TANGENTS AND SECANTS OF ALGEBRAIC VARIETIES

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CONTENTS

Index of Notations iii
Introduction 1

Chapter I. Theorem on Tangencies and Gauss Maps 14
1 Theorem on tangencies and its applications 15
2 Gauss maps of projective varieties 21
3 Subvarieties of complex tori 28

Chapter II. Projections of Algebraic Varieties 35
1 A criterion for existence of good projections 36
2 Hartshorne’s conjecture on linear normality and its relative analogues 41

Chapter III. Varieties of Small Codimension Corresponding to Orbits of Algebraic Groups 47
1 Orbits of algebraic groups, null-forms and secant varieties 48
2 HV-varieties of small codimension 54
3 HV-varieties as birational images of projective spaces 64

Chapter IV. Severi Varieties 69
1 Reduction to the nonsingular case 70
2 Quadrics on Severi varieties 73
3 Dimension of Severi varieties 79
4 Classification theorems 84
5 Varieties of codegree three 90

Chapter V. Linear Systems of Hyperplane Sections on Varieties of Small Codimension 101
1 Higher secant varieties 102
2 Maximal embeddings of varieties of small codimension 109

Chapter VI. Scorza Varieties 116
1 Properties of Scorza varieties 117
2 Scorza varieties with $\delta = 1$ 122
3 Scorza varieties with $\delta = 2$ 126
4 Scorza varieties with $\delta = 4$ 131
5 End of classification of Scorza varieties 145

Bibliography 149
CHAPTER I

THEOREM ON TANGENCIES AND GAUSS MAPS
1. Theorem on tangencies and its applications

Let \( X^n \subset \mathbb{P}^N \) be an irreducible nondegenerate (i.e. not contained in a hyperplane) \( n \)-dimensional projective variety over an algebraically closed field \( K \), and let \( Y^r \subset \mathbb{P}^N \) be a non-empty irreducible \( r \)-dimensional variety. We set

\[
\Delta_Y = (Y \times X) \cap \Delta_X = \{(y, x) \in Y \times X \mid x = y\},
\]
where \( \Delta_X \) is the diagonal in \( X \times X \),

\[
S^0_{Y, X} \subset (Y \times X \setminus \Delta_Y) \times \mathbb{P}^N, \quad S^0_{Y, X} = \{(y, x, z) \mid z \in \langle x, y \rangle\},
\]
where \( \langle x, y \rangle \) denotes the chord joining \( x \) with \( y \). We denote by \( S_{Y, X} \) the closure of \( S^0_{Y, X} \) in \( Y \times X \times \mathbb{P}^N \), by \( p_i^Y \) the projection of \( S_{Y, X} \) onto the \( i \)th factor of \( Y \times X \times \mathbb{P}^N \) \((i = 1, 2)\), and by \( \varphi^Y : S_{Y, X} \rightarrow \mathbb{P}^N \) the projection onto the third factor, and put

\[
p_{12}^Y = p_1^Y \times p_2^Y : S_{Y, X} \rightarrow Y \times X, \quad S(Y, X) = \varphi^Y(S_{Y, X}),
\]

\[
T_{Y, X}^r = (p_{12}^Y)^{-1} (\Delta_Y), \quad \psi^Y = \varphi^Y|_{T_{Y, X}^r}, \quad T'(Y, X) = \psi^Y(T_{Y, X}^r).
\]

1.1. Definition. The variety \( S(Y, X) \) is called the join of \( Y \) and \( X \) or, if \( Y \subset X \), the secant variety of \( X \) with respect to \( Y \).

We observe that in the case when \( Y = X \) the above definition reduces to the usual definition of secant variety of \( X \); we shall denote \( S(X, X) \) simply by \( SX \).

In what follows we shall assume that \( Y^r \subset X^n \) is a subvariety of \( X \).

1.2. Definition. The variety \( T'(Y, X) \) is called the variety of (relative) tangent stars of \( X \) with respect to the subvariety \( Y \).

We observe that \( T'(X, X) = T'X \) is the usual variety of tangent stars (cf. [45; 97]).

1.3. Definition. The cone \( T_{Y, X, y}^r = \psi^Y \left((p_{12}^Y)^{-1} (y \times y)\right) \) is called the (projective) tangent star to \( X \) with respect to \( Y \subset X \) at a point \( y \in Y \).

From this definition it is evident that \( T_{Y, X, y}^r \) is a union of limits of chords

\[
(y', x'), \quad y' \in Y, \; x' \in X, \quad y', x' \rightarrow y.
\]

It is also clear that \( T_{Y, X, y}^r \subset T_{X, y}^r \subset T_{X, y} \), where \( T_{X, y}^r = T_{X, X, y}^r \) is the (projective) tangent star to \( X \) at \( y \) (cf. [45; 97]) and \( T_{X, y} \) is the (embedded) tangent space to \( X \) at \( y \). On the other hand, \( T_{Y, X, y}^r \supset T_{y, X}^r \), where \( T_{y, X}^r = T_{y, X, y}^r \) is the (projective) tangent cone to \( X \) at the point \( y \).

By definition, \( T'(Y, X) = \bigcup_{y \in Y} T_{Y, X, y}^r \). If \( X \) is nonsingular along \( Y \), i.e. \( Y \cap \text{Sing} \; X = \emptyset \) and \( Y \subset \text{Sm} \; X = X \setminus \text{Sing} \; X \), then \( T'(Y, X) = T(Y, X) = \bigcup_{y \in Y} T_{X, y} \) is the usual variety of tangents.
1.4. Theorem. An arbitrary irreducible subvariety $Y^r \subset X^n$, $r \geq 0$ satisfies one of the following two conditions:

a) $\dim T'(Y, X) = r + n, \quad \dim S(Y, X) = r + n + 1$;

b) $T'(Y, X) = S(Y, X)$.

Proof. Let $t = \dim T'(Y, X)$. It is clear that $t \leq r + n$. In the case when $t = r + n$ the theorem is obvious since $S(Y, X)$ is an irreducible variety, $S(Y, X) \supset T'(Y, X)$ and $\dim S(Y, X) \leq r + n + 1$.

Suppose that $t < r + n$, and let $L^{N-t-1}$ be a linear subspace of $\mathbb{P}^N$ such that $L \cap T'(Y, X) = \emptyset$.

We denote by $\pi: \mathbb{P}^N \setminus L \rightarrow \mathbb{P}^t$ the projection with center at $L$ and put $X' = \pi(X)$, $Y' = \pi(Y)$. Since $\pi|_X$ is a finite morphism, we have $\dim (Y' \times X') = r + n > t$, and from the connectedness theorem of Fulton and Hansen (cf. [26] and [27, 3.1]) it follows that $Y \times X' = (\pi|_Y \times \pi|_X)^{-1}(\Delta_{\mathbb{P}^t})$ is a connected scheme.

I claim that

$$\operatorname{Supp}(Y \times X') = \Delta_Y.$$ (1.4.2)

In fact, suppose that this is not so. Then by definition for all $(y, x) \in (Y \times X') \setminus \Delta_Y$ we have

$$\varphi^Y\left((p_{12})^{-1}(y, x)\right) \cap L \neq \emptyset,$$

and therefore for each point $(y, y) \in \Delta_Y \cap \overline{(Y \times X') \setminus \Delta_Y}$

$$T'(Y, X) \cap L \supset T'_{Y, X, y} \cap L = \varphi^Y\left((p_{12})^{-1}(y, y)\right) \cap L \neq \emptyset$$

contrary to (1.4.1). This proves (1.4.2).

From (1.4.2) it follows that $L \cap S(Y, X) = \emptyset$. Hence

$$t \leq \dim S(Y, X) \leq N - \dim L - 1 = t,$$

i.e. condition b) holds. $\square$

1.5. Corollary. $\operatorname{codim}_{S(Y, X)} T'(Y, X) \leq 1$.

1.6. Definition. Let $L \subset \mathbb{P}^N$ be a linear subspace. We say that $L$ is tangent to a variety $X \subset \mathbb{P}^N$ along a subvariety $Y \subset X$ (resp. $L$ is $J$-tangent to $X$ along $Y$, resp. $L$ is $J$-tangent to $X$ with respect to $Y$) if $L \supset T_{Y, X, y}$ (resp. $L \supset T^*_{Y, X, y}$, resp. $L \supset T'_{Y, X, y}$) for all points $y \in Y$.

It is clear that if $L$ is tangent to $X$ along $Y$, then $L$ is $J$-tangent to $X$ along $Y$ and if $L$ is $J$-tangent to $X$ along $Y$, then $L$ is $J$-tangent to $X$ with respect to $Y$. If $X$ is nonsingular along $Y$, then all the three notions are identical.
1.7. Theorem. Let \( Y^r \subset X^n \) and \( Z^b \subset Y^r \) be closed subvarieties, and let \( L^m \subset \mathbb{P}^N, n \leq m \leq N-1 \) be a linear subspace which is \( J \)-tangent to \( X \) with respect to \( Y \) along \( Y \setminus Z \) (i.e. \( L \supset T_{Y,X,y} \) for all points \( y \in Y \setminus Z \)). Then \( r \leq m - n + b + 1 \).

Proof. It is clear that Theorem 1.7 is true (and meaningless) for \( r \leq b + 1 \).

Suppose that \( r > b + 1 \). Without loss of generality we may assume that \( Y \) is irreducible. Let \( M \) be a general linear subspace of codimension \( b + 1 \) in \( \mathbb{P}^N \). Put \( X' = X \cap M, \quad Y' = Y \cap M, \quad L' = L \cap M. \)

It is clear that
\[
\begin{align*}
n' &= \dim X' = n - b - 1, \\
r' &= \dim Y' = r - b - 1, \\
m' &= \dim L' = m - b - 1
\end{align*}
\]

and \( L' \) is \( J \)-tangent to \( X' \) with respect to \( Y' \) along \( Y' \). In other words,
\[
T'(Y', X') \subset L'.
\]

In particular, from (1.7.2) it follows that
\[
\dim T'(Y', X') \leq m'.
\]

Since \( n > r > b + 1 \), from the Bertini theorem it follows that the varieties \( X' \) and \( Y' \) are irreducible. By [58, Lema 1, Corolario 1], the variety \( X' \) is nondegenerate, and so the relative secant variety \( S(Y', X') \) containing \( X' \) does not lie in the subspace \( L' \). From (1.7.2) it follows that
\[
S(Y', X') \neq T'(Y', X').
\]

In view of (1.7.4) Theorem 1.4 yields
\[
\dim T'(Y', X') = r' + n'.
\]

Combining (1.7.3) and (1.7.5) we see that \( r' + n' \leq m' \), and in view of (1.7.1) \( r \leq m - n + b + 1 \). □

1.8. Corollary (Theorem on tangencies). If a linear subspace \( L^m \subset \mathbb{P}^N \) is tangent to a nondegenerate variety \( X^n \subset \mathbb{P}^N \) along a closed subvariety \( Y^r \subset X^n \), then \( r \leq m - n \).

1.9. Remark. It is clear that if \( Z \) does not contain components of \( Y \), then in the statement of Theorem 1.7 we may assume that \( Z \subset Y \cap \text{Sing} X \).

We give an example showing that the bound in Theorem 1.7 is sharp.

1.10. Example. Let \( X^n \subset \mathbb{P}^N \), \( N = 2n - b - 2 \) be a cone with vertex \( \mathbb{P}^b \) over the Segre variety \( \mathbb{P}^1 \times \mathbb{P}^{n-b-2} \subset \mathbb{P}^{2n-2b-3}, n > b + 2 \). Then
\[
X^* = (X')^* = \mathbb{P}^1 \times \mathbb{P}^{n-b-2} \subset (\mathbb{P}^b)^* = \mathbb{P}^{2n-2b-3},
\]
and a subspace \( L^m \subset \mathbb{P}^N, \, n \leq m \leq N - 1 \) is tangent to \( X \) at a point \( x \in SmX \) (and all points of the \((b+1)\)-dimensional affine linear space \((x, \mathbb{P}^b) \setminus \mathbb{P}^b)\) if and only if the \((N-m-1)\)-dimensional linear subspace \( L^* \) of \( X^* \) contained in the \((N-m-1)\)-dimensional linear subspace \( T_{X,x}^* \subset X^* \) (here and in what follows asterisk denotes dual variety and \( \langle A \rangle \) denotes the linear span of a subset \( A \subset \mathbb{P}^N \)). It is easy to see that an arbitrary \((n-b-3)\)-dimensional linear subspace lying in \( X^* \) coincides with \( T_{X,x}^* \) for some \( x \in X \).

Let \( \mathbb{P}^{n-b-2} \subset X^* = \mathbb{P}^1 \times \mathbb{P}^{n-b-2} \) be a linear subspace, and let \( L^* \) be an arbitrary \((N-m-1)\)-dimensional linear subspace of \( \mathbb{P}^{n-b-2} \). Then the \( m \)-dimensional linear subspace \( L = (L^*)^* \) is tangent to \( X \) at all points of \( Y = \mathbb{P}^{n-b+1} \subset \mathbb{P}^{n-1} \subset X \) and \( Z = \text{Sing } X = \mathbb{P}^b \) the inequality in Theorem 1.7 turns into equality.

**1.11. Proposition.** Let \( X^n \subset \mathbb{P}^N \) be a nondegenerate variety satisfying condition \( R_k \) (cf. [30, Chapter IV2, (5.8.2)]) (in other words, \( X \) is regular in codimension \( k \), i.e. \( b = \dim (\text{Sing } X) < n - k \)), and let \( L \) be an \( m \)-dimensional linear subspace of \( \mathbb{P}^N \). Put \( X' = X \cdot L \), and let \( b' = \dim (\text{Sing } X') \). Then \( b' \leq 2N - m - n + b - 1 = b + c + \varepsilon - 1 \), i.e. \( X' \) satisfies condition \( R_{k-c-2\varepsilon+1} \), where \( c = \text{codim}_{\mathbb{P}^N} X = N - n, \varepsilon = \text{codim}_{\mathbb{P}^N} L = N - m \).

**Proof.** For an arbitrary point \( \lambda \) of the \((\varepsilon - 1)\)-dimensional linear subspace \( L^* \subset \mathbb{P}^N \) we put \( X_\lambda = X \cdot \lambda^* \), where \( \lambda^* \) is the hyperplane corresponding to \( \lambda \). It is clear that \( X' = \bigcap_{\lambda \in L^*} X_\lambda \). Let \( Y = \text{Sing } X' \), \( Y_\lambda = \text{Sing } X_\lambda \), \( \lambda \in L^* \). It is easy to see that \( Y \subset \bigcup_{\lambda \in L^*} Y_\lambda \), so that

\[
\exists_{\lambda \in L^*} b'_\lambda = \dim Y \leq \max_{\lambda \in L^*} b_\lambda + \varepsilon - 1, \tag{1.11.1}
\]

where \( b_\lambda = \dim Y_\lambda \). It is clear that the hyperplane \( \lambda^* \) is tangent to \( X \) at all points of \( Y_\lambda \setminus \text{Sing } X \). Hence from Theorem 1.7 it follows that

\[
b_\lambda \leq b + c. \tag{1.11.2}
\]

Combining (1.11.1) and (1.11.2) we obtain the desired bound for \( b' \). \( \square \)

The following simple example shows that the bound in Proposition 1.11 is sharp.

**1.12. Example.** Let \( X^{N-1} \subset \mathbb{P}^N \) be a quadratic cone with vertex \( \mathbb{P}^b \), and let \( \left[ \frac{N+b}{2} \right] + 1 \leq m \leq N - 1 \) (here and in what follows \( [a] \) is the largest integer not exceeding a given number \( a \in \mathbb{R} \)). Then \( X^* \) is a nonsingular quadric in the \((N - b - 1)\)-dimensional linear subspace \( \mathbb{P}^b^* \subset \mathbb{P}^{N^*} \). It is well known (cf. [28, Volume II, Chapter 6; 37, Chapter XIII]) that \( X^* \) contains a linear subspace of dimension \( \left[ \frac{N-b-2}{2} \right] \). Let \( L^* \) be its linear subspace of dimension \( N - m - 1 \). Put \( L = (L^*)^* \), \( X' = X \cdot L \). Then \( \dim L = m \), and it is easy to see that \( Y = \text{Sing } X' \) is an \((N - m + b)\)-dimensional linear subspace.
1.13. Corollary. Suppose that a variety $X^n \subset \mathbb{P}^N$ satisfies conditions $S_{r+1} = S_{N-m+1}$ and $R_{e+2c-1} = R_{3N-2m-n-1}$, and let $L^m \subset \mathbb{P}^N$ be a linear subspace for which $\dim (X \cdot L) = n - \varepsilon = m + n - N$. Then the scheme $X \cdot L$ is reduced. In particular, if $X$ is nonsingular, $N < \frac{3}{2}(m + n + 1)$, and $\dim (X \cdot L) = m + n - N$, then $X \cdot L$ is a reduced scheme.

Proof. From Proposition 1.11 it follows that in the conditions of Corollary 1.13 $X' = X \cdot L$ satisfies condition $R_0$. Since $\dim X' = n - \varepsilon$, $X'$ satisfies condition $S_1$ (cf. [61, §17]). Hence to prove Corollary 1.13 it suffices to apply Proposition 5.8.5 from [30, Chapter IV.2].

