - 1)$-dimensional linear subspace ${}^\perp T_{Z,z}$.*

(iii) *$Z^* \subset \Sigma$. Furthermore, if $z \in Z$ is a general point, then $\mathcal{F}_z \cap X'^* = \mathcal{F}_z \cap \Sigma = \mathcal{F}_z \cap Z^* = {}^\perp T_{Z,z}$.*

Proof. (i) Since the ground field has characteristic zero, $\phi = \phi_{\mathcal{J}}$ is generically smooth (or submersive), i.e., $r = \operatorname{rk} \mathrm{d}_\xi \phi$ for a general point $\xi \in \mathbb{P}^{N'*}$. On the other hand, it is clear that $\operatorname{rk} \mathrm{d}_\xi \phi = \operatorname{rk} H_\xi - 1$, which proves the first assertion of (i).

The second assertion easily follows from the observation that $\operatorname{rk} \mathrm{d}_\xi \phi$ is lower semicontinuous as a function of ξ and, by virtue of (4.2.1) and (4.2.3), $\operatorname{rk} \mathrm{d}_\alpha \phi = \operatorname{rk} \mathrm{d}_\alpha \gamma + 1$ for a general point $\alpha \in X'^*$.

(ii) Let $\xi \in \mathcal{F}_z \setminus \Sigma$, and let H_ξ be the Hessian matrix at the point ξ. Since the matrix H_ξ is symmetric and $\phi(\xi) = z$, one sees that, in the obvious notations,

$$T_{\mathcal{F}_z,\xi} = \big\langle \xi, \mathbb{P}\big(\operatorname{Ker} H_\xi\big)\big\rangle = \big\langle \xi, {}^{\perp}\mathbb{P}\big(\operatorname{Im} H_\xi\big)\big\rangle \supseteq \big\langle \xi, {}^{\perp}T_{Z,z}\big\rangle. \tag{4.9.1}$$

For $\xi \notin \Sigma$, the Euler formula (4.2.8) shows that

$$\xi \notin {}^{\perp}\mathbb{P}(\operatorname{Im} H_\xi). \tag{4.9.2}$$

If the fibre $\mathcal{F}_z$ is reduced, then, since the ground field has characteristic zero, from (4.9.1) it follows that $\mathcal{F}_z$ is a cone with vertex ${}^{\perp}T_{Z,z}$. If, moreover, $z \in \operatorname{Sm} Z$ and $\mathcal{F}_z$ is equidimensional of dimension $N' - r$, then (4.9.1) shows that $\mathcal{F}_\xi$ is a union of $(N'-r)$-dimensional linear subspaces of $\mathbb{P}^{N'*}$ meeting along ${}^{\perp}T_{Z,z}$.

Considering the graph of ϕ obtained, at a general point $z \in Z = (Z^*)^*$, by blowing up Z (so that z is replaced by the linear subspace ${}^{\perp}T_{Z,z}$ along which the hyperplane ${}^{\perp}z$ is tangent to Z^*) and restricting the projection corresponding to ϕ on the inverse image of Z, we see that, for a general point $\zeta \in Z^*$, the fibre $\mathcal{F}_\zeta$ has only one component.

(iii) To prove that $Z^* \subset \Sigma$ it suffices to verify that, for a general point $z \in Z$, ${}^{\perp}T_{Z,z} \subset \Sigma$. Suppose that this is not so, and let $\xi \in {}^{\perp}T_{Z,z} \setminus \Sigma$. By (ii), ${}^{\perp}T_{Z,z} \subset \mathcal{F}_z$ and thus $\phi(\xi) = z$. But then ${}^{\perp}T_{Z,z} \subset {}^{\perp}\mathbb{P}(\operatorname{Im} H_\xi)$ and from (4.9.2) it follows that, contrary to our assumption, $\xi \notin {}^{\perp}T_{Z,z}$. The resulting contradiction shows that $Z^* \subset \Sigma$.