1.14. Corollary. If $X^n \subset \mathbb{P}^N$ satisfies conditions $S_{e+2} = S_{N-m+2}$ and $R_{e+2c} = R_{3N-2m-n}$ and $L^m \subset \mathbb{P}^N$ is a linear subspace such that $\dim (X^n \cdot L^m) = n - \varepsilon = m + n - N$, then the scheme $X \cdot L$ is normal (and therefore irreducible and reduced). In particular, if $X$ is nonsingular, $N \leq \frac{3}{2}(m + n)$ and $\dim (X \cdot L) = m + n - N$, then $X \cdot L$ is a normal scheme.

Proof. From Proposition 1.11 it follows that in the conditions of Corollary 1.14 $X' = X \cdot L$ satisfies condition $R_1$. Since $\dim X' = n - \varepsilon$, $X'$ satisfies condition $S_2$ (cf. [61, §17]). Hence to prove Corollary 1.14 it suffices to apply Serre's normality criterion (cf. [30, Chapter IV.2, (5.8.6)]).

Of special importance to applications is the case when $L$ is a hyperplane. We formulate our results in this case.

1.15. Corollary. a) If a variety $X^n \subset \mathbb{P}^N$ is nondegenerate and normal and $N \leq 2n - b - 1$, where $b = \dim (\text{Sing} \, X)$, then all hyperplane sections of $X$ are reduced. In particular, if $X$ is nonsingular and $N < 2n$, then all hyperplane sections of $X$ are reduced.

b) If a nondegenerate variety $X^n \subset \mathbb{P}^N$ has properties $S_3$ and $R_{N-n+2}$ (the last assumption means that $N < 2n - b - 2$), then all hyperplane sections of $X$ are normal (and therefore irreducible and reduced). In particular, if $X$ is nonsingular and $N < 2n - 1$, then all hyperplane sections of $X$ are normal.

1.16. Remark. Corollary 1.15 gives a much more precise information than Bertini type theorems describing properties of generic hyperplane sections (cf. e.g. [80]), but, as shown by Examples 1.18 and 1.19 below, the assumptions in its statement cannot be weakened.

1.17. Remark. If $K = \mathbb{C}$ and $b = -1$, then in the assumptions of Corollary 1.15 b) irreducibility of hyperplane sections follows from the Barth-Larsen theorem according to which for $N < 2n - 1$ the Picard group $\text{Pic} \, X \simeq \mathbb{Z}$ is generated by the class of hyperplane section of $X$ (cf. [54; 60; 65]).

We give examples showing that the bounds in Corollary 1.15 are sharp.

1.18. Example. Let $X_0 = \mathbb{P}^1 \times \mathbb{P}^{n-b-1} \subset \mathbb{P}^{2n-2b-1}$, $n > b + 1$, and let $Y_0 = x \times \mathbb{P}^{n-b-2} \subset X_0$ be a linear subspace. We denote by $X' \subset \mathbb{P}^{2(n-b-1)}$ the section of $X_0$ by a general hyperplane passing through $Y_0$. It is easy to see that $X'$ is a nonsingular projectively normal variety (cf. e.g. [73]). Let $X^n \subset \mathbb{P}^N$
I. THEOREM ON TANGENCIES AND GAUSS MAPS

,..., \( N = 2n - b - 1 \) be the projective cone with vertex \( \mathbb{P}^b \) over \( X' \). It is clear that \( X \) is a normal variety and \( \dim(\text{Sing} \ X) = b \), so that \( X \) satisfies conditions \( S_2 \) and \( R_{n-b-1} = R_{N-n} \). However \( X \) has a non-reduced hyperplane section corresponding to the hyperplane in \( \mathbb{P}^{2n-2b-1} \) which is tangent to \( X_0 \) along \( Y_0 \) (cf. Example 1.10).

1.19. Example. Let \( X_0 = \mathbb{P}^{n-b-2} \subset \mathbb{P}^{2n-2b-3} \), \( n > b + 2 \), and let \( X \) be the projective cone with vertex \( \mathbb{P}^b \) over \( X_0 \). Then \( X^n \subset \mathbb{P}^N \), \( N = 2n - b - 2 \) is a Cohen-Macaulay variety (cf. e.g. [47; 73]) and \( \dim(\text{Sing} \ X) = b \), so that \( X \) satisfies conditions \( S_3 \) and \( R_{n-b-1} = R_{N-n+1} \). However for each hyperplane \( L \) such that \( L^* \in X^* \), \( L \cdot X \) is a reducible and therefore non-normal variety, viz. \( L \cdot X = H_1 \cup H_2 \), where \( H_1 = \mathbb{P}^{n-1} \) and \( H_2 \) is the cone with vertex \( \mathbb{P}^b \) over \( \mathbb{P}^1 \times \mathbb{P}^{n-b-3} \), is a reducible and therefore non-normal variety, and \( \text{Sing}(L \cdot X) = H_1 \cap H_2 = \mathbb{P}^{n-2} \) (cf. Example 1.10).
2. Gauss maps of projective varieties

Let \( X^n \subset \mathbb{P}^N \) be an irreducible nondegenerate variety. For \( n \leq m \leq N - 1 \) we put
\[
P_m = \{ (x, \alpha) \in \text{Sm} X \times G(N, m) \mid L_\alpha \supset T_{X,x} \},
\]
where \( G(N, m) \) is the Grassmann variety of \( m \)-dimensional linear subspaces in \( \mathbb{P}^N \), \( L_\alpha \) is the linear subspace corresponding to a point \( \alpha \in G(N, m) \), and the bar denotes closure in \( X \times G(N, m) \). We denote by \( p_m : P_m \to X \) (resp. \( \gamma_m : P_m \to G(N, m) \)) the projection map to the first (resp. second) factor.

2.1. Definition. The map \( \gamma_m \) is called the \( m \)th Gauss map, and its image \( X_m^* = \gamma_m(P_m) \) is called the variety of \( m \)-dimensional tangent subspaces to the variety \( X \).

2.2. Remark. Of special interest are the two extreme cases, viz. \( m = n \) and \( m = N - 1 \). For \( m = n \) we get the ordinary Gauss map \( \gamma : X \dashrightarrow G(N, n) \), and for \( m = N - 1 \) we see that \( X_{N-1}^* = X^* \subset \mathbb{P}^N \) is the dual variety and if \( X \) is nonsingular, then \( P_{N-1}^N = \mathbb{P}(N_{\mathbb{P}^N/X^*}(-1)) \), where \( N_{\mathbb{P}^N/X^*} \) is the normal bundle to \( X \) in \( \mathbb{P}^N \) (cf. [16, Exposé XVII]).

2.3. Theorem. Let \( \dim (\text{Sing} X) = b \geq -1 \). Then
\begin{enumerate}[a)]
\item for each point \( \alpha \in \gamma_m (p_m^{-1} (\text{Sm} X)) \), \( \dim \gamma_m^{-1}(\alpha) \leq m - n + b + 1 \);
\item for a general point \( \alpha \in X_m^* \), \( \dim \gamma_m^{-1}(\alpha) \leq \max \{ b + 1, m + n - N - 1 \} \);
\item \( \dim X_m^* \geq \min \{ (m-n)(N-m)+n-b-1, (m-n+1)(N-m)+1 \} \);
\item if \( \text{char} K = 0 \) and \( \gamma_m = \nu_m \circ \tilde{\gamma}_m \) is the Stein factorization of the morphism \( \gamma_m \), then \( \nu_m \) is a birational isomorphism and the generic fiber of the morphism \( \gamma_m \) (and \( \tilde{\gamma}_m \)) is a linear subspace of \( \mathbb{P}^N \) of dimension \( \dim P_m - \dim X_m^* \).
\end{enumerate}

Proof. a) immediately follows from Theorem 1.7, and since
\[
\dim P_m = \dim X + \dim G(N - n - 1, m - n - 1) \\
= n + (m-n)(N-m),
\]
(2.3.1) \( a' \) follows from a).

b) Suppose first that \( m = N - 1 \). It is clear that \( \dim \gamma_m^{-1}(\alpha) \leq n - 1 \), and it suffices to verify that if \( n - 1 \geq b + 2 \), i.e. \( n \geq b + 3 \), then for a general point \( \alpha \in X^* \) we have \( \dim \gamma_m^{-1}(\alpha) \neq n - 1 \). Suppose that this is not so, and let \( x \) be a general point of \( X \). Since \( n - 1 > b + 1 \), from Theorem 1.7 it follows that the system of divisors
\[
Y_\alpha = p_{N-1} (\gamma_m^{-1}(\alpha)), \quad \alpha \in T_{X,x}^X
\]
is not fixed, and therefore \( X = \overline{\bigcup_\alpha Y_\alpha} \), where \( \alpha \) runs through the set of general points of \( T_{X,x}^X \). Hence for a general point \( y \in X \) there exists a hyperplane \( \Lambda_y \subset T_{X,x}^X \) such that for a general point \( \beta \in \Lambda_y \) we have \( L_\beta \supset T_{X,y} \). But then
\[
(T_{X,x}, T_{X,y}) \subset (\Lambda_y)^+ = \mathbb{P}^{n+1},
\]
i.e. for a general pair of points \( x, y \in X \) we have
\[
\dim (T_{X,x} \cap T_{X,y}) = n - 1.
\]
From this it follows that either all \( n \)-dimensional linear subspaces from \( \gamma_n(X) \) are contained in an \((n + 1)\)-dimensional linear subspace \( \mathbb{P}^{n+1} \subset \mathbb{P}^N \) or they all pass through an \((n - 1)\)-dimensional subspace \( \mathbb{P}^{n-1} \subset \mathbb{P}^N \). But in the first case \( X \) is a hypersurface and by Theorem 1.7 \( \dim Y_\alpha = n - 1 \leq b + 1 \), contrary to our assumption, and in the second case the intersection of \( X \) with a general linear subspace \( \mathbb{P}^{N-n+1} \subset \mathbb{P}^N \) is a nonsingular strange curve (we recall that a projective curve of degree \( \geq 2 \) is called strange if all its tangent lines pass through a fixed point). It is well known (cf. [59; 34, Chapter IV; 39 or 75]) that the only nonsingular strange curves are conics in characteristic 2. Therefore in the second case \( X \) is a quadric, and we again come to a contradiction. Thus assertion b) holds for \( m = N - 1 \) (if \( \text{char} \, K = 0 \), then one can simplify the proof using the reflexivity theorem according to which \( (X^*)^* = X \) (cf. [96]).

Next we prove assertion b) for \( m = k \) under the assumption that it holds for \( m = k + 1 \). It is clear that for general points \( \alpha_k \in X_k^* \), \( \alpha_{k+1} \in X_{k+1}^* \) we have
\[
\dim Y_{\alpha_k} \leq \dim Y_{\alpha_{k+1}}.
\]
(2.3.2)

If \( b + 1 \geq k + n - N \), then from the induction hypothesis it follows that
\[
\dim Y_{\alpha_k} \leq \dim Y_{\alpha_{k+1}} \leq b + 1.
\]
Suppose that
\[
\dim Y_{\alpha_{k+1}} \leq k + n - N > b + 1.
\]
(2.3.3)
If \( \dim Y_{\alpha_k} < \dim Y_{\alpha_{k+1}} \), then assertion b) immediately follows from (2.3.3). Otherwise from (2.3.2) and (2.3.3) it follows that for a general point \( x \in X \) and a general point \( \alpha_{k+1} \in X_{k+1}^* \) for which \( Y_{\alpha_{k+1}} \ni x \) each hyperplane in \( L_{\alpha_{k+1}} \) containing \( T_{X,x} \) is tangent to \( X \) at all points of a \((\dim Y_{\alpha_{k+1}})\)-dimensional component of \( Y_{\alpha_{k+1}} \) that are nonsingular on \( X \), and by Theorem 1.7 \( \dim Y_{\alpha_{k+1}} \leq b + 1 \). But then
\[
\dim Y_{\alpha_k} = \dim Y_{\alpha_{k+1}} \leq b + 1,
\]
so that inequality b) holds also in this case. Assertion b) is proved.

b') immediately follows from b) in view of (2.3.1).

c) Let \( \alpha_m \) be a general point of \( X_m^* \). The linear subspace \( L_m \subset \mathbb{P}^N \) is tangent to \( X \) at all points of the subvariety
\[
Y_m \cap \text{Sm} \, X,
\]
and it is easy to see that
\[
Y_m \cap \text{Sm} \, X = \bigcap_{L \supset L_m} (Y_\alpha \cap \text{Sm} \, X),
\]
(2.3.4)
where $\alpha$ runs through the set of points of $X^*$ for which $L_\alpha \supset L_{\alpha_m}$. From the reflexivity theorem (cf. e.g. [49]) it follows that if char $K = 0$, then for a general point $\alpha \in X^*$ we have

$$Y_\alpha = p_{N-1}(\gamma_{N-1}^1(\alpha)) = (T_{X^*, \alpha})^*$$  \hspace{1cm} (2.3.5)$$

is a linear subspace of $\mathbb{P}^N$ of dimension $N - \dim X^* - 1$. From (2.3.4) and (2.3.5) it follows that

$$Y_{\alpha_m} = \bigcap_{L_\alpha \supset L_{\alpha_m}} (T_{X^*, \alpha})^*$$

is also a linear subspace of $\mathbb{P}^N$. Since char $K = 0$, the morphism $\gamma_m$ is separable and therefore smooth at a general point. Hence $\nu_m$ is a birational isomorphism. This completes the proof of assertion c) and Theorem 2.3. □

We observe that if char $K = p > 0$, then assertion c) of Theorem 2.3 is no longer true. As an example, it suffices to consider the hypersurface in $\mathbb{P}^{n+1}$ defined by equation $\sum_{i=0}^{n+1} x_i^{p+1} = 0$ (in this case $\gamma$ is the Frobenius map). The case of positive characteristic is treated in [50].

2.4. Corollary. If char $K = 0$, $X^n \subset \mathbb{P}^N$ is a nonsingular variety, and $N - n + 1 \leq m \leq N - 1$, then a general $m$-dimensional tangent subspace is tangent to $X$ along a linear subspace of dimension at most $m + n - N - 1$ (for $N \geq 2n$ this bound is better than the one given in Theorem 1.7). For $n \leq m \leq N - n + 1$ a general $m$-dimensional tangent subspace is tangent to $X$ at a single point.

2.5. Corollary. Let $X^n \subset \mathbb{P}^N$, $X^n \neq \mathbb{P}^n$, $n^* = \dim X^*$, $b = \dim (\text{Sing } X)$. Then $n^* \geq n - b - 1$. In particular, for a nonsingular variety $n^* \geq n$. If $n \geq b + 3$, then $n \geq N - n + 1$ (this bound is better than the preceding one if $N \geq 2n - b - 1$).

The following example shows that both bounds in Corollary 2.5 are sharp.

2.6. Example. Let $X_0 = \mathbb{P}^1 \times \mathbb{P}^{n-b-2} \subset \mathbb{P}^{2n-2b-3}$, $n > b + 2$, and let $X$ be a projective cone with vertex $\mathbb{P}^b$ and base $X_0$. Then $X^n \subset \mathbb{P}^N$, $N = 2n - b - 2$, $\dim (\text{Sing } X) = b$, $X^* = X_0^* \simeq X_0$, and $n^* = n - b - 1 = N - n + 1$.

2.7. Remark. In the case when char $K = 0$ and $b = -1$, the inequality $n^* \geq N - n + 1$ was independently proved by Landman (cf. [50]). Another proof was earlier given by the author (cf. [96, Proposition 1] for $n = 2$; the general case is quite similar).

2.8. Corollary. Let $X^n \subset \mathbb{P}^N$, $X^n \neq \mathbb{P}^n$, $b = \dim (\text{Sing } X)$. Then $\dim \gamma(X) \geq n - b - 1$. In particular, for a nonsingular variety, $\dim \gamma(X) = \dim X$ and $\gamma$ is a finite morphism. If in addition char $K = 0$, then $\gamma$ is a birational isomorphism (i.e. $\gamma$ is the normalization morphism).

2.9. Remark. In the case when $K = \mathbb{C}$ and $b = -1$, Griffiths and Harris [29] proved that $\dim \gamma_\alpha(X) = \dim X$. Different proofs of finiteness of $\gamma_\alpha$ in this case were later given by Ein [18] and Ran [68]. In our first proof of Corollary 2.8 (and Theorem 1.7) we used methods of formal geometry. Since related techniques is used in §3, we give this proof here.
As in the proof of Theorem 1.7, considering the intersection of $X$ with a general $(N-b-1)$-dimensional linear subspace of $\mathbb{P}^N$ we reduce everything to the case when $b = -1$. Suppose that the $n$-dimensional linear subspace $L$ corresponding to a point $\alpha_L \in G(N,n)$ is tangent to $X$ along an irreducible subvariety $Y$, $\dim Y > 0$, i.e. $Y \subset \gamma^{-1}(\alpha_L)$. Let $\mathfrak{X} = X_{/Y}$ be the completion of $X$ along $Y$, and let $\mathfrak{G} = \gamma(X)/\alpha_L$ be the formal neighborhood of the point $\alpha_L$ in the variety $\gamma(X) \subset G(N,n)$. Since $X^n \neq \mathbb{P}^n$, $\dim \gamma(X) > 0$. Hence $H^0(\mathfrak{G}, \mathcal{O}_\mathfrak{G})$ and $H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \supset H^0(\mathfrak{G}, \mathcal{O}_\mathfrak{G})$ are infinite-dimensional vector spaces over the field $K$.

On the other hand, let $M \subset \mathbb{P}^N$ be a linear subspace, $\dim M = N - n - 1$, $M \cap L = \emptyset$, and let $\pi: X \dashrightarrow \mathbb{P}^n$ be the projection with center at $M$. Then $\pi_{/Y}: \mathfrak{X} \rightarrow \mathbb{P}^n_{/\pi(Y)}$ is an isomorphism of formal spaces, and therefore

$$H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \simeq H^0(\mathcal{L}, \mathcal{O}_\mathcal{L}),$$

(2.9.1)

where $\mathcal{L} = L_{/Y} \simeq \mathbb{P}^n_{/\pi(Y)}$ is the completion of $L$ along $Y$. But by the well-known theorem on formal functions (cf. [31, Chapter V; 36]), $H^0(\mathcal{L}, \mathcal{O}_\mathcal{L}) = K$ which is impossible since $H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is infinite-dimensional in view of (2.9.1).

The above contradiction shows that $\dim Y = 0$, i.e. $\gamma$ is a finite morphism.

Although, as we have already seen, the bounds in Theorem 2.3 are sharp, one can still prove stronger results for certain special classes of projective varieties. An important example is given by complete intersections.

2.10. Proposition. Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular complete intersection. Then all Gauss maps $\gamma_m, n \leq m \leq N - 1$ are finite and $\dim X_m^n = \dim P^n_m = n + (m-n)(N-m)$. If in addition $\operatorname{char} K = 0$, then all $\gamma_m, n \leq m \leq N-1$ are birational isomorphisms.

Proof. Let $\alpha_m \in X_m^n, \alpha \in X^n$ be points for which there is an inclusion of the corresponding linear subspaces $L_{\alpha_m} \subset L_\alpha$. Then it is clear that $\gamma_m^{-1}(\alpha_m) \subset \gamma_{N-1}^{-1}(\alpha)$. Hence it suffices to prove Proposition 2.10 in the case when $m = N - 1$.

We recall that $P_{N-1} = \mathbb{P} \left( N_{P_{N-1}/X^n}(-1) \right)$ (cf. Remark 2.2). Furthermore, the morphism $\gamma_{N-1}: P_{N-1} \rightarrow X_{N-1}$ is defined by a linear subsystem without fixed points of the complete linear system $[\mathcal{O}_{P_{N-1}}(1)]$, where $\mathcal{O}_{P_{N-1}}(1)$ is the tautological sheaf on $\mathbb{P} \left( N_{P_{N-1}/X^n}(-1) \right)$ (cf. [16, ExposÉ XVII]). In view of [30, Chapter II, 6.6.3] and [31, Chapter III, to show that $\gamma_{N-1}$ is finite it suffices to verify that $N_{P_{N-1}/X^n}(-1)$ is an ample vector bundle. But if $X$ is complete intersection of hypersurfaces $F_i$,

$$\deg F_i = a_i \geq 2, \quad i = 1, \ldots, N - 1,$$

then

$$N_{P_{N-1}/X^n}(-1) = \bigoplus_{i=1}^{N-n} \mathcal{O}_X(a_i - 1),$$

and by [31, Chapter III] $N_{P_{N-1}/X^n}(-1)$ is an ample bundle.

The remaining assertions of Proposition 2.10 follow from (2.3.1) and assertion c) of Theorem 2.3. □

2.11. Remark. The above proof of Proposition 2.10 can also be interpreted in elementary terms; cf. [42].
The Gauss map $\gamma: X \to G(N,n)$, where $X^n \subset \mathbb{P}^N$, $X^n \neq \mathbb{P}^n$ is a nonsingular variety, can also be interpreted in another way. To begin with, $\gamma$ is the map corresponding to the vector bundle $N_{\mathbb{P}^N/X^n}(-1)$ with a distinguished $(N+1)$-dimensional vector subspace of sections corresponding to points of $K^{N+1}$ (where $\mathbb{P}^N = (K^{N+1} \setminus 0)/K^*$; cf. [28]).

Furthermore, let $L \subset \mathbb{P}^N$, dim $L = N-n-1$ be a general linear subspace, and let $\pi_L: X \to \mathbb{P}^n$ be the projection with center in $L$. We denote by $R_L$ the ramification divisor of the finite covering $\pi_L$,

$$R_L = \{x \in X \mid T_{X,x} \cap L \neq \emptyset\}.$$  

The Gauss map $\gamma$ is defined by the linear system $|R_L|$ generated by the divisors $R_L$, $L \in G(N,N-n-1)$. This linear system does not have fundamental points, and ramification divisors $R_L$ corresponding to various linear subspaces $L^{N-n-1} \subset \mathbb{P}^N$ are preimages of Schubert divisors on $G(N,n)$ (cf. [28, Chapter 1; 37, Chapter XIV, §8]).

2.12. Proposition. The linear system $|R_L|$ is ample.

Proof. Proposition 2.12 immediately follows from Corollary 2.8 in view of [30, Chapter II, 6.6.3].

2.13. Remark. In the case when char $K = 0$ Ein [18] proved that ramification divisor is ample for an arbitrary nonsingular finite covering of $\mathbb{P}^n$ of degree greater than one.

Let $X^n \subset \mathbb{P}^N$, $X^n \neq \mathbb{P}^n$ be a nonsingular variety. The exact sequences

$$0 \to T_X \to \mathcal{O}_{X}^{N+1} \to N(-1) \to 0,$$

$$0 \to \mathcal{O}_X(-1) \to T_X \to \Theta_X(-1) \to 0,$$

where $\Theta_X$ is the tangent bundle to $X$ and $T_X = \gamma^*(S)$ is the preimage of the standard vector subbundle $S$ of rank $n+1$ on $G(N,n)$ (so that projectivizations of fibres of $T_X$ naturally correspond to projective tangent spaces to $X$), show that

$$\gamma^*(\mathcal{O}_{G(N,n)}(1)) \simeq \text{det } T_X \simeq K_X(n+1) = K_X \otimes \mathcal{O}_X(n+1),$$

where $K_X$ is the canonical line bundle on $X$ (cf. [64, 6.19]; we denote by the same symbol a bundle and the corresponding sheaf of sections). We remark that the property that a section of the line bundle $K_X(n+1)$ vanishes along a divisor from $|R_L|$ lies in the basis of the classical definition of canonical class. An immediate consequence of Proposition 2.12 is the following

2.14. Corollary. Let $X^n \subset \mathbb{P}^N$, $X^n \neq \mathbb{P}^n$ be a nonsingular variety. Then $K_X(n+1)$ is an ample line bundle.

2.15. Remark. It is worthwhile to compare Corollary 2.14 with some known results on the index of Fano varieties [51]. In general the role of very ampleness versus ampleness in such type of results is still to be investigated. However in the conditions of Corollary 2.14 the bundle $K_X(n+1)$ is actually very ample, at least if
26 I. THEOREM ON TANGENCIES AND GAUSS MAPS

char $K = 0$ (cf. [18]). This is easily shown by induction on $n$ using the fact that $X$ has sufficiently many nonsingular hyperplane sections, and by Kodaira’s vanishing theorem, for such a section $H^{n-1} \subset X^n$ the complete linear system

$$|K_H + nH^2| = |K_X + (n+1)H|$$

is cut by the linear system $|K_X + (n+1)H|$ (here $K_H$ is the canonical class of $H$; we denote by the same symbol the canonical divisor class and the canonical line bundle).

2.16. Proposition. Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety, and let $Y^r \subset X^n$ be a subvariety of $X$ for which $m - n = \dim L < \dim \mathbb{P}^N X = N - n$, where $L^m = \langle Y \rangle$ is the linear span of $Y$. Then $r \leq \min \{n-1, \left[\frac{N+b}{2}\right]\}$, where $b = \dim (\text{Sing} X)$.

Proof. Without loss of generality we may assume that $Y \not\subset \text{Sing} X$. From our assumption it follows that for an arbitrary point $y \in Y$

$$\dim (T_{X,y} \cap L) \geq \dim T_{y,y} \geq r. \quad (2.16.1)$$

Hence

$$\gamma(Y) = \gamma(Y \cap \text{Sm} X) \subset \{\alpha \in \text{G}(N,n) \mid \dim L_\alpha \cap L \geq r\} = S(L, r) \subset \text{G}(N,n),$$

where $S(L, r)$ is the corresponding Schubert cell and $\gamma: X \dasharrow \text{G}(N,n)$ is the Gauss map. Since by our assumption $m - r < N - n$, i.e. $n + m - r < N$, from (2.16.1) it follows that for each point $y \in Y \cap \text{Sm} X$ there exists a hyperplane $L$ containing $Y$ which is tangent to $X$ at $y$. Put

$$S(M, L, r) = \{\alpha \in \text{G}(N,n) \mid L_\alpha \in M, \dim L_\alpha \cap L \geq r\}.$$ 

Then $S(M, L, r) \subset S(L, r)$ and

$$\dim S(M, L, r) = (r+1)(m-r) + (n-r)(N-n-1),$$

$$\dim S(L, r) = (r+1)(m-r) + (n-r)N - n,$$

$$\text{codim}_S S(M, L, r) = n - r = \text{codim}_X Y.$$ 

Replacing if necessary $r$ by $\min \{\dim (T_{X,y} \cap L) \}$ we may assume that

$$\gamma(Y) \cap S(M, L, r) \cap \text{Sm} (S(L, r)) \neq \emptyset.$$ 

Then

$$\dim (\gamma(Y) \cap S(M, L, r)) \geq \dim \gamma(Y) - \text{codim}_{S(L, r)} S(M, L, r)$$

$$= (r-f) - (n-r) = 2r - n - f, \quad (2.16.2)$$

where $f$ is the dimension of general fiber of $\gamma|_Y$. On the other hand

$$\gamma(Y) \cap S(M, L, r) = \gamma(\{y \in Y \cap \text{Sm} X \mid T_{X,y} \subset M\}), \quad (2.16.3)$$
and from Theorem 1.7 it follows that
\[
\dim (\gamma(Y) \cap S(M, L, r)) \leq N - n + b - f. \tag{2.16.4}
\]

Combining (2.16.3) and (2.16.4), we conclude that \(2r - n - f \leq N - n + b - f\), i.e. \(r \leq \left\lfloor \frac{N + b}{3} \right\rfloor\). Proposition 2.16 is proved. \(\Box\)

We observe that \(\left\lfloor \frac{N + b}{2} \right\rfloor < n - 1\) for \(N < 2n - b - 2\).

2.17. Remark. For \(K = \mathbb{C}\), \(b = -1\) Proposition 2.16 can be also deduced from the Barth-Larsen theorem on the structure of integral cohomology of \(X\) (cf. [54]).

2.18. Remark. It is worthwhile to compare Proposition 2.16 with the known classical result the first rigorous proof of which was probably given by Lluis (cf. [58, Lema 1, Corolario 1]) in which \(r\) is arbitrary, but \(L\) is a general linear subspace.

2.19. Example. Let \(X_0^{n-b-1}, n \geq b + 5, n + b \equiv 1 \pmod{2}\) be a general linear projection of the Grassmann variety \(G(\frac{n-b+1}{2}, 1)\) in \(\mathbb{P}^{2n-2b-5}\), and let \(X_0 \subset \mathbb{P}^N, N = 2n - b - 4\) be a cone with vertex \(\mathbb{P}^b\) and base \(X_0\). Then \(X_0 \simeq G(\frac{n-b+1}{2}, 1)\) (cf. [33; 38]) and \(\dim (\text{Sing} Y) = b\). Furthermore, \(X_0 \cap Y^r\), where \(Y, b < r < n, r \equiv n \pmod{2}\) is the cone with vertex \(\mathbb{P}^b\) over \(Y_0^{r-b-1}\), and \(Y_0\) is the projection of a Grassmann subvariety \(G(\frac{r-b+1}{2}, 1) \subset G(\frac{n-b+1}{2}, 1)\). Then \(m = b + 1 + 2(r-b-1) - 3 = 2r - b - 4\), and \(m - r = r - b - 4 < N - n = n - b - 4\). On the other hand, for \(r = n - 2\) we have an equality in Proposition 2.16, viz. \(r = \left\lfloor \frac{N + b}{2} \right\rfloor = 2n - 4\).

2.20. Corollary. If \(X^n \not\subset \mathbb{P}^n\), then \(X\) does not contain linear subspaces of dimension greater than \(\left\lfloor \frac{N + b}{2} \right\rfloor\). If \(X\) is not a hypersurface (i.e. \(N > n + 1\)), then \(X\) does not contain projective hypersurfaces of dimension greater than \(\left\lfloor \frac{N + b}{2} \right\rfloor\).

The following examples show that the bound in Corollary 2.20 is sharp.

2.21. Example. a) Let \(X_0^{n-b-1}, n \geq b + 2\) be a nonsingular quadric, and let \(X^n \subset \mathbb{P}^N, N = n + 1\) be a cone with vertex \(\mathbb{P}^b\) and base \(X_0\). Then \(\dim (\text{Sing} Y) = b\), and \(X\) contains a linear subspace \(Y^r = \mathbb{P}^r\), where \(r = b + 1 + \left\lfloor \frac{n-b-1}{2} \right\rfloor = \left\lfloor \frac{N + b}{2} \right\rfloor\) (cf. [28, Volume 2, Chapter 6; 37]).

b) Let \(X_0 = \mathbb{P}^1 \times \mathbb{P}^{n-b-2}, b \geq b + 3\) be a Segre variety, and let \(X^n \subset \mathbb{P}^N, N = 2n - b - 2\) be a cone with vertex \(\mathbb{P}^b\) and base \(X_0\). Then \(\dim (\text{Sing} Y) = b\), and \(X\) contains a linear subspace \(Y^{n-1} = \mathbb{P}^{n-1}\). In this case \(r = n - 1 = \frac{N + b}{2}\).

b) In the assumptions of Example 2.19, let \(n = b + 7\). Then \(X^n \subset \mathbb{P}^{n+3}\) contains the quadratic cone \(Y^{n-2}\) with vertex \(\mathbb{P}^{b}\) whose base is a nonsingular four-dimensional quadric \(G(3, 1)\). Here \(n - 2 = \frac{n + b + 3}{2} = \frac{N + b}{2}\).

Apparently, it is hard to construct examples of multi-dimensional varieties containing a hypersurface of dimension \(\left\lfloor \frac{N + b}{2} \right\rfloor\).
3. Subvarieties of complex tori

Besides subvarieties of projective space there is another important class of varieties for which it is natural to introduce Gauss maps, viz. subvarieties of complex tori.

Let $A^N$ be an $n$-dimensional complex torus, and let $X^n \subset A^N$ be an analytic subset. Let $C^N$ be the universal covering of $A^N$, and let $C^N \to A^N$ be the corresponding homomorphism of abelian groups. Using shifts, one can identify the tangent space to $A^N$ at an arbitrary point $z \in A^N$ with $C^N$, and the tangent space to $X$ at a point $x \in X$ can be identified with a vector subspace $\Theta_{X,x} \subset C^N$.

3.1. Definition. Let $A$ be a complex torus, and let $Y \subset A$ be a connected analytic subset. The smallest subtorus of $A$ containing all the differences $y - y'$, $y, y' \in Y$ (in the sense of group structure on $A$) is called the toroidal hull of $Y$ and is denoted by $\langle Y \rangle$.

We observe that for an arbitrary point $y \in Y$ we have $Y \subset y + \langle Y \rangle$.

3.2. Lemma. Let $Y \subset A^N$ be a connected compact analytic subset whose tangent subspaces at smooth points are contained in a vector subspace $C^m \subset C^N$. Then $\dim \langle Y \rangle \leq m$.

Proof. It is easy to see that there exist an $N$-dimensional torus $\tilde{A}^N$ and an $m$-dimensional subtorus $T^m \subset \tilde{A}^N$, $T^m \supset Y$ such that $\tilde{A}$ is locally isomorphic to $A$ in a neighborhood of $Y$. It is clear that in a suitable neighborhood of $T$ in $\tilde{A}$ and therefore in sufficiently small neighborhoods of $Y$ in $A$ and $Y$ in $\tilde{A}$ there exist $N - m$ analytically independent holomorphic functions. On the other hand, from [5] and [36] it follows that in a small neighborhood of $Y$ in $A$ there exist exactly $\dim A - \dim \langle Y \rangle$ analytically independent holomorphic functions. Hence $\dim A - \dim \langle Y \rangle \geq N - m$, i.e. $\dim \langle Y \rangle \leq m$. □

3.3. Definition. Let $X^n \subset A^N$ be an $n$-dimensional analytic subset of an $N$-dimensional torus $A$, and let $Y^r$ be an $r$-dimensional analytic subset of $X$. We say that a vector subspace $C^m \subset C^N$ is tangent to $X$ along an analytic subset $Y \subset X$ if $C^m \supset \Theta_{X,y}$ for all $y \in Y$.

3.4. Lemma. Let $X^n \subset A^N$ be an analytic subset, and let $C^m \subset C^N$ be a vector subspace which is tangent to $X$ along a connected compact analytic subset $Y^r \subset X^n$. Then there exist an $N$-dimensional complex torus $\tilde{A}^N$, an $m$-dimensional complex subtorus $T^m \subset \tilde{A}^N$, $T^m \supset Y^r$, neighborhoods $U \subset A^N$, $U \supset Y$, $U + \langle Y \rangle = U$, $\tilde{U} \subset \tilde{A}^N$, $\tilde{U} \supset Y$, $\tilde{U} + \langle Y \rangle = \tilde{U}$, and an analytic subset $\tilde{X} \subset T \cap \tilde{U}$ such that $\tilde{U} \simeq U$, $\langle Y \rangle = \langle Y \rangle$, $\tilde{X} \simeq X$, and the mappings $U \simeq \tilde{U}$, $T \hookrightarrow \tilde{A}$ and $X \cap U \hookrightarrow T \cap \tilde{U}$ are compatible with the action of $\langle Y \rangle$.