If $z \notin X'$, then, clearly, $\mathcal{F}_z \cap X'^* = \mathcal{F}_z \cap \Sigma$. From the first assertion of (iii) it follows that ${}^{\perp}T_{Z,z} \subset \mathcal{F}_z \cap Z^* \subset \mathcal{F}_z \cap \Sigma$. If, furthermore, z is general in the sense of (ii), then from (4.9.1) and (ii) it follows that $\mathcal{F}_z \cap \Sigma = {}^{\perp}T_{Z,z}$. $\square$

Describing forms with vanishing hessian and the dual varieties of the corresponding hypersurfaces is an interesting geometric problem which will be dealt with elsewhere. In particular, one can obtain natural proofs and far going generalizations of the relevant results in [GN], [Fr], [C] and [Il]. However in the present paper we only need the properties listed in Proposition 4.9.

We proceed with showing that if the hessian $h = h_F$ vanishes identically on $\mathbb{P}^{N'*}$, then we get an even better bound for $d^* = \deg F$. To this end, we first recall some useful classical notions.

Let $G(m,l)$ denote the Grassmann variety of l-dimensional linear subspaces in $\mathbb{P}^m$, let $\mathcal{U}$ be the universal bundle over $G(m,l)$, and let $p : \mathcal{U} \to \mathbb{P}^m$ and $q : \mathcal{U} \to G(m,l)$ denote the natural projections. For a subvariety $\Theta \subset G(m,l)$, we denote by $\mathcal{U}_\Theta$ the restriction of $\mathcal{U}$ on Θ and put $p_\Theta = p\big|_{\mathcal{U}_\Theta}$, $q_\Theta = q\big|_{\mathcal{U}_\Theta}$.

Definition 4.10. An irreducible subvariety $\Theta \subset G(m,l)$ is called *congruence* if $\dim \Theta = m - l$.

The number of points in a general fibre of the map $p_\Theta : \mathcal{U}_\Theta \to \mathbb{P}^m$ (equal to $\deg p_\Theta$) is called the *order* of Θ (usually there is no danger of confusing this notion with the one introduced in Definition 1.3). In particular, the map p_Θ is birational if and only if Θ is a congruence of order one.

The subvariety $B \subset \mathbb{P}^m$ of points in which p_Θ fails to be biholomorphic is called the *focal* or *branch locus* of Θ. In particular, if Θ is a congruence of order one, then $B = \{w \in \mathbb{P}^m \mid \dim p_\Theta^{-1}(w) > 0\}$ and $\operatorname{codim}_{\mathbb{P}^m} B > 1$; in this case B is also called the *jump locus* of the congruence Θ.

Let $\Theta \subset G(m,l)$ be a congruence of order one, and let $L \subset \mathbb{P}^m$, $\dim L = m-l$ be a general linear subspace. Intersection with L gives rise to a rational map $\rho_L : \Theta \dashrightarrow \mathbb{P}^{m-l}$ with fundamental locus $B = B_L = \Theta \cap \Gamma_L$, where $\Gamma_L = \{\ell \in G(m,l) \mid \dim \ell \cap L > 0\}$.

Lemma 4.11. *Either Θ is the congruence formed by all the l-dimensional subspaces passing through a fixed $(l-1)$-dimensional linear subspace of $\mathbb{P}^m$ or $B \neq \varnothing$ and* $\operatorname{codim}_\Theta B = 2$.

Proof. Let $\widetilde{L} \subset \mathbb{P}^m$ be a general $(m-l+1)$-dimensional linear subspace containing L. Intersection with $\widetilde{L}$ gives rise to a rational map $\rho_{\widetilde{L}} : \Theta \dashrightarrow G(m-l+1,1)$, and it is clear that ρ_L factors through $\rho_{\widetilde{L}}$. Now it suffices to prove the claim for $\widetilde{\Theta} = \rho_{\widetilde{L}}(\Theta)$, which is clearly a congruence of lines of order one. In other words, it is enough to prove the lemma for $l=1$.