Proof. The tori $\tilde{A}$ and $T$ are constructed as in Lemma 3.2. To construct $\tilde{X}$ it suffices to take the preimage of $X$ in $\tilde{C}^N$ and to project it to the universal cover $C^m$ of the torus $T^m$. Considering the quotient tori, it is easy to verify that this can be done equivariantly. □
3.5. Theorem. Let \( X^n \subset A^n \) be an analytic subset of a complex torus, and let \( \mathbb{C}^m \subset \mathbb{C}^N \) be a vector subspace which is tangent to \( X \) along a connected compact analytic subset \( Y^r \subset X^n \). Then for some neighborhood \( Y \subset U \subset A \) we have \( X \cap U \subset X_U \), where \( X_U \) is a product of the torus \( \langle Y \rangle \subset A \), \( \dim \langle Y \rangle = k \) and a (local) analytic subset of an \( (m-k) \)-dimensional complex torus \( B^{m-k} \), and there is a natural isomorphism \( \mathbb{C}^m \simeq \mathbb{C}^k \times \mathbb{C}^{m-k} \), where \( \mathbb{C}^k \subset \mathbb{C}^N \) is the universal cover of the torus \( \langle Y \rangle \) and \( \mathbb{C}^{m-k} \) is the universal cover of the torus \( B \).

Proof. In the notations of Lemma 3.4 we consider the canonical holomorphic mappings

\[
\pi: A \to A/\langle Y \rangle_A, \\
\bar{\pi}: \tilde{A} \to \tilde{A}/\langle Y \rangle_{\tilde{A}}, \\
\pi_T: T \to T/\langle Y \rangle_T.
\]

From Lemma 3.4 it follows that

\[
X'_U = \pi(X \cap U) \simeq \bar{\pi}(\tilde{X}) \simeq \pi_T(\tilde{X}),
\]

where

\[
\pi(Y) = y \in X'_U \subset X' = \pi(X).
\]

In particular, the neighborhood \( X'_U \) of the point \( y \) in \( X' \) embeds as an analytic subset in the \( (m-k) \)-dimensional torus \( B = T/\langle Y \rangle_T \). We put

\[
\tilde{X} = \pi^{-1}(X') \subset A, \quad \tilde{X}_U = \tilde{X} \cap U = \pi^{-1}(X'_U).
\]

Then \( \tilde{X}_U \) is the desired analytic subset of \( U \), and for an arbitrary point \( z \in \tilde{X}_U \) the tangent space to \( \tilde{X}_U \) at \( z \) has dimension not exceeding \( m \) and is tangent to \( X \) along the analytic subset

\[
X \cap \pi^{-1}(\pi(z)) = X \cap (z + \langle Y \rangle_A).
\]

\[\square\]

3.6. Corollary (Theorem on tangencies for subvarieties of complex tori). Let \( X^n \subset A^N \) be an analytic subset of a complex torus, and let \( \mathbb{C}^m \subset \mathbb{C}^N \) be a vector subspace which is tangent to \( X \) along a compact analytic subset \( Y^r \subset X^n \). Then \( r \leq k_m \), where \( k_m \) is the maximal dimension of complex subtorus \( C \subset A \) such that \( \dim(X + C) \leq m \).

3.7. Remark. In contrast to the case of subvarieties of projective spaces (cf. Corollary 1.8), in Corollary 3.6 we do not assume that \( X \) is nondegenerate (an analytic subset \( X \subset A \) is called nondegenerate if \( \langle X \rangle = A \)). However if \( \dim X \leq m \), then \( k \geq \dim\langle X\rangle \geq n \) and Corollary 3.6 is trivial.

Let \( X^n \subset A^N \) be an analytic subset of a complex torus, let \( n \leq m \leq N-1 \), and let

\[
P = \{(x, \alpha) \in \text{Sm}X \times \text{Gras}(N, m) \mid L_\alpha \supset \Theta_{x,x}\},
\]

where \( \text{Gras}(N, m) \simeq G(N-1, m-1) \) is the Grassmann variety of \( m \)-dimensional vector subspaces in \( \mathbb{C}^N \), \( L_{\alpha} \subset \mathbb{C}^N \) is the vector subspace corresponding to a point \( \alpha \in \text{Gras}(N, m) \), and the bar denotes closure in \( X \times \text{Gras}(N, m) \). We denote by \( p_m: \mathcal{P}_m \to X \) (resp. \( \gamma_m: \mathcal{P}_m \to \text{Gras}(N, m) \)) the projection map to the first (resp. second) factor.
3.8. Definition. The mapping \( \gamma_m \) is called the \( m \)th Gauss map, and its image \( X_m^* = \gamma_m(P_m) \subset \text{Gras}(N, m) \) is called the variety of tangent \( m \)-spaces to the variety \( X \).

In particular, for \( m = n \) we obtain the usual Gauss map \( \gamma: X \to \text{Gras}(N, n) \), and for \( m = N - 1 \) we get a map \( \gamma_{N-1}: \mathcal{P}^{N-1} \to \mathbb{P}^{N-1} \).

3.9. Proposition. Let \( X^n \subset A^N \) be an irreducible compact analytic subset. Then there exists an analytic subtorus \( C^k \subset A^N \) such that

\[
\begin{align*}
(i) & \quad X + C = C; \\
(ii) & \quad \gamma = \gamma'\pi\big|_X, \text{ where } \pi: A \to B, B = A/C \text{ is the canonical holomorphic map and } \gamma: X \to \text{Gras}(N, n), \text{ and } \gamma': X' \to \text{Gras}(N - k, n - k), X' = \pi(X) \subset B \text{ are the Gauss maps}; \\
(iii) & \quad \text{the map } \gamma': X' \to \gamma'(X') \subset \text{Gras}(N - k, n - k) \text{ is generically finite.}
\end{align*}
\]

Proof. Arguing by induction, we assume that Proposition 3.9 is already verified for \( N' < N \) and prove it in the case \( \dim A = N \). If the map \( \gamma \) is generically finite, then it suffices to put \( C = 0, X' = X \). Suppose that for a general point \( x \in X \) we have \( \dim \gamma^{-1}(\gamma(x)) > 0 \), and let \( Y \) be a positive-dimensional component of \( N', \gamma(x) \). By Lemma 3.2 \( 0 < k = \dim(Y) \leq n \). Since a continuous family of complex analytic subtori of \( A \) is constant, we conclude that if \( \tilde{x} \) is another general point of \( X \) and \( \tilde{Y} \) is a positive-dimensional component of the fiber \( \gamma^{-1}(\gamma(\tilde{x})) \), then \( \langle \tilde{Y} \rangle = \langle Y \rangle \). We put

\[
\begin{align*}
C &= \langle Y \rangle, & X' &= \pi(X) \subset B, \\
B &= A/C, & x' &= \pi(x) = \pi(x).
\end{align*}
\]

Since the tangent space to \( X \) is constant along \( Y \cap \text{Sm}X \) and the kernel of the differential \( d_x(\pi\big|_X) \) coincides with \( \Theta_{\pi^{-1}(x'), x} \), we see that \( Y \) lies in a fiber of the Gauss map for the subvariety \( \pi^{-1}(x') \subset C \). But \( Y \) spans \( C \) and \( \dim C \leq n < N \) (otherwise \( Y = X = C = A \) and Proposition 3.9 is obvious), so that from the induction hypothesis it follows that \( Y = C \).

Thus a general and therefore each fiber of the map \( \pi\big|_X \) coincides with the corresponding fiber of the map \( \pi: A \to B \); moreover, \( X + C = C \) and \( X \) is a locally trivial analytic fiber bundle over \( X' \) with fiber \( C \). Furthermore,

\[
\begin{align*}
\text{Sing } X &= \pi^{-1}(\text{Sing } X'), \\
\Theta_{X, x} &= \Theta_{X', x'} \oplus \mathbb{C}^k,
\end{align*}
\]

where \( \mathbb{C}^k \subset \mathbb{C}^N \) is the universal covering of \( C \) and \( \gamma = \gamma'\pi\big|_X \). \( \square \)

3.10. Corollary. Let \( X^n \subset A^N \) be a compact complex submanifold. Then the Gauss map \( \gamma: X \to \text{Gras}(N, n) \) can be represented in the form \( \gamma = \gamma'\pi \), where \( \pi: X \to X' \) is a locally trivial analytic fiber bundle whose fiber is a complex subtorus \( C^k \subset A^N \), \( X' \) is a compact complex subvariety of the torus \( B = A/C \), and the Gauss map \( \gamma': X' \to \text{Gras}(N - k, n - k) \) is finite. In particular, if \( X \) does
not contain complex subtori (e.g. if $A$ is a simple torus), then the Gauss map $\gamma$ is finite.

**Proof.** Corollary 3.10 is an immediate consequence of Theorem 3.5 and Proposition 3.9. □

Our results also allow to describe the structure of Gauss maps $\gamma_m$ for arbitrary $n \leq m \leq N - 1$.

**3.11. Theorem.** Let $X^n \subset A^N$ be a compact analytic submanifold, $n \leq m \leq N - 1$. Then

a) there exist finitely many subtori $C_1, \ldots, C_l \subset A$ such that if $\bar{X}_i = X + C_i$, $i = 1, \ldots, l$, $\alpha \in X_m^*$, $L_\alpha$ is the $m$-dimensional vector subspace of $\mathbb{C}^N_{\alpha}$ corresponding to $\alpha$, and $Y$ is a connected component of $p_m(\gamma^{-1}_m(\alpha))$, then for some $1 \leq i \leq l$ we have $(Y) = C_i$, $L_\alpha$ is tangent to $\bar{X}_i$ along a torus $y + C_i$, $y \in Y$ (so that in particular $\alpha \in (\bar{X}_i)_m = \gamma_m(p_m(\bar{X}_i)))$, and $Y$ is a connected component of the analytic subset $X \cap (y + C_i)$;

b) the components of general fibers of the Gauss maps are the same. More precisely, if $n \leq m, m' \leq N - 1$ and $x \in X$, $\alpha_m \in \gamma_m(p_m^{-1}(x))$, $\alpha_{m'} \in \gamma_{m'}(p_{m'}^{-1}(x))$ are general points, then in a neighborhood of $x$ we have

$$p_m(\gamma^{-1}_m(\alpha_m)) = p_{m'}(\gamma^{-1}_{m'}(\alpha_{m'})).$$

If $C \subset X$ is the maximal analytic subtorus of $A$ for which $X + C = X$, then a general subspace $L_\alpha \subset \mathbb{C}^N_{\alpha}$, $\alpha \in X_m^*$ is tangent to $X$ along a union of tori of the form $x + C$, $x \in X$.

**Proof.** Theorem 3.11 is an immediate consequence of Theorem 3.5 and Corollary 3.10. □

**3.12. Corollary.** If $X$ does not contain complex subtori, then for an arbitrary $n \leq m \leq N - 1$ the $m$th Gauss map $\gamma_m$ is generically finite. In particular,

$$\dim X_m^* = \dim \mathcal{P}_m = n + \dim (\text{Gras}(N-n,m-n)) = n + (m-n)(N-m)$$

(compare with (2.3.1)) and $X_{N-1}^* = \mathbb{P}^{N-1}$. If $A$ is a simple torus (i.e. $A$ does not contain proper analytic subtori), then all Gauss maps $\gamma_m\colon \mathcal{P}_m \to X_m^*$ are finite.

**3.13.** Let $X^n \subset A^N$ be an analytic submanifold. Then the tangent bundle $\Theta_X$ naturally embeds in the restriction of the tangent bundle $\Theta_A$ on $X$ (which is a trivial bundle on $X$ with fiber $\mathbb{C}^N$), and we can consider the normal bundle $\mathcal{N}_{A/X} = (\Theta_A|_X)/\Theta_X$. It is clear that if $S$ (resp. $Q$) is the canonical vector sub- (resp. quotient-) bundle on $\text{Gras}(N,n)$ and $\gamma\colon X \to \text{Gras}(N,n)$ is the Gauss map, then $\Theta_X = \gamma^*(S)$ and $\mathcal{N}_{A/X} = \gamma^*(Q)$. In other words, the Gauss map $\gamma$ is induced by the normal bundle $\mathcal{N}_{A/X}$ and the linear map

$$\Gamma(A,\mathcal{N}_A) \to \Gamma(X,\mathcal{N}_{A/X})$$

of the corresponding vector spaces of sections (cf. [28, Volume 1, Chapter I, §5]). Similarly, the map $\gamma_{N-1}\colon \mathbb{P}(\mathcal{N}_{A/X}) \to \mathbb{P}^{N-1}$ is induced by the invertible sheaf $\mathcal{O}_X(1)$ on $\mathbb{P}(\mathcal{N}_{A/X}) = \mathcal{P}_{N-1}$.
The exact sequence
\[ 0 \to \Theta_X \to \Theta_A \big|_X \to N_{A/X} \to 0 \]
shows that
\[ \det N_{A/X} = -\det \Theta_X = K_X, \]
where \( K_X \) is the canonical line bundle on \( X \). Since for the Plücker embedding we have \( \det Q = O_{Gras(N,n)}(1) \), the map \( \gamma \) is also defined by a (base point free) linear subsystem of the canonical linear system \( |K_X| \), viz. by the linear system spanned by the ramification divisors
\[ R_L = \{ x \in X \mid \dim (\Theta_{X,x} \cap L) > 0 \}, \]
where \( L \) runs through the set of general \((N-n)\)-dimensional vector subspaces of \( \mathbb{C}^N \) (compare with Section 2).

3.14. Proposition. Let \( X^n \subset A^N \) be an analytic submanifold.

a) The following conditions are equivalent:
   (i) The bundle \( N_{A/X} \) is ample;
   (ii) The mappings \( \gamma_m, n \leq m \leq N - 1 \) are finite;
   (iii) \( X_{N-1}^N = \mathbb{P}^{N-1} \) and \( \gamma_{N-1}: \mathbb{P}(N_{A/X}) \to \mathbb{P}^{N-1} \) is a finite covering.

b) Suppose that condition (iii) from a) holds. Then either \( n = N - 1 \) or \( \deg \gamma_{N-1} = c_n (\Omega_X^1) = (-1)^n c_n (X) = |e(X)| \geq N - 1 \), where \( e(X) \) is the (topological) Euler-Poincaré characteristic of \( X \) and \( \Omega_X^1 \) is the sheaf of differential forms of rank one.

Proof. a) (i)⇔(iii) in view of the definition of ampleness of vector bundle (cf. [31, Chapter III]), Corollary 6.6.3 from [31, Chapter II] and Proposition 2.6.2 from [30, Chapter III], (ii)⇒(iii) is obvious, and (iii)⇒(ii) follows from the fact that for \( m < N - 1 \) the fibers of \( \gamma_m \) (or, more precisely, their projections to \( X \)) are contained in the fibers of \( \gamma_{N-1} \).

b) From the description of the map \( \gamma_{N-1} \) given in 3.13 it immediately follows that
\[ \deg \gamma_{N-1} = c_n (\Theta_X^1) = c_n (\Omega_X^1) = (-1)^n c_n (X) = |e(X)|. \]
In [55, 3.1] it is shown that if \( Y \) is a complex manifold and \( \pi: Y \to \mathbb{P}^k \) is a finite covering of degree \( \leq k - 1 \), then \( \text{Pic} Y = \mathbb{Z} \). To verify b) it suffices to put \( k = N - 1 \), \( Y = \mathbb{P}(N_{A/X}) \) and to observe that for \( N - n - 1 > 0 \) we have
\[ \text{rk} (\text{Pic} (\mathbb{P}(N_{A/X}))) \geq 2. \]
\[ \square \]

We observe that in view of Corollary 3.12 assertions (i)–(iii) hold in the case when \( A \) is a simple torus.

3.15. Proposition. Let \( X^n \subset A^N \) be an analytic submanifold. Then the canonical linear system \( |K_X| \) is base point free, and its suitable multiple defines a holomorphic mapping \( \pi: X \to X' \) making \( X \) a locally trivial analytic fiber bundle over a complex manifold \( X' \); the fiber of \( \pi \) is the maximal analytic subtorus \( C \subset A \) for which \( X + C = X \) (where \( X' \) embeds isomorphically in \( B = A/C \)).

Proof. In view of the above description of Gauss map (cf. 3.13), Proposition 3.15 immediately follows from Corollary 3.10 and Corollary 6.6.3 from [30, Chapter II].
3.16. **Corollary.** Let \( X^n \subset A^N \) be a nondegenerate complex submanifold (i.e. \( \langle X \rangle = A \)). Then there exists an analytic subtorus \( C \subset A \) such that if \( \pi: A \rightarrow B = A/C \) is the projection map, then

1. \( \pi|_X: X \rightarrow X' \subset B \) is a locally trivial analytic fiber bundle with fiber \( C \) (so that \( X = \pi^{-1}(X') \));
2. the mapping \( \pi|_X \) is equivalent to the mapping defined by a sufficiently high multiple of the canonical class \( K_X \);
3. the canonical class \( K_X \) is ample;
4. \( B = \langle X' \rangle \) is an abelian variety.

3.17. **Corollary.** An analytic submanifold \( X^n \subset A^N \) is a variety of general type (i.e. the canonical dimension of \( X \) coincides with its dimension) if and only if the canonical class \( K_X \) is ample.

3.18. **Remark.** From Corollary 2.16 it follows that for a nonsingular variety \( X^n \neq \mathbb{P}^n \) over an algebraically closed field of characteristic zero the Gauss map \( \gamma \) is birational, and according to Remark 2.15, the map defined by the complete linear system \( |K_X + (n + 1)H| \), where \( H \) is a hyperplane section of \( X \), is an isomorphism. However for submanifolds of complex tori the map \( \gamma \) and the canonical map defined by the complete linear system of canonical divisors can be finite maps of degree greater than one. As an example, it suffices to consider a hyperelliptic curve \( X \) of genus \( g > 1 \) embedded in its Jacobian variety \( J_X \). In this case the Gauss map coincides with the canonical map which clearly has degree two (it is clear that the normal bundle \( N_{J_X/X} \) is ample, and all the Gauss maps \( \gamma_m, 1 \leq m \leq g-1 \) are finite; cf. Proposition 3.14). In [83] it is shown that in the conditions of Proposition 3.14 \( \deg \gamma_X \leq \frac{c(X)}{N-n} \).

3.19. **Remark.** The study of submanifolds of complex tori was begun by Hartshorne [32] and continued by Sommese [84] who revealed the role of complex subtori using the notion of \( k \)-ampleness. At the same time Ueno [93, §10] undertook a thorough investigation of properties of the canonical dimension of submanifolds of complex tori (his results easily follow from ours, but are stated in different terms) and announced in [92] our Corollary 3.17, but his proof turned out to be erroneous (cf. [93, 10.13]). Griffiths and Harris [29, §4 b)] showed that the map \( \gamma' \) from our Corollary 3.10 is generically finite, and basing on their result Ran [68] gave a different proof of Corollary 3.17 and of Proposition 3.20 below in the case \( c = 0 \).

The following two results are analogs of Proposition 2.16 for submanifolds of complex tori.

3.20. **Proposition.** Let \( X^n \subset A^N \) be a complex submanifold, and let \( Y^r \subset X^n \) be a complex subtorus. Then \( r \leq \left\lceil \frac{n(N-n)+c}{N-n+c} \right\rceil \), where \( c \) is the maximum of dimensions of complex subtori \( C \subset A \) such that \( X+C = X \) (this bound is nontrivial for \( c < 2n-N-1 \)). In particular, if \( X \) is a hypersurface (i.e. \( n = N-1 \)) containing a complex subtorus \( Y^r \) of dimension \( r > \frac{N}{2} \), then \( X \) is a locally trivial analytic bundle whose fiber is a complex torus and whose base is a hypersurface in a complex torus of smaller dimension.

**Proof.** It is clear that for an arbitrary point \( y \in Y \) we have \( \Theta_{X,y} \supset \Theta_{Y,y} = \mathbb{C}^r \),
where $C^r \subset \mathbb{C}^N$ is the universal covering of the torus $Y$. Hence

$$\gamma_X(Y) \subset S_Y = \{ \alpha \in \text{Gras}(N, n) \mid L_\alpha \supset C^r \}$$

and by Corollary 3.10

$$r - c \leq \dim \gamma_X(Y) \leq \dim S_Y = \dim (\text{Gras}(N - r, n - r)) = (n - r)(N - n)$$

which implies the assertion of the proposition. □

3.21. Proposition. Let $X^n \subset A^N$ be a complex submanifold, and let $Y^r \subset X^n$ be an analytic subset for which $\dim \langle Y \rangle = m$, where $m - r = \text{codim} \langle Y \rangle Y < \text{codim} A X = N - n$. Denote by $d$ the maximal dimension of complex subtori $D \subset A$ for which $X + D \neq A$. Then $r \leq \left[ \frac{n + d}{2} \right]$. In particular, if $A$ is a simple torus, then $r \leq \left[ \frac{n}{2} \right]$.

Proof. Proposition 3.21 can be proved in essentially the same way as Proposition 2.16. In the notations corresponding to those of 2.16 we have

$$\dim (\gamma(Y) \cap S(M, L, r)) \geq \dim \gamma(Y) - \text{codim}_{S(L, r)} S(M, L, r) = (r - f) - (n - r) = 2r - n - f, \quad (3.21.1)$$

where $f$ is the dimension of general fiber of $\gamma |_Y$ (compare with (2.16.2)). On the other hand, from Corollary 3.6 it follows that

$$\dim (\gamma(Y) \cap S(M, L, r)) \leq d - f. \quad (3.21.2)$$

Combining (3.21.1) and (3.21.2) we get $2r - n - f \leq d - f$, i.e. $r \leq \left[ \frac{n + d}{2} \right]$ as required. □

We observe that $\left[ \frac{n + d}{2} \right] < n - 1$ for $d < n - 2$.

3.22. Corollary. If $X \neq A$, then $X$ does not contain complex subtori of dimension greater than $\left[ \frac{n + d}{2} \right]$. If $X$ is not a hypersurface (i.e. $N > n + 1$), then $X$ does not contain hypersurfaces (in complex tori) of dimension greater than $\left[ \frac{n + d}{2} \right]$.

3.23. Remark. In contrast to the case of subvarieties of projective spaces (cf. Proposition 2.16), in Proposition 3.21 we do not assume that $X$ is nondegenerate. However if $\langle X \rangle \neq A$, then $d \geq \dim \langle X \rangle \geq n$, so that in the degenerate case our results are trivial.
II. PROJECTIONS OF ALGEBRAIC VARIETIES

1. An existence criterion for good projections

Let \( Y^r \subset X^n \) be a nonempty irreducible \( r \)-dimensional subvariety of an irreducible \( n \)-dimensional variety \( X \) defined over an algebraically closed field \( K \), and let \( \Delta_Y \subset Y \times Y \subset Y \times X \) be the diagonal. Denote by \( I_Y \) the Ideal of \( \Delta_Y \) in \( Y \times X \) and put

\[
\Theta'_{Y,X} = \text{Spec} \left( \bigoplus_{j=0}^{\infty} I^j/I^{j+1} \right),
\]

\[
\Theta'_{Y,X,y} = \Theta'_{Y,X} \otimes K(y), \quad y \in Y.
\]

1.1. Definition. We call \( \Theta'_{Y,X,y} \) the (affine) tangent star to \( X \) with respect to \( Y \subset X \) at the point \( y \in Y \).

It is easy to see that

\[
\Theta'_{y,X} \subset \Theta'_{Y,X,y} \subset \Theta'_{X,y} \subset \Theta_{X,y},
\]

where \( \Theta'_{y,X} = \Theta'_{Y,X,y} \cap \Theta_{X,y} \) is the (affine) tangent cone to \( X \) at the point \( y \), \( \Theta'_{Y,X,y} = \Theta'_{X,y} \) is the (affine) tangent star to \( X \) at \( y \), and \( \Theta_{X,y} \) is the Zariski tangent space to \( X \) at \( y \). Furthermore, if \( X^n \subset \mathbb{P}^N \) and the bar denotes projective closure, then in the notations of Section 1 of Chapter 1 we have

\[
\overline{\Theta'_{y,X}} = T'_{y,X}, \quad \overline{\Theta'_{Y,X,y}} = T'_{Y,X,y},
\]

\[
\overline{\Theta_{X,y}} = \Theta_{X,y}.
\]

(cf. [45]).

1.2. Definition. Let \( f : X \to X' \) be a morphism of algebraic varieties. We say that \( f \) is unramified in the sense of Johnson (J-unramified) with respect to \( Y \subset X \) at a point \( y \in Y \) if the morphism \( d_y f \mid_{\Theta'_{Y,X,y}} \) is quasifinite. If \( f \) is J-unramified with respect to \( Y \) at all points \( y \in Y \), then we say that \( f \) is J-unramified with respect to \( Y \). If moreover \( Y = X \), then the morphism \( f \) is called J-unramified.

1.3. Definition. In the notations of Definition 1.2 we say that \( f \) is an embedding in the sense of Johnson (J-embedding) with respect to \( Y \subset X \) if \( f \) is J-unramified with respect to \( Y \) and is one-to-one on \( f^{-1}(f(Y)) \). If moreover \( Y = X \), then the morphism \( f \) is called J-embedding.

1.4. Remark. If \( X \) is nonsingular along \( Y \), i.e. \( Y \cap \text{Sing} X = \emptyset \) and \( Y \subset \text{Sm} X \), then \( f \) is unramified with respect to \( Y \) if and only if \( f \) is unramified at all points \( y \in Y \); \( f \) is a J-embedding with respect to \( Y \) if and only if \( f \) is a closed embedding in some neighborhood of \( Y \) in \( X \).

1.5. Proposition. Let \( X^n \subset \mathbb{P}^N \) be a projective algebraic variety, let \( Y^r \subset X^n \) be a nonempty irreducible subvariety, let \( L^{N-m-1} \subset \mathbb{P}^N \), \( L \cap X = \emptyset \) be a linear subspace, and let \( \pi : X \to \mathbb{P}^m \) be the projection with center in \( L \).

a) The following conditions are equivalent:

(i) The morphism \( \pi \) is J-unramified with respect to \( Y \);

(ii) \( L \cap T Y, X = \emptyset \).
b) The following conditions are equivalent:
   (i) The morphism $\pi$ is unramified at the points of $Y$;
   (ii) $L \cap T(Y, X) = \emptyset$.

c) The following conditions are equivalent:
   (i) The morphism $\pi$ is a $J$-embedding with respect to $Y$;
   (ii) $L \cap S(Y, X) = \emptyset$.

d) The following conditions are equivalent:
   (i) The morphism $\pi$ is an isomorphic embedding;
   (ii) $L \cap S(Y, X) = L \cap T(Y, X) = \emptyset$.

Proof. Most of the assertions of the proposition are obvious. To verify a) it suffices to use the fact that $\pi|_{T'_Y(X, y)}$ is quasifinite iff $\pi|_{T'_Y(X, y)}$ is finite or equivalently $L \cap T'_Y(X, y) = \emptyset$ (we recall that $T'_Y(X, y)$ is a projective cone with vertex $y$).

1.6. Proposition. a) In the conditions of Proposition 1.5 suppose that the morphism $\pi: X^n \to \mathbb{P}^m$ is $J$-unramified with respect to an irreducible subvariety $Y^r \subset X^n$, where $m < r + n$ (i.e. $\dim L \geq N - n - r$). Then $\pi$ is a $J$-embedding with respect to $Y$.

b) In the conditions of Proposition 1.5 suppose that the morphism $\pi: X^n \to \mathbb{P}^m$ is unramified at all points $y \in Y^r$, where $Y^r \subset X^n$ is an irreducible subvariety and $m < r + n$ (i.e. $\dim L \geq N - n - r$). Then $\pi$ is an isomorphism in a neighborhood of $Y$.

Proof. In view of Proposition 1.5 a), our condition means that

$$L \cap T'(Y, X) = \emptyset. \quad (1.6.1)$$

Therefore

$$\dim T'(Y, X) < \text{codim}_{\mathbb{P}^m} L \leq r + n. \quad (1.6.2)$$

In view of Theorem 1.4 of Chapter I, from (1.6.2) it follows that

$$S(Y, X) = T'(Y, X). \quad (1.6.3)$$

In view of Proposition 1.5 c), assertion a) of Proposition 1.6 now follows from (1.6.1) and (1.6.3).

b) According to Proposition 1.5 b), our condition means that

$$L \cap T(Y, X) = \emptyset. \quad (1.6.4)$$

Therefore

$$\dim T'(Y, X) \leq \dim T(Y, X) < \text{codim}_{\mathbb{P}^m} L \leq r + n. \quad (1.6.5)$$

By Theorem 1.4 of Chapter I, from (1.6.5) it follows that

$$S(Y, X) = T'(Y, X). \quad (1.6.6)$$

In view of Proposition 1.5 d), our assertion now follows from (1.6.4), (1.6.6), and the obvious inclusion $T'(Y, X) \subset T(Y, X)$. \qed
1.7. **Corollary.** Let \( X^n \subset \mathbb{P}^N \) be a projective variety, let \( L^{N-m-1} \subset \mathbb{P}^N \), \( L \cap X = \emptyset \) be a linear subspace, and let \( \pi : X \to \mathbb{P}^m \) be the projection with center at \( L \). Suppose that \( m \leq 2n - 1 \). Then

a) The morphism \( \pi \) is J-unramified if and only if \( \pi \) is a J-embedding;

b) The morphism \( \pi \) is unramified if and only if \( \pi \) is an embedding.

1.8. **Remark.** Corollary 1.7 was proved by Johnson [45] by means of formal computations involving characteristic classes under the assumption \( N \leq 2n \). For nonsingular varieties Corollary 1.7 was proved in [26], and the general case was settled in [27, §5] and [97, §2] (in these papers the authors actually consider various special cases of Theorem 1.4 of Chapter I for \( Y = X \)).

1.9. **Lemma.** Let \( X \subset \mathbb{P}^N \) be a nondegenerate variety, \( x \in X, y \in \mathbb{P}^N, y \neq x, z \in (y,x), z \neq y \). Then

a) \( T_{S(y,X),y} = \mathbb{P}^N \);

b) \( T_{S(y,X),z} \supset (y,T_{X,x}) \);

c) \( d_{y\times x\times z} \varphi^y (\Theta_{S_{y,X},y\times x\times z}) = (y,T_{X,x}) \), where \( S(y,X) \) is the cone with vertex \( y \) and base \( X \) (cf. Section 1 of Chapter I), \( d\varphi^y \) is the differential of the map \( \varphi^y \), and the bar denotes closure in the Zariski topology.

**Proof.** a) It is clear that \( T_{S(y,X),y} \supset S(y,X) \supset X \). Therefore \( \langle X \rangle \subset T_{S(y,X),y} \). But by definition for a nondegenerate variety \( X \subset \mathbb{P}^N \), we have \( \langle X \rangle = \mathbb{P}^N \).

b) It suffices to consider the affine case. Furthermore, we may assume that \( y \) coincides with the origin. Since the affine part of \( S(y,X) \) is a cone, the (embedded) tangent spaces at the points \( z \) and \( \mu z \) \( (\mu \in K^* = K \setminus 0) \) coincide with each other and contain the origin. To verify b) it suffices to choose \( \mu \) so that \( \mu z = x \).

c) As in the proof of b), we consider the affine case and assume that \( y \) coincides with the origin. Then the restriction of \( p^y_x \) on the affine part of \( S_{y,X} \) admits a section \( \sigma \),

\[
\sigma(x') = (y,x',\lambda x'), \quad x' \in X, \quad z = \lambda x
\]

and

\[
d_{y\times x\times z} \varphi^y (\Theta_{S_{y,X},y\times x\times z}) = \langle d_{y\times x\times z} \varphi^y (\Theta_{S_{y,X},y\times x\times z}) \rangle = \langle 0, x, \Theta_{X,x} \rangle = \langle 0, \Theta_{X,x} \rangle
\]

which implies c).

\[\square\]

1.10. **Proposition.** a) Let \( y \in Y \subset X \subset \mathbb{P}^N, x \in X, y \neq y, z \in \langle y,x \rangle \). Then

\[
T_{S(Y,X),z} \supset \langle Y,y,T_{X,x} \rangle.
\]

b) Suppose in addition that \( \text{char } K = 0 \). Then for general points \( y \in Y, x \in X, z \in \langle y,x \rangle \) we have \( T_{S(Y,X),z} = \langle Y,y,T_{X,x} \rangle \).
1. AN EXISTENCE CRITERION FOR GOOD PROJECTIONS

Proof. a) It is clear that

\[ S(Y, X) \supset S(y, X), \]
\[ S(Y, X) \supset S(x, Y). \]

Therefore

\[ T_{S(Y, X), z} \supset \langle T_{S(y, X), z}, T_{S(x, Y), z} \rangle. \] (1.10.1)

According to statements a) and b) of Lemma 1.6,

\[ T_{S(y, X), z} \supset T_{X, x}, \]
\[ T_{S(x, Y), z} \supset T_{Y, y}. \] (1.10.2)

Assertion a) immediately follows from (1.10.1) and (1.10.2).

b) First of all we observe that if \( z \in \langle y, x \rangle \), then in the notations of Section 1 of Chapter 1

\[ \Theta_{S(y, x) y x x z} = \left\langle \Theta_{(p_Y)^{-1}(y) y x x z}, \Theta_{(p_X)^{-1}(x) y x x z} \right\rangle \]
\[ = \left\langle \Theta_{S(y, x) y x x z}, \Theta_{S(y, x) y x x z} \right\rangle. \] (1.10.3)

From Lemma 1.9 c) it follows that

\[ d_{y x x z}^Y (\Theta_{S(y, x) y x x z}) = \langle T_{Y, y}, T_{X, x} \rangle. \] (1.10.4)

If char \( K = 0 \), then the map

\[ d_{y x x z}^Y : \Theta_{S(y, x) y x x z} \rightarrow \Theta_{S(Y, X), z} \]

is surjective for a general point \( y x x z \in S_{Y, X} \), and assertion b) follows from (1.10.3) and (1.10.4).

\[ \square \]

1.11. Remark. The arguments used in the proof of Lemma 1.9 also work in a slightly more general situation. In particular, in Chapter IV we shall need the following variant of Lemma 1.9 b): for \( x \in \text{Sm} X \) we have \( T'_{z, (y, X)} \supset T_{X, x} \) (we recall that, in the notations of Section 1 of Chapter 1, \( T'_{z, S(y, X)} \) is the tangent cone to \( S(y, X) \) at the point \( z \)). Similarly, if in the conditions of Proposition 1.10 \( x, y \in \text{Sm} X \), then \( T'_{z, S(Y, X)} \supset T_{Y, y}, T_{X, x} \).