Suppose now that $l=1$ and $B = \varnothing$, so that $V = V_\Theta = \bigcup_{\vartheta \in \Theta} {}^\perp\ell_\vartheta \subsetneq \mathbb{P}^{m*}$, where $\ell_\vartheta = p_\Theta(q_\Theta^{-1}(\vartheta))$. Since each ${}^\perp\ell_\vartheta$ is a linear subspace of codimension two in $\mathbb{P}^{m*}$, V is a hypersurface, and a general point $v \in V$ is contained in at least an $(m-2)$-dimensional family of ${}^\perp\ell_\vartheta$'s. Thus, for $m > 2$, the tangent hyperplane $T_{V,v}^{m-1}$ meets V along a positive-dimensional family of linear subspaces of dimension $m-2$. Hence V is a hyperplane, and all lines ℓ_ϑ, $\vartheta \in \Theta$ pass through one and the same point $z = V^\perp \in \mathbb{P}^m$. The remaining assertions of the lemma, as well as the cases when $m \leq 2$ are now clear. $\square$

Returning to our setup, we recall that $h_F = 0$ if and only if $Z \subsetneq \mathbb{P}^{N'}$. Suppose that this is so, let $l = N' - r$, and let $\Theta = \Theta_F \subset G(N',l)$ be the corresponding congruence of order one formed by the fibres of the Jacobian map $\phi : \mathbb{P}^{N'*} \dashrightarrow Z$ (cf. Proposition 4.9, (ii)). Cutting X'^* with a general r-dimensional linear subspace $\Lambda \subset \mathbb{P}^{N'*}$ and restricting ϕ on Λ, we obtain a rational map $\Lambda \dashrightarrow \mathbb{P}^{N'}$. Let $\varpi = \varpi_\Lambda : \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^r$ denote the projection with center at the $(N'-r-1)$-dimensional linear subspace ${}^\perp\Lambda$, and put $X'' = \varpi(X')$. Then, by Proposition 2.4, $X''^* = X'^* \cap \Lambda$. We denote by F' the restriction of F on Λ and by $\mathcal{J}'$ the Jacobian system of F'. Then $\mathcal{J}'$ defines a rational map $\phi' = \phi_{\mathcal{J}'} : \Lambda \dashrightarrow \mathbb{P}^r$ and $\phi' = \varpi \circ \phi\big|_\Lambda$. Furthermore, it is clear that ϕ' is dominant and the hessian $h_{F'}$ does not vanish identically on Λ. More precisely, we have the following

Proposition 4.12. *In the above notations* $(h_{F'}) \geq (r-n'-1)(F') + E_\Lambda + W_\Lambda$, *where parentheses denote the divisor of form,* $E_\Lambda = p_\Theta(q_\Theta^{-1}(B)) \cap \Lambda$ *is the* *exceptional divisor of ϕ' outside of X''^*, $W_\Lambda = (\phi\big|_\Lambda)^{-1}(R_\Lambda)$, and R_Λ is the ramification divisor of the finite covering $\varpi_\Lambda : Z \to \mathbb{P}^r$. Furthermore,* $d^* = \operatorname{codeg} X = \operatorname{codeg} X' = \operatorname{codeg} X'' \geq \dfrac{2(r+1) + \deg E_\Lambda + \deg W_\Lambda}{n'+2}$.

Proof. The fact that the divisor $(h_{F'})$ contains $(r-n'-1)(F')$ was proved in Corollary 4.4, and from the above discussion it is clear that $(h_{F'})$ also contains E_Λ and W_Λ. Computing the degrees, we get $(r+1)(d^*-2) \geq (r-n'-1)d^* + \deg E_\Lambda + \deg W_\Lambda$, so that $d^* \geq \dfrac{2(r+1) + \deg E_\Lambda + \deg W_\Lambda}{n'+2}$. $\square$

Proposition 4.12 shows that to give a lower bound for the codegree it suffices to obtain lower bounds for $\deg W_\Lambda$ and $\deg E_\Lambda$.

Proof of Theorem 4.7. In view of Proposition 4.12, to prove Theorem 4.7 it suffices to show that $\deg E_\Lambda + \deg W_\Lambda \geq 2(N'-r)$. The case when $h_F \neq 0$, i.e., $Z = \mathbb{P}^{N'}$ was dealt with in Corollary 4.5; so, we may assume that $r < N'$.