1.12. Remark. Roberts showed (cf. [70]) that for each prime number \( p > 0 \) there exists an irreducible (singular) curve \( X \subset \mathbb{P}^K \), char \( K = p \) such that for \( Y = X \) and a general point \( z \in \mathbb{P}^1 = SX \) the inclusion in statement a) of Proposition 1.10 is strict. An example of such curve is given by the projective closure of the image of affine line \( \mathbb{A}^1_K \) under the embedding \( t \mapsto (t, t^p, t^{p^2}) \).
1.13. Theorem. Let $X^n \subset \mathbb{P}^N$ be a projective variety, and let $Y^r \subset X^n$ be an irreducible subvariety. Consider the following conditions:

a) For an arbitrary linear subspace $L^{N-m-1} \subset \mathbb{P}^N$ the projection $X \to \mathbb{P}^m$ with center at $L$ is a $J$-embedding with respect to $Y$;

b) There exists a linear subspace $L^{N-m-1} \subset \mathbb{P}^N$ such that the projection $X \to \mathbb{P}^m$ with center at $L$ is a $J$-embedding with respect to $Y$;

c) $\dim S(Y, X) \leq m$;

d) There exists a Zariski open subset $U \subset Y \times X$ such that for $y \times x \in U$ we have $\dim \langle T_{Y,y}, T_{X,x} \rangle \leq m$;

e) For all points $y \in SmY$, $x \in SmX$ we have $\dim \langle T_{Y,y}, T_{X,x} \rangle \leq m$.

Then $a) \Leftrightarrow b) \Leftrightarrow c) \Rightarrow d) \Leftrightarrow e)$. If in addition $\text{char } K = 0$, then all conditions $a)\text{--}e)$ are equivalent to each other.

Proof. From Proposition 1.5 c) it immediately follows that $a) \Rightarrow b) \Rightarrow c) \Rightarrow a)$. $c) \Rightarrow d)$. In the notations of Section 1 of Chapter I we set

$$U = p_{12}^{-1}((p_Y)^{-1}(Sm S(Y, X))) \setminus \Delta_Y. \tag{1.13.1}$$

It is clear that $U$ is a Zariski open subset in $Y \times X$. Let $y \times x \in U$. In view of (1.13.1) there exists a point $z \in (y, x) \cap Sm S(Y, X)$. From Proposition 1.10 a) it follows that $T_{S(Y,X), z} \supset \langle T_{Y,y}, T_{X,x} \rangle$. Hence

$$\dim \langle T_{Y,y}, T_{X,x} \rangle \leq \dim S(Y, X) \leq m.$$ 

d) $\Rightarrow e)$. It is easy to see that the function

$$y \times x \mapsto \dim (T_{Y,y} \cap T_{X,x})$$

is upper semicontinuous on $Y \times X$. Since the functions $y \mapsto \dim T_{Y,y}$, $x \mapsto \dim T_{X,x}$ are constant on Sm $Y$ and Sm $X$ (and are equal to $r$ and $n$ respectively), the function

$$s(y, x) = \dim \langle T_{Y,y}, T_{X,x} \rangle = \dim T_{Y,y} + \dim T_{X,x} - \dim (T_{Y,y} \cap T_{X,x})$$

is lower semicontinuous on Sm $Y \times Sm X$. From d) it follows that $s(y, x) \leq m$ for $y \times x \in U$. Hence from semicontinuity of $s$ and irreducibility of Sm $Y \times Sm X$ it follows that $s(y, x) \leq m$ for all $y \in SmY$, $x \in SmX$.

e) $\Rightarrow d)$ is obvious; it suffices to set

$$U = Sm Y \times Sm X.$$ 

Next we verify the implication $e) \Rightarrow c)$ under the assumption that $\text{char } K = 0$. It is clear that there exist points $y \in Sm Y$, $x \in Sm X$, $z \in \langle y, x \rangle$ such that $y \times x \times z$ is a general point of $S_{Y,X}$ in the sense of proposition 1.10 b), i.e. the differential $d_{y \times x \times z}$ is a surjective map. From Proposition 1.10 b) it follows that

$$\dim S(Y, X) \leq \dim T_{S(Y,X), z} = \dim \langle T_{Y,y}, T_{X,x} \rangle \leq m.$$ 

\[ \square \]

1.14. Corollary. Let $X^n \subset \mathbb{P}^N$ be a nonsingular variety over an algebraically closed field $K$ of characteristic zero. Then $X$ can be isomorphically projected to a projective space of smaller dimension if and only if for a general (and hence each) pair of points of $X$ there exists a hyperplane which is tangent to $X$ at these points.

1.15. Remark. Griffiths and Harris [29] proved Corollary 1.14 in the special case when $N \geq 2n + 1$. However Landman discovered that in this case Corollary 1.14 was already proved by Terracini [90].
2. Hartshorne’s conjecture on linear normality and its relative analogs

2.1. Theorem. Let $X^n \subset \mathbb{P}^N$ be a nondegenerate variety, and let $Y^r \subset X^n$ be a closed subvariety. Suppose that there exists a point $u \in \mathbb{P}^N \setminus X$ such that the projection $\pi: X \to \mathbb{P}^{N-1}$ with center at $u$ is a $J$-embedding with respect to $Y$. Then $\text{codim}_{\mathbb{P}^N} X^n = N - n \geq \frac{1}{2}(r - b) + 1$, where $b = \dim (Y \cap \text{Sing} X)$.

Proof. Clearly it suffices to consider the case when $Y$ is irreducible. From Proposition 1.5(c) it follows that $S(Y, X) \neq \mathbb{P}^N$. Let $s = \dim S(Y, X)$, and let $z$ be a general point of $S(Y, X)$. In the notations of Section 1 of Chapter I we put

$L = T_{S(Y, X), z}$, $Q_z = p^Y_Y \left( (\psi_Y)^{-1}(z) \right)$.

From Theorem 1.4 of Chapter I it follows that either $T'(Y, X) = S(Y, X)$ or $s = n + r + 1$. In the last case

$N \geq s + 1 = n + r + 2$,

$\text{codim}_{\mathbb{P}^N} X^n = N - n \geq r + 2 > \frac{1}{2}(r - b) + 1$.

Thus we may assume that

$T'(Y, X) = S(Y, X)$, $Q_z \neq \emptyset$, $\dim Q_z = r + n - s$.

It is clear that

$L \supset T_{X, x}$ $\forall x \in Q_z \setminus \text{Sing} X$.

(2.1.1)

Let $M^{N-b-1} \subset \mathbb{P}^N$ be a general linear subspace of codimension $b + 1$, and let

$X' = X \cap M$, $Q'_z = Q_z \cap M$,

$Y' = Y \cap M$, $L' = L \cap M$.

Then the variety $X' \subset \mathbb{P}^{N-b-1}$ is nonsingular along $Y'$, and from (2.1.1) it follows that

$T'(Q'_z, X') = T(Q'_z, X') \subset L'$.

However it is clear that $X' \not\subset L'$.

Hence

$S(Q'_z, X') \neq T'(Q'_z, X')$.

(2.1.2)

From (2.1.2) and Theorem 1.4 of Chapter I it follows that

$\dim S(Q'_z, X') = \dim Q'_z + \dim X' + 1$

$= (r + n - s - b - 1) + (n - b - 1) + 1$

$= 2n + r - s - 2b - 1$. (2.1.3)

On the other hand,

$\dim S(Q'_z, X') \leq \dim (S(Y, X) \cap M) = s - b - 1$. (2.1.4)

From (2.1.3) and (2.1.4) it follows that

$2n + r - s - 2b - 1 \leq s - b - 1$, $2s \geq 2n + r - b$, $2N \geq 2s + 2 \geq 2n + r - b + 2$,

i.e.

$\text{codim}_{\mathbb{P}^N} X^n = N - n \geq \frac{1}{2}(r - b) + 1$.

□
2.2. Corollary. Let \( Y^r \subset X^n \subset \mathbb{P}^N \), where \( X \) is nonsingular in a neighborhood of \( Y \). Suppose that there is a point \( u \in \mathbb{P}^N \setminus X \) such that the projection \( \pi : X \to \mathbb{P}^{N-1} \) with center at \( u \) is a closed embedding in a neighborhood of \( Y \). Then \( N \geq n + \frac{1}{2}(r + 3) \).

2.3. Remark. From Proposition 1.6 a) it follows that for \( N < n + r \) theorem 2.1 (resp. Corollary 2.2) is true if instead of assuming that \( \pi \) is a \( J \)-embedding with respect to \( Y \) (resp. an embedding in a neighborhood of \( Y \)) we assume that \( \pi \) is \( J \)-unramified with respect to \( Y \) (resp. unramified at all points \( y \in Y \)).

The following examples show that the bounds in Corollary 2.2 and Theorem 2.1 are sharp.

2.4. Example. To simplify arguments, in the following examples we assume that \( \text{char } K = 0 \).

a) Let \( X^2 \subset \mathbb{P}^4 \) be the rational surface \( F_1 \) of degree three. Let \( Y_1 \subset X^2 \) be the minimal section, so that \( Y \) is an exceptional curve of the first kind. Then \( Y \subset \mathbb{P}^4 \) is a projective line, and the embedding \( X \hookrightarrow \mathbb{P}^4 \) is defined by the complete linear system \( |Y + 2F| \), where \( F_1 \subset X^2 \) is a fiber of the ruled surface \( F_1 \). Since \( F \subset \mathbb{P}^4 \) is a projective line, the tangent plane at an arbitrary point of \( X \) contains the fiber passing through this point and therefore intersects \( Y \).

From Proposition 1.10 b) it follows that \( \dim S(Y, X) = r + n + \frac{1}{2}(r + 1) = 3 \).

Hence \( S(Y, X) = T(Y, X) \neq \mathbb{P}^4 \)
and by Proposition 1.5 d) there exists a projection \( \pi : X \to \mathbb{P}^3 \) which is an isomorphic embedding in a neighborhood of \( Y \) (in a suitable coordinate system \( \pi(X) \subset \mathbb{P}^3 \) is defined by the equation \( u_0u_3^2 = u_1u_2^3 \)). In this example \( N = 4 = n + \frac{1}{2}(r + 3) \), i.e. the inequality in Corollary 2.2 turns into equality.

b) Let \( X^6 = G(4, 1) \subset \mathbb{P}^9 \) (the Plücker embedding), and let \( Y = \mathbb{P}^3 \) be the linear subspace of lines passing through a fixed point of \( \mathbb{P}^4 \). Then for general points \( y \in Y, x \in X \) the line \( T_{Y,y} \cap T_{X,x} \) parametrizes lines in \( \mathbb{P}^4 \) passing through a fixed point and intersecting a fixed line. From Proposition 1.10 b) it follows that

\[ \dim S(Y, X) = \dim T(Y, X) = 8 = r + n - 1 = n + \frac{1}{2}(r + 1) = N - 1, \]

so that again the inequality in Corollary 2.2 turns into equality.

To show that the bound given in Theorem 2.1 is also sharp for \( b > -1 \) it suffices to consider the cone with vertex \( \mathbb{P}^b \) over one of the varieties constructed in Example 2.4.

Let \( X^n \subset \mathbb{P}^N \) be a nonsingular variety, and let \( D_m \) (resp. \( R_m \)) be the double point (resp. ramification) locus with respect to a general projection \( \mathbb{P}^N \to \mathbb{P}^m \), \( n \leq m \leq N - 1 \). In other words, if \( L^{N-m-1} \subset \mathbb{P}^N \) is a general linear subspace, then

\[
D_m = \{ x \in X \mid \langle x, x' \rangle \cap L \neq \emptyset \},
\]

\[
R_m = \{ x \in X \mid T_{X,x} \cap L \neq \emptyset \}.
\]
2.5. **Corollary.** Let \( r \geq 2(m - n) \). Then \( Y^r \cap D_m \neq \emptyset \) for an arbitrary subvariety \( Y^r \subset X^n \). If in addition \( r > 0 \), then \( Y^r \cap R_m \neq \emptyset \) for an arbitrary subvariety \( Y^r \subset X^n \).

**Proof.** Suppose that \( Y^r \cap D_m = \emptyset \) (resp. \( Y^r \cap R_m = \emptyset \)). Then for a general linear subspace \( L^{N-m-1} \subset \mathbb{P}^N \) we have \( S(Y, X) \cap L = \emptyset \) (resp. \( T(Y, X) \cap L = \emptyset \)), and therefore \( \dim S(Y, X) \leq m \) (resp. \( \dim T(Y, X) \leq m \)) (compare with Proposition 1.5b, d)). On the other hand, from Theorem 2.1 it follows that

\[
s = \dim S(Y, X) \geq n + \frac{1}{2}(r + 1),
\]

(cf. (2.1.5)), and by Theorem 1.4 of Chapter I either

\[
T(Y, X) = S(Y, X)
\]

or

\[
\dim T(Y, X) = r + n,
\]

so that for \( r > 0 \)

\[
\dim T(Y, X) \geq n + \frac{1}{2}(r + 1).
\]

Thus under our assumptions \( m \geq n + \frac{1}{2}(r + 1) \), i.e. \( r < 2(m - n) \). Hence for \( r \geq 2(m-n) \) (resp. \( r \geq \max \{1, 2(m-n)\} \)) we have \( Y^r \cap D_m \neq \emptyset \) (resp. \( Y^r \cap R_m \neq \emptyset \)). \( \square \)

2.6. **Remark.** Corollary 2.5 is nontrivial only if \( m \leq \frac{1}{2}(3n - 1) \). It is clear that this corollary holds if we only require that \( X \) be nonsingular along \( Y \) (we considered the case of nonsingular \( X \) in order not to introduce definition of ramification locus and double point locus in the general situation; cf. e.g. [39; 45; 48]). Example 2.4 shows that the bound in Corollary 2.5 is sharp.

2.7. **Remark.** If \( m = n \), then from Corollary 2.5 it is easy to deduce that the linear system \( |R_n| \) generated by ramification divisors \( R_L \), where \( L \) runs through the set of general \( (N-n-1) \)-dimensional linear subspaces of \( \mathbb{P}^N \), is ample on \( X \). This result was already proved in Proposition 2.12 of Chapter I.

Example 2.4 shows that the bound given in Theorem 2.1 is sharp. However in one important special case, viz. when \( Y = X \), this bound (which takes the form \( n \leq \frac{1}{2}(2N+b-2) \), where \( b = \dim (\text{Sing } X) \)) can be somewhat improved. This is due to the fact that for \( Y = X \) the subvariety \( Q_z \) involved in the proof of Theorem 2.1 can be replaced by the subvariety \( Y_z = p_1(\varphi^{-1}(z)) \), where \( \dim Y_z = \dim Q_z + 1 \).

2.8. **Theorem.** Let \( X^n \subset \mathbb{P}^N \) be a nondegenerate variety, \( b = \dim (\text{Sing } X) \). Suppose that there exists a point \( u \in \mathbb{P}^N \setminus X \) such that the projection \( \pi: X \to \mathbb{P}^{N-1} \) with center at \( u \) is a \( J \)-embedding. Then \( n \leq \frac{1}{2}(2N+b) - 1 \) (i.e. \( \text{codim}_{\mathbb{P}^N} X \geq \frac{1}{2}(n - b + 3) \)).

**Proof.** As we already pointed out, the proof is parallel to the proof of Theorem 2.1. In the notations of 2.1, from Proposition 1.10a) it follows that

\[
L \supset T_{X, x} \forall x \in Y_z \setminus \text{Sing } X, \quad Y_z = p_1(\varphi^{-1}(z)). \tag{2.8.1}
\]
II. PROJECTIONS OF ALGEBRAIC VARIETIES

Let \( M^{N-b-1} \subset \mathbb{P}^N \) be a general linear subspace of codimension \( b+1 \), and let

\[
X' = X \cap M, \quad Y'_z = Y_z \cap M, \quad L' = L \cap M.
\]

Then the variety \( X' \subset \mathbb{P}^{N-b-1} \) is nonsingular, and from (2.8.1) it follows that

\[
T'(Y'_z, X') = T(Y'_z, X') \subset L'.
\]

However it is obvious that \( X' \not\subset L' \). Hence

\[
S(Y'_z, X') \neq T'(Y'_z, X'). \quad (2.8.2)
\]

From (2.8.2) and Theorem 1.4 of Chapter I it follows that

\[
s - b - 1 = \dim (SX \cap M) \geq \dim SX' \geq \dim S(Y'_z, X') = \dim X'_z + \dim X' + 1
\]

\[
= ((2n+1-s)-(b+1)) + (n-b-1) + 1 = 3n - s - 2b,
\]

i.e.

\[
3n \leq 2s + b - 1 \leq 2N + b - 3.
\]

\[ \Box \]

2.9. Remark. In [97] we gave a proof of Theorem 2.8 based on Theorem 1.7 of Chapter I. In the case when \( K = \mathbb{C}, b = -1 \) Lazarsfeld showed that Theorem 2.8 can be derived directly from the connectedness theorem of Fulton and Hansen (cf. [27, §7]).

2.10. Remark. For \( n > 1 \) in the statement of Theorem 2.8 it suffices to assume that there exists a point \( u \in \mathbb{P}^N \setminus X \) such that the projection \( \pi: X \to \mathbb{P}^{N-1} \) with center in \( u \) is \( J \)-unramified. In fact, from Proposition 1.5 a) and Theorem 1.4 of Chapter I it follows that if \( \pi \) is \( J \)-unramified, then \( T'X \neq \mathbb{P}^N \) and either \( N \geq \dim SX = 2n + 1, n \leq \frac{1}{2}(N-1) \leq \frac{1}{3}(2N + b) - 1 \) or \( SX = T'X \ni u \) and \( \pi \) is a \( J \)-embedding, so that in both cases the conditions of Theorem 2.8 are satisfied.

2.11. Corollary. If a nondegenerate nonsingular variety \( X^n \subset \mathbb{P}^N \) can be isomorphically projected to a projective space \( \mathbb{P}^m, m < N \), then \( n \leq \frac{2}{3}(m-1) \). If \( n > 1 \), then it suffices to require existence of unramified projection \( X \to \mathbb{P}^m \).

2.12. Remark. For \( m = N - 1 \) the bound given in Corollary 2.11 is sharp, but for \( m < N - 1 \) this bound can be improved (cf. [100] or Corollary 2.16 in Chapter V).

2.13. Remark. The varieties for which the inequality in Theorem 2.8 or Corollary 2.11 turns into equality will be classified in Chapter IV (cf. Chapter IV, Theorems 1.4 and 4.7). We observe that from the proof of Theorem 2.8 it follows that if \( n = \frac{1}{3}(2N + b) - 1 \), then for a general point \( z \in SX \) we have \( \dim (Y_z \cap \text{Sing} X) = b \), i.e. \( Y_z \) contains a component of \( \text{Sing} X \).
2.14. Theorem. Let $X^n \subset \mathbb{P}^N$, $b = \dim (\text{Sing} \ X)$, $n > \frac{1}{3}(2N + b - 1)$. Then $X$ is not projection of a variety of the same dimension and degree nontrivially embedded in a projective space of larger dimension.

Proof. Suppose the converse. Then there exist a variety

$$X' \subset \mathbb{P}^{N'}$$

and a linear subspace

$$L \subset \mathbb{P}^{N'}, \quad \dim L = N' - N - 1$$

such that $X'$ is nondegenerate and if $\pi : \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^N$ is the projection with center $L$, then $\pi(X') = X$. From our assumptions on the dimension and degree of $X'$ it follows that $L \cap X' = \emptyset$.

We may assume that $N' = N + 1$. In fact, if $N' > N + 1$, then we pick a general linear subspace

$$L' \subset L, \quad \dim L' = N' - N - 2$$

and denote by $\pi' : \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^{N+1}$ the projection with center $L'$. Put

$$X'' = \pi'(X'), \quad L'' = \pi'(L),$$

and let $\pi'' : \mathbb{P}^{N+1} \dashrightarrow \mathbb{P}^N$ be the projection with center $L''$ (here $L''$ is a point in $\mathbb{P}^{N+1}$). Then it is clear that $L'' \not\in X''$ and

$$\pi''(X'') = \pi(X') = X,$$

$$\dim X'' = \dim X' = n,$$

$$\deg X'' = \deg X' = \deg X.$$

Thus we may assume that $N' = N + 1$ and $L$ is a point in $\mathbb{P}^{N+1}$. Let $b' = \dim (\text{Sing} \ X')$. Since for $L \not\in X'$

$$\deg X' = \left( K(X') : K(X) \right) \cdot \deg X,$$

from (2.14.1) it follows that $\pi$ is a finite birational map, so that

$$\pi(\text{Sing} \ X') \subset \text{Sing} \ X, \quad b' \leq b. \quad (2.14.2)$$

From the condition of the theorem and inequality (2.14.2) it follows that

$$\dim X' = n > \frac{2N + b - 1}{3} = \frac{2(N + 1) + b - 3}{3} \geq \frac{2N' + b'}{3} - 1. \quad (2.14.3)$$

In view of Theorem 2.8 and Proposition 1.5 c), from (2.14.3) it follows that

$$SX' = \mathbb{P}^{N'} \quad (2.14.4)$$

and therefore $L \in SX'$. 
Let 

\[ \varphi': S_{X'} \to SX', \quad p'_1: S_{X'} \to X' \]

be the canonical projections. Put

\[ D_L = p'_1((\varphi')^{-1}(L)) \]

It is easy to see that

\[ \pi(D_L) \subset \text{Sing} X. \]  \hspace{1cm} (2.14.5)

On the other hand, from (2.14.4) it follows that

\[ \dim D_L = \dim ((\varphi')^{-1}(L)) \geq 2n + 1 - N' = 2n - N. \]  \hspace{1cm} (2.14.6)

By our assumption,

\[ 2n - N > b + (N - n - 1). \]  \hspace{1cm} (2.14.7)

Hence from (2.14.5) and (2.14.6) it follows that

\[ b = \dim (\text{Sing} X) \geq \dim (\pi(D_L)) = \dim D_L \geq 2n - N \]

which contradicts (2.14.7) for \( N \geq n + 1 \). For \( N = n \) we have \( X = \mathbb{P}^N \), and the assertion of the theorem is obvious. \( \Box \)

2.15. Corollary. For \( n \geq \frac{2}{3}(N - 1) \) a nonsingular variety \( X^n \subset \mathbb{P}^N \) cannot be obtained by projecting a variety of the same dimension and degree nontrivially embedded in a projective space of larger dimension.

2.16. Definition. A variety \( X^n \subset \mathbb{P}^N \) is called linearly normal if the linear system of hyperplane sections of \( X \) is complete, which means that the restriction map \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1)) \) is surjective (if \( X \) is nondegenerate, this condition is equivalent to \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \cong H^0(X, \mathcal{O}_X(1)) \)).

Thus a variety \( X^n \subset \mathbb{P}^N \) is not linearly normal if and only if there exists a nondegenerate variety \( X' \subset \mathbb{P}^{N'}, N' > N \) and a projection \( \pi: \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^N \) such that \( \pi|_{X'}: X' \sim X \). The following corollary immediately follows from Corollary 2.15.

2.17. Corollary. For \( n > \frac{2}{3}(N - 1) \) any nonsingular variety \( X^n \subset \mathbb{P}^N \) is linearly normal.

The simplest case when Corollary 2.17 can be applied and yields a nontrivial result is the case of threefolds in \( \mathbb{P}^5 \); until now it was unknown whether or not they are linearly normal.

2.18. Remark. Corollary 2.17 was stated as conjecture by R. Hartshorne in 1973 (cf. [33, 4.2]).

2.19. Remark. A variety \( X^n \subset \mathbb{P}^N \) is called projectively normal if all its Veronese embeddings \( v_k(X) \), \( k \geq 1 \) are linearly normal (or, in other words, if the restriction maps \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, \mathcal{O}_X(k)) \) are surjective for all \( k \geq 1 \)). Rao [69] constructed threefolds in \( \mathbb{P}^3 \) which are not projectively normal. We already observed that all such varieties are linearly normal.
CHAPTER III

VARIETIES OF SMALL CODIMENSION CORRESPONDING TO ORBITS OF ALGEBRAIC GROUPS
1. Orbits of algebraic groups, null-forms, and secant varieties

1.1. Let $K$ be an algebraically closed field, char $K = 0$, and let $G$ be a linear algebraic group over $K$ acting on a vector space $V = K^{N+1}$. Let $v \in V$ be a vector for which $Gv$ is a punctured cone in $V$, and let

$$X^n = Gv/K^* \hookrightarrow \mathbb{P}^N$$

be the corresponding projective variety.

Let $H$ be the stabilizer of $v$. By Corollary 2 from [67, no 4], $H$ contains a maximal unipotent subgroup of $G$ (in our case this simply reflects the fact that each parabolic subgroup contains a Borel subgroup; cf. [7]). In particular, $H$ contains the unipotent radical of $G$, and without loss of generality we may assume that the group $G$ is reductive (this is a classical theorem of E. Cartan; cf. [12]). Moreover, since we are interested only in the variety $X$ corresponding to the orbit $Gv$, we may assume that $G$ is a semisimple group (perhaps in this situation it would be more natural to consider groups with one-dimensional center, but the notion of semisimple group is universally accepted, and it seems inconvenient to use notations in which everything should be tensored by $GL_1$).

Fixing a Borel subgroup $B$ corresponding to a maximal unipotent subgroup contained in $H$ we can represent $v$ in the form

$$v = v_1 + \cdots + v_r,$$

where for $b \in B$

$$bv_i = \Lambda_i(b)v_i, \quad i = 1, \ldots, r,$$

$\Lambda_i$ is the highest weight of the restriction of action of $G$ on an invariant subspace $V_i \subset V$, and $v_i$ is a highest weight vector (primitive element) in $V_i$. It is clear that

$$Gv \subset \bigoplus_{i=1}^r V_i,$$

and without loss of generality we may assume that

$$V = \bigoplus_{i=1}^r V_i.$$

Since $X$ is a projective variety, $Gv$ consists of two orbits, viz. $Gv$ and 0, and therefore all $\Lambda_i$ are collinear (cf. [17], [85]). On the other hand, since $Gv$ is a cone, all $\Lambda_i$ lie in an affine hyperplane (cf. [17]). Thus we may assume that $r = 1$ and $Gv$ is the orbit of highest weight vector of an irreducible representation of a semisimple group $G$ (varieties of such type were considered in [95] and were called $HV$-varieties). In particular, from this it follows that the variety $X = Gv/K^* \subset \mathbb{P}^N$ is rational (cf. [72] and 1.3 below) and is defined in $\mathbb{P}^N$ by quadratic equations (cf. [57]).
Let $\Lambda$ be the highest weight of our representation, let $v_\Lambda$ be the corresponding highest weight vector, and let $\mathfrak{g}$ be the Lie algebra of the group $G$. It is clear that the variety of tangents

$$TX = \bigcup_{x \in X} T_{X,x}$$

corresponds to the affine cone

$$G_{v_\Lambda} \subset V.$$ (1.1.1)

Furthermore, if $P_\Lambda \subset G$ is the parabolic subgroup stabilizing the line $Kv_\Lambda$ (or, which is the same, the point $x_\Lambda \in X$ corresponding to $v_\Lambda$), then the stabilizer of $v_\Lambda$ in $G$ coincides with $P_\Lambda$ (this follows from Corollary 2.8 in Chapter I).

Similarly, let $N_\Lambda \subset V^*$ be the subspace of points corresponding to hyperplanes passing through $v_\Lambda$ (i.e. the ‘normal’ subspace). Then the variety corresponding to the cone $G_{N_\Lambda}$ coincides with the dual variety $X^* \subset (P N^*)^*$ (here we consider the contragredient representation of $G$ in $V^*$). Moreover, the stabilizer of $N_\Lambda$ coincides with that of $v_\Lambda$. It is well known that from this it follows that the varieties $TX$ and $X^*$ (as well as $X$) are rational and arithmetically Cohen-Macaulay (cf. [47]).

We proceed with finding out which of the varieties corresponding to orbits of highest weight vectors are complete intersections. If $X$ is a complete intersection, then, according to Proposition 2.10 of Chapter I,

$$X^* = \gamma_{N-1} \left( \mathbb{P} \left( N_{P^N/X}(-1) \right) \right),$$

where $N_{P^N/X}$ is the normal bundle and

$$\gamma_{N-1}: \mathbb{P} \left( N_{P^N/X}(-1) \right) \to X^*$$

is a finite birational morphism. Since, as we have already observed, in our case the variety $X^*$ is normal, from this it follows that $X^* \subset (\mathbb{P}^N)^*$ is a nonsingular hypersurface. On the other hand, $X^* \simeq \mathbb{P} \left( N_{P^N/X}(-1) \right)$ is a projective bundle over $X$ with fiber $\mathbb{P}^{N-n-1}$. Hence $N-n-1 \leq 0$, i.e. either $X = \mathbb{P}^n$ or $X$ is a hypersurface. In the last case it is clear that $X$ is a quadric. Summing up, we obtain the following result.

1.2. Theorem. Let $X^n = Gv/K^* \subset (V \setminus 0)/K^* = \mathbb{P}^N$ be the projective variety corresponding to an orbit $Gv \subset V$ of an irreducible representation of an algebraic group $G$ which is a punctured cone. Then $X = Gx$, where $x$ is the point of $\mathbb{P}^N$ corresponding to a highest weight vector $v$. Furthermore, $X$, $TX$, and $X^*$ are rational arithmetically Cohen-Macaulay varieties. The variety $X$ is defined in $\mathbb{P}^N$ by quadratic equations. Moreover, either $X = \mathbb{P}^n$ (i.e. $G$ acts transitively on $V \setminus 0$) or $X$ is a nonsingular quadric, or $X$ is not a complete intersection.

By analogy with [95], we call projective varieties satisfying the conditions of Theorem 1.2 HV-varieties.

1.3. We turn to secant varieties. In the above notations, let $M$ be the lowest weight of the representation of $G$ in $V$, and let $v_M$ be the corresponding weight vector. Although the orbit $G_{v_M}$ does not necessarily contain all weight vectors, we have

$$v_M \in G_{v_\Lambda}$$
since
\[ M = w_0(\Lambda), \]
where \( w_0 \) is the involution in the Weyl group \( W \) of the group \( G \) transforming the positive Weyl chamber to the negative one (cf. [9, ch. VI, §1, no. 6, Corollary 3]), and we may assume that \( w_0 \) is contained in the normalizer of a maximal torus of \( G \). Let \( x_\Lambda \) (resp. \( x_M \)) be the point in \( X \) corresponding to \( v_\Lambda \) (resp. \( v_M \)), and let \( P_\Lambda \) (resp. \( P_M \)) be the stabilizer of \( x_\Lambda \) (resp. \( x_M \)). Consider the orbit of the point \( x_\Lambda \times x_M \in X \times X \) under the natural action of \( G \) on \( X \times X \). It is clear that the stabilizer of \( x_\Lambda \times x_M \) coincides with \( P_\Lambda \cap P_M \). Since \( P_\Lambda \) contains the ‘upper’ and \( P_M \) the ‘lower’ Borel subgroup of \( G \), we have
\[ \dim(P_\Lambda \cdot P_M) = \dim G \]
(cf. [7, ch. IV, Theorem 14.1]). Hence
\[ \dim (G \cdot (x_\Lambda \times x_M)) = \dim G - \dim (P_\Lambda \cap P_M) = \]
\[ (\dim G - \dim P_\Lambda) + (\dim P_\Lambda - \dim (P_\Lambda \cap P_M)) = \]
\[ \dim X + (\dim (P_\Lambda \cdot P_M) - \dim P_M) = 2\dim X = 2n \]
(similar computation shows that \( \dim (B_M/B_M \cap P_\Lambda) = n \), so that \( B_M \cdot x_\Lambda = X \) and from [72] it follows that \( X \) is a rational variety). Hence the orbit \( G \cdot (x_\Lambda \times x_M) \)
is dense in \( X \times X \) and
\[ SX = G(x_\Lambda, x_M). \]  
(1.3.1)

Let \( U \subset V \) be the plane spanned by the vectors \( v_\Lambda \) and \( v_M \), let \( \mathfrak{N} \) be the cone of null-forms in \( V \) (i.e. \( \mathfrak{N} \) is the subset in \( V \) defined by vanishing of all \( G \)-invariant polynomials), and let \( Z \subset \mathbb{P}^N \) be the projective variety corresponding to the cone \( \mathfrak{N} \). It is clear that \( X \subset Z \). Consider the action of the maximal torus \( T \subset B \) on \( U \). Let
\[ v = \alpha_\Lambda v_\Lambda + \alpha_M v_M, \]
where \( \alpha_\Lambda, \alpha_M \neq 0 \). Then
\[ Tv \subset U \]
and there are two possibilities: either
\[ \Lambda + M \neq 0, \quad \dim Tv = 2 \]
or
\[ \Lambda + M = 0, \quad \dim Tv = 1. \]
In the first case
\[ \overline{Gv} \supset \overline{Tv} \ni 0 \]
and therefore
\[ \overline{GU} = \overline{Gv} \subset \mathfrak{N}, \quad SX \subset Z \]  
(1.3.2)
(cf. [17], [85]). In the second case
\[ \overline{GU} = G(Ke); \]  
(1.3.3)
examples show that in this case \( SX \) may lie or not lie in \( Z \).