We claim that under this assumption, i.e., when the hessian vanishes, one has a strict inequality

$$\deg E_\Lambda + \deg W_\Lambda > 2(N'-r), \tag{4.7.1}$$

which, by virtue of 4.12, yields a strict inequality

$$d^* > 2\frac{N+1}{n+2}. \tag{4.7.2}$$

To prove (4.7.1) we recall that from Lemma 4.11 it follows that $E_\Lambda \neq \varnothing$, and so

$$\deg E_\Lambda > 0 \tag{4.7.3}$$

(as a matter of fact, it is very easy to show that E_Λ is nonlinear, and so $\deg E_\Lambda \geq 2$). On the other hand, from Proposition 4.9, (i) it follows that Z is nondegenerate, and, applying Proposition 3.1, (ii), we conclude that

$$\deg W_\Lambda \geq 2(N'-r). \tag{4.7.4}$$

The inequality (4.7.1) (hence (4.7.2), hence Theorem 4.7) is an immediate consequence of the inequalities (4.7.3) and (4.7.4). $\square$

Remarks 4.13. (i) The bounds in Theorems 2.7 and 4.7 appear to be quite different, and the methods used to prove these bounds are very different as well. However, there seems to be a connection between these bounds. To wit, one may conjecture that, at least for nondegenerate smooth varieties $X \subset \mathbb{P}^N$ of dimension n, one always has

$$\frac{N+1}{n+1} \leq \operatorname{ord} X \leq 2\frac{N+1}{n+2} \tag{4.13.1}$$

(we recall that the first inequality in (4.13.1) was proved in Proposition 1.8). Thus, at least for smooth varieties, Theorem 2.7 should follow from Theorem 4.7. The upper bound for order in (4.13.1) can be viewed as a generalization of the theorem on linear normality. Indeed, if the secant variety SX is a proper subvariety of $\mathbb{P}^N$ or, equivalently, $\operatorname{ord} X \geq 3$, then from (4.13.1) it follows that $N \geq \frac{3n}{2} + 2$, which coincides with the bound for linear normality (cf. [Z2, Chapter 2, §2]). Similarly, for $\operatorname{ord} X \geq 4$, (4.13.1) yields $N \geq 2n + 3$, which can be easily proven directly.

(ii) From (4.7.2) and Corollary 4.5 it follows that if the bound in Theorem 4.7 is sharp (i.e., $d^* = 2\frac{N+1}{n+2}$) and X is not a quadratic cone, then $\operatorname{def} X = 0$ and $h_F \neq 0$ (i.e., $r = N$); see Conjecture 4.15 for a more precise statement.

(iii) The lower bound in (4.7.1) can be considerably improved. To wit, as in the case of W_L, one can obtain a lower bound for $\deg E_L$ in terms of the codimension of Z. This yields a better lower bound for d^* in the case of vanishing hessian. However, we do not need it here.

(iv) The techniques of studying Jacobian maps and Hessian matrices that we started developing in this section can also be useful in studying other problems, such as classification of Jacobian (or polar) Cremona transformations (cf. [EKP], [Dolg]) or, more generally, classification of forms F for which the general fibre of the Jacobian map $\phi_{\mathcal{J}_F}$ is linear (i.e., classification of homaloidal and subhomaloidal forms). This topic will be dealt with elsewhere.

Proposition 4.14. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety. Then the following conditions are equivalent*:

(i) $d^* = \operatorname{codeg} X = 2\frac{N+1}{n+2}$;

(ii) *Either $N = n+1$, $h_F = 0$ and X is a quadratic cone with vertex $\mathbb{P}^{\operatorname{def} X - 1}$ or $\operatorname{def} X = 0$ and $h_F = F^{N-n-1}$, where F is a suitably chosen equation of the hypersurface X^*.*

Proof. In Remark 4.13, (ii) we already observed that, unless X is a quadratic cone, (i) implies $\operatorname{def} X = 0$ and $h_G \neq 0$, where G is an equation of X^*. Thus from Corollary 4.4 it follows that $G^{N-n-1} \mid h_G$. Since $\deg G^{N-n-1} = d^*(N - n - 1)$ and $\deg h_G = (N+1)(d^* - 2)$, (i) implies that $\deg G^{N-n-1} = \deg h_G$, and so $h_G = c \cdot G^{N-n-1}$, where $c \in \mathbb{C}$ is a nonzero constant. Hence, replacing G by $F = c^{-\frac{1}{n+2}}$, we get $h_F = F^{N-n-1}$. Thus (i)$\Rightarrow$(ii).

Conversely, in the non-conic case from (ii) it follows that $\deg F^{N-n-1} = \deg h_F$, hence $d^*(N - n - 1) = (N+1)(d^* - 2)$, whence (ii)$\Rightarrow$(i). □

Having obtained a bound for codegree in Theorem 4.7, it is natural to proceed with describing the varieties for which this bound is sharp, just as in Theorem 2.12 we classified the varieties on the boundary of Theorem 2.7. However, we have not proved classification theorem for these varieties yet.

Conjecture 4.15. A nondegenerate variety $X^n \subset \mathbb{P}^N$ satisfies the equivalent conditions of Proposition 4.14 if and only if X is one of the following varieties:

I. X is a quadric, $N = n + 1$, $d^* = 2$;

II. X is a Scorza variety (cf. [Z2, Chapter VI]). More precisely, in this case there are the following possibilities:

II.1. $X = v_2(\mathbb{P}^n)$ is a Veronese variety, $N = \frac{n(n+3)}{2}$, $d^* = n + 1$;

II.2. $X = \mathbb{P}^a \times \mathbb{P}^a$, $a \geq 2$ is a Segre variety, $n = 2a$, $N = a(a+2) = \frac{n(n+4)}{4}$, $d^* = a + 1$;

II.3. $X = G(2m + 1, 1)$ is the Grassmann variety of lines in $\mathbb{P}^{2m+1}$, $m \geq 2$, $n = 4m$, $N = m(2m + 3) = \frac{n(n+6)}{8}$, $d^* = m + 1$;

II.4. $X = E$ is the variety corresponding to the orbit of highest weight vector in the lowest dimensional nontrivial representation of the group of type E_6, $n = 16$, $N = 26$, $d^* = 3$.

Remarks 4.16. (i) As in Conjectured Theorem 2.12, it seems reasonable to argue by induction. Let $x \in X$ be a general point, and let $\mathcal{P}_x \subset X^*$ be the $(N-n-1)$-dimensional linear subspace of $\mathbb{P}^{N*}$ which is the locus of tangent hyperplanes to X at x. It is clear that the restriction of the Jacobian linear system $\mathcal{J}_F$ on $\mathcal{P}_x$ is a hypersurface in $\mathcal{P}_x$ defined by vanishing of $\frac{\partial F}{\partial x_i}$ for any $i = 0, \dots, N$ (since $\mathcal{P}_x$ is a fibre of the Gauß map $\gamma : X^* \dashrightarrow X$, the nonvanishing partial derivatives are proportional to each other on $\mathcal{P}_x$). In this way one obtains a homogeneous polynomial $F' = \frac{\partial F}{\partial x_i}$ of degree $d^{*\prime} = d^* - 1$ on $\mathcal{P}_x = \mathbb{P}^{N-n-1\,*}$ and a map $\phi' : \mathbb{P}^{N-n-1\,*} \dashrightarrow \mathbb{P}^{N-n-1}$ defined by any row of the Hessian matrix of F restricted on $\mathcal{P}_x$.

(ii) It is worthwhile to observe that the varieties in Conjecture 4.15 are the same as those in Conjectured theorem 2.12, i.e., although the bounds in Theorem 2.7 and Theorem 4.7 are quite different, the extremal varieties are expected to be the same (and, in particular, Remark 2.13, (iii) should also apply to the varieties on the boundary of Theorem 4.7). This comes as no big surprise in view of Remark 4.13, (i). Assuming the bound (4.13.1), we observe that Conjecture 4.15 implies Theorem 2.12.

(iii) As in Remark 2.13, (iv), we observe that, having classified varieties of minimal codegree, it is natural to proceed with giving classification of varieties of “next to minimal codegree”. However, this notion is not so easy to define. In particular, one should exclude projected varieties and varieties with positive defect. This having been said, we observe that, unlike the bound for minimal degree in Theorem 3.4 which is additive in $n = \dim X$ and $N = \dim \langle X \rangle$, the bound in Theorem 4.7 is of a multiplicative nature. Thus, varieties of next to minimal degree in the sense of Theorem 4.7 should have codegree $d^* = 2\frac{N+1}{n+1}$. An example of such varieties is given by the homogeneous Legendrean varieties $X^n \subset \mathbb{P}^{2n+1}$ mentioned in Remark 3.6, in which case $d^* = 4$.

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