We observe that from (1.3.2) and (1.3.3) it follows that if
\[
z \in (x, x)\Lambda, \quad z \neq x, x,
\]
then
\[
Gz = SX.
\]

The involution \( w_0 \) for simple Lie groups is described in the tables in [9]. In particular, \( w_0 = -1 \) (and therefore \( \Lambda + M = 0 \) for all representations) if and only if \( G \) is a simple group of one of the following types: \( A_1, B_r \) \((r \geq 2),\) \( C_r \) \((r \geq 2),\) \( D_{2l} \) \((l \geq 2),\) \( E_7, E_8, F_4, G_2.\)

From (1.3.2), (1.3.3), and (1.3.4) it follows that if \( SX = P^N \), then either \( N = V \), i.e. \( I_G[V] = K \), where \( I_G[V] \) is the algebra of polynomials on \( V \) invariant with respect to the action of \( G \), or \( Gv \) is a hypersurface in \( V \) and, since \( N \supset Gv \), \( \dim N \geq \dim Gv \), \( I_G[V] = K[F] \), where \( F \) generates the ideal of \( V \) in \( K[V] \). All representations for which the algebra of invariants has such form have been classified (cf. [46], [85 (addendum)]), viz. if \( G \) is a simple group, then \( I_G[V] = K \) if and only if \( G = SL_r, Sp_{2r}, \wedge^2 (SL_{2r+1}) \) \((r > 1),\) \( Spin_10; \) (1.3.5)

\[
\wedge^3 (Sp_6), S^3 (SL_2), Spin_r \) \((r = 7, 9, 11, 12, 14),\) \( E_6, E_7, G_2,\) (1.3.6)

where \( \wedge \) denotes exterior power of representation and \( \wedge_0 \) is the ‘principal part’ of decomposition of \( \wedge \) into irreducible summands. From the lists (1.3.5) and (1.3.6) it is easy to select representations for which \( SX = P^N \). It turns out that if \( SX = P^N \) for a variety \( X^n \subset P^N \) corresponding to representation of a simple group \( G \), then there are the following possibilities:

\[
X = P^n, \quad X = Q^n \subset P^{n+1}, \quad X = v_3(P^1) \subset P^3, \\
X = G(4,1)^6 \subset P^9, \quad X = G(5,2)^9 \subset P^{19}, \quad X = S^{10} \subset P^{15}, \\
X = S^{15} \subset P^{11}, \quad X = C^6 \subset P^{13}, \quad X = E^{27} \subset P^{55},
\]

where \( Q^n \) is a nonsingular quadric, \( v_3(P^1) \) is a rational cubic curve, \( S^{10} \) and \( S^{15} \) are the spinor varieties parametrizing linear subspaces on a nonsingular eight- and ten-dimensional quadric respectively (cf. [11], [35], [87]), \( C^6 \) is the variety corresponding to the orbit of highest weight vector of representation \( \wedge^3 (Sp_6) \), and \( E^{27} \) is the variety corresponding to the orbit of highest weight vector of the standard representation of the group \( E_7 \).

Summing up, we obtain the following result.
1.4. Theorem. If \( X = Gx_A = Gx_M \), then \( SX = G(x_A, x_M) \). If \( A + M \neq 0 \), then \( SX \subset Z \) and in particular \( SX \neq P^N \) if the representation of \( G \) in \( V \) has at least one nontrivial invariant. If \( SX = P^N \), then \( I_G[V] = K[F] \) (here \( F \) is a form which may belong to \( K \)). Furthermore, if \( G \) is a simple group, then \( X \) is one of the following nine varieties: \( P^n, Q^n, v_3(P^1), G(4, 1), G(5, 2), S^{10}, S^{15}, C^6, E^{27} \).

1.5. By Theorem 1.2 we may assume that

\[
G = G_1 \times \cdots \times G_d, \quad V = V_1 \otimes \cdots \otimes V_d, \quad d \geq 1,
\]

where \( G_i (i = 1, \ldots, d) \) is a simple group and representation \( G \to \text{Aut} V \) is a tensor product of nontrivial irreducible representations \( G_i \to \text{Aut} V_i \) with highest weights \( \Lambda_i \) and highest weight vectors \( v_i \in V_i \). It is clear that the highest weight \( \Lambda \) of the representation \( G \to \text{Aut} V \) is equal to \( \Lambda_1 + \cdots + \Lambda_d \) and the corresponding highest weight vector can be represented in the form \( v = v_1 \otimes \cdots \otimes v_d \).

Let \( X_i \subset P^{N_i} = P(V_i) \) be the projective variety corresponding to the orbit \( G_i v_i \), and let \( n_i = \dim X_i \), \( i = 1, \ldots, d \). Then it is clear that the variety \( X^n \subset P^N = P(V) \) corresponding to the orbit \( Gv \) is projectively isomorphic to the Segre embedding of \( X_1 \times \cdots \times X_d \). Usually, it is one of the semisimple (but not simple) algebraic groups. In view of 1.5, it suffices to prove the following general theorem in whose statement we no longer assume that \( X \) corresponds to group actions.

1.6. Theorem. Let \( \{ X^n \subset P^N \} = \{ X_1^{n_1} \times \cdots \times X_d^{n_d} \subset P^N \} \), where \( X_i^{n_i} \subset P^N \), \( 0 < n_1 \leq \cdots \leq n_d \), \( d \geq 2 \) (the Segre embedding). Then \( \dim SX = 2n + 1 \) except in the case when \( d = 2 \), \( X_1 = P^{n_1} \), \( X_2 = P^{n_2} \); in this last case \( \dim SX = 2n - 1 \). Furthermore, \( SX = P^N \) if and only if \( \{ X^n \subset P^N \} = \{ P^1 \times P^{n-1} \subset P^{2n-1} \} \) or \( \{ X^n \subset P^N \} = \{ P^1 \times V^{n-1} \subset P^{2n+1} \} \), where \( n \geq 2 \) and \( V^{n-1} \subset P^n \) is a hypersurface (we observe that the variety \( P^1 \times P^1 \times P^1 \subset P^7 \) belongs to this last case).

Proof. Arguing by induction, it is easy to reduce everything to the case \( d = 2 \). Let \( x = (x_1, x_2) \); \( x' = (x_1', x_2') \) be a general pair of points of the variety

\[
X = X_1 \times X_2 \subset P^{N_1} \times P^{N_2} = P,
\]

and let \( z \) be a general point of the chord \( \langle x, x' \rangle \). If \( \dim SX \leq 2n \), then

\[
\dim Y_z \geq 1, \tag{1.6.1}
\]

where, in the notations of Section 1 of Chapter I and Theorem 2.8 of Chapter II,

\[
Y_z = \{ p_X \} \subset \langle \varphi_{X, z} \rangle = \{ x \in X \mid \exists y \in X, y \neq x, z \in \langle x, y \rangle \}. \tag{1.6.2}
\]

On the other hand, it is clear that

\[
Y_z \subset \hat{Y}_z.
\]
where
\[ \tilde{Y}_z = (p_P)_1(\varphi^{-1}_P(z)) = \{ x \in P \mid \exists y \in P, y \neq x, z \in \langle x, y \rangle \}. \] (1.6.3)

But it is well known (cf. e. g. \cite{33, 38}) that if \( x'_1 \neq x_1, x'_2 \neq x_2 \), then
\[ \tilde{Y}_z = \langle x_1 \times x'_1 \rangle \times \langle x_2, x'_2 \rangle \] (1.6.4)
is a nonsingular two-dimensional quadric and
\[ Y_z \subset \tilde{Y}_z \cap X = \{ (y_1, y_2) \mid y_1 \in \langle x_1, x'_1 \rangle \cap X_1, y_2 \in \langle x_2, x'_2 \rangle \cap X_2 \}. \] (1.6.5)

Suppose that at least one of the varieties \( X_1, X_2 \), say \( X_1 \) is not a projective space. Then from (1.6.5) it follows that \( Y_z \) consists of a finite number of points and a finite number of lines from one of the two families of lines on \( \tilde{Y}_z \). But from (1.6.2), (1.6.3), and (1.6.4) it follows that, along with each line \( l \subset Y_z \subset \tilde{Y}_z \) from one family of lines on \( \tilde{Y}_z \), \( Y_z \) contains a line from the other family, viz. the line intersecting \( l \) at the point \( l \cap \psi^{-1}_P(z) \). In view of (1.6.1) we come to a contradiction. Thus if \( \dim SX < 2n + 1 \), then \( X^n = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \) is a Segre variety.

Suppose now that \( SX = \mathbb{P}^N \). Then
\[ N = (N_1 + 1) \ldots (N_d + 1) - 1 \leq 2n + 1 = 2(n_1 + \cdots + n_d) + 1, \] (1.6.6)
and it is clear that \( d = 3, n_1 = n_2 = n_3 = N_1 = N_2 = N_3 = 1 \) or \( d = 2 \). In the last case from (1.6.6) it follows that there are three possibilities: \( n_1 = n_2 = N_1 = N_2 = 2; n_1 = N_1 = 2, n_2 = N_2 = 3; n_1 = N_1 = 1, N_2 = n_2 + 1 \). \( \square \)

1.7. Corollary. In the conditions of Theorem 1.2 suppose that the semisimple group \( G \) is not simple. Then \( \dim SX \leq 2n \) if and only if \( X^n \subset \mathbb{P}^N = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \subset \mathbb{P}^{n_1 + n_2 + n_1 + n_2} \) (in which case \( \dim SX = 2n - 1 \)). Furthermore, \( SX = \mathbb{P}^N \) if and only if \( \{X^n \subset \mathbb{P}^N\} = \{\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}\} \) or \( \{X^n \subset \mathbb{P}^N\} = \{\mathbb{P}^1 \times Q^{n-1} \subset \mathbb{P}^{2n+1}\}, \) where \( Q^{n-1} \subset \mathbb{P}^n \) is a nonsingular quadric.

Proof. Corollary 1.7 immediately follows from Theorems 1.6 and 1.2. \( \square \)
2. $HV$-varieties of small codimension

In this section we study those varieties $X^n \subset \mathbb{P}^N$ corresponding to orbits of highest weight vectors of irreducible representations of semisimple algebraic groups for which $N \leq 2n + 1$.

2.1. Proposition. If $X^n = Gx_\Lambda \subset \mathbb{P}^N = \mathbb{P}(V)$, where $G \to \text{Aut} V$ is an irreducible representation of a semisimple, but not simple group $G$ such that $N \leq 2n + 1$, then $X$ is of one of the following types:

a) $X^n = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ ($n \geq 2$);

b) $X^n = \mathbb{P}^1 \times Q^{n-1} \subset \mathbb{P}^{2n+1}$, where $Q^{n-1}$ is a nonsingular quadric ($n \geq 2$);

c) $X^4 \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;

d) $X^5 = \mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$.

Furthermore, in cases a) and b) $SX = \mathbb{P}^N$ and in cases c) and d) $SX \neq \mathbb{P}^N$.

Proof. Proposition 2.1 immediately follows from Corollary 1.7. □

2.2. Proposition. If $X^n = Gx_\Lambda \subset \mathbb{P}^N = \mathbb{P}(V)$, where $G \to \text{Aut} V$ is an irreducible representation of a simple group $G$ such that $N \leq 2n + 1$, then $\dim V < \dim G$ except in the following cases:

a) $\text{rk} G = 2$, $\dim G = \dim V$, $G \to \text{Aut} V$ is the adjoint representation;

b) $G = SL_2$, $G \to \text{Aut} V$ is the adjoint representation, $X = v_2(\mathbb{P}^1) = Q^1 \subset \mathbb{P}^2$ is a conic;

c) $G = SL_2$, $X = v_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is a rational cubic curve.

Proof. Since the parabolic subgroup $P_\Lambda$ stabilizing the point $x_\Lambda$ contains a Borel subgroup $B_\Lambda \subset G$, we have

$$n = \dim X = \dim G - \dim P_\Lambda \leq \dim G - \dim B_\Lambda = \frac{1}{2}(\dim G - \text{rk} G),$$
i.e.

$$\dim G \geq 2n + \text{rk} G. \quad (2.2.1)$$

From (2.2.1) it follows that for $N \leq 2n + 1$

$$\dim G \geq 2n + \text{rk} G \geq \dim V + (\text{rk} G - 2). \quad (2.2.2)$$

The inequality (2.2.2) shows that if $\text{rk} G > 2$, then $\dim V < \dim G$. The proof of Proposition 2.2 is completed by a direct check. □

All irreducible representations $G \to \text{Aut} V$ of simple algebraic groups $G$ for which $\dim V < \dim G$ were classified in [2] and [20]. Using tables from these papers (which for our purposes should be complemented by adding the adjoint representations of the groups $SL_3$, $Sp_4$, and $G_2$ and the second and third symmetric powers of the standard representation of $SL_2$) we select those representations for which $N \leq 2n + 1$.

First we describe those varieties $X^n \subset \mathbb{P}^N$ for which $SX \neq \mathbb{P}^N$. Special attention will be devoted to the case of Severi varieties which is important for what follows.
2.3. Definition. A nonsingular nondegenerate (not necessarily homogeneous) variety $X^n \subset \mathbb{P}^N$ is called Severi variety if $n = \frac{2}{3}(N-2)$ and $SX \neq \mathbb{P}^N$.

We recall that from Corollary 2.11 in Chapter II it follows that for $n > \frac{2}{3}(N-2)$ we have $SX = \mathbb{P}^N$, so that for a fixed $N$ Severi varieties have maximal dimension among the varieties which can be isomorphically projected to a projective space of smaller dimension. Complete classification of Severi varieties will be given in Chapter IV, and here we restrict ourselves to classifying homogeneous Severi varieties (a posteriori all Severi varieties turn out to be homogeneous).

First we consider the case when the group $G$ is not simple.

2.4. Theorem. If $X^n \subset \mathbb{P}^N$ is a projective variety corresponding to the orbit of highest weight vector of an irreducible representation of a simple group $G$ in a vector space $V^{N+1}$, where $N \leq 2n+1$ and $SX \neq \mathbb{P}^N$, then either $G$ is a simple group or $X$ is a Segre variety of the form $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \subset \mathbb{P}^{n_1+n_1+n_2}$, where $n_1 = 2$, $2 \leq n_2 \leq 3$. In the last case $X$ is a Severi variety if and only if $n_1 = n_2 = 2$, $N = n_1 n_2 + n_1 + n_2 = 8$; for this Severi variety $V$ can be identified with the space $M_3$ of $3 \times 3$-matrices over the field $K$, $X$ corresponds to the cone of matrices of rank less than or equal to one, and $SX$ is the cubic hypersurface corresponding to the cone of degenerate matrices defined by equation $\det M = 0$, $M \in M_3$. Furthermore, for the above variety $X = \text{Sing} SX$, $(SX)^* \simeq X$, $X^* \simeq SX$.

Proof. The first assertion of the theorem follows from Proposition 2.1. Furthermore, it is clear that the variety $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ corresponds to the standard representation of the group $SL_{n_1+1} \times SL_{n_2+1}$ in the space of matrices of order $(n_1+1) \times (n_2+1)$, and the orbit of highest weight vector consists of matrices of rank one. In particular, for the Severi variety $\mathbb{P}^2 \times \mathbb{P}^2$ we have $V \simeq M_3$, $I_{SL_3 \times SL_3}[V] = K[\det]$. By Theorem 1.4, in this case $SX \subset Z \simeq (\mathfrak{g} \setminus 0)/K^*$.

Hence $SX = Z$ is a cubic hypersurface in $\mathbb{P}^8$ (corresponding to the cone of matrices of rank less than or equal to 2), $X = \text{Sing} SX$, and from the structure of orbits of the representation of $G$ in $V$ and the contragredient representation of $G$ in $V^*$ it immediately follows that there exist natural $G$-isomorphisms $$(SX)^* \simeq X, \quad X^* \simeq SX.$$ □

2.5. Now let $G$ be a simple group. Proposition 2.2 and analysis of the results of [41] and tables from [2] and [20] yield the following list of irreducible representations for which $\dim SX < N \leq 2n+1$ (in the statement of our results we denote by $\varphi_i$ the $i$-th fundamental weight in the notations of [9] and [94]):
A_1) \ G = SL_3, \ \Lambda = 2\varphi_1 \ (or \ \Lambda = 2\varphi_2), \ n = 2, \ N = 5, \ X = \nu_2(\mathbb{P}^2) \ is the 
Veronese surface. \ The \ space \ \mathbb{P}^5 \ is \ identified \ with \ projectivization \ of \ the \ vector 
space \ \text{of} \ symmetric \ 3 \times 3 \ matrices, \ and \ the \ algebra \ of \ invariants \ has \ the \ form 
I_G[V] = K[\det], \ so \ that \ Z \ is \ a cubic \ hypersurface \ in \ \mathbb{P}^5. \ From \ Theorem \ \ref{thm:nonsingular} \ it \ follows 
that \ SX \subset Z, \ and \ therefore \ SX = Z, \ so \ that \ X \ is \ a \ Severi \ variety. \ The \ surface \ X 
-corresponds \ to \ the \ cone \ of \ symmetric \ matrices \ of \ rank \ less \ than \ or \ equal \ to \ one, 
and \ the \ variety \ SX \ corresponds \ to \ the \ cone \ of \ degenerate \ symmetric \ matrices.
Moreover, \ X = \text{Sing}SX, \ and \ from \ the \ structure \ of \ orbits \ of \ the \ representation \ of 
G \ in \ V \ and \ the \ contragredient \ representation \ of \ G \ in \ V^* \ it \ immediately \ follows 
that \ there \ exist \ natural \ G-isomorphisms \ (SX)^* \simeq X, \ X^* \simeq SX. \ The \ variety \ X^* 
is \ normal, \ and \ for \ \alpha \in \text{Sing}X^* \simeq X \ the \ hyperplane \ L_\alpha \ is \ tangent \ to \ X \ along \ a 
nonsingular \ conic \ Q^1 \subset X.

A_2) \ G = SL_3, \ \Lambda = \varphi_1 = \varphi_2 \ (adjoint \ representation), \ n = 3, \ N = 7. \ The 
space \ V \ is \ identified \ with \ the \ vector \ space \ of \ 3 \times 3 \ matrices \ with \ trace \ zero, \ and \ the 
group \ G = SL_3 \ is \ identified \ with \ a \ subgroup \ of \ the \ group \ SL_3 \times SL_3 \ acting \ in \ the 
-space \ \text{of} \ all \ 3 \times 3 \ matrices \ as \ tensor \ product \ of \ the \ representations \ of \ SL_3 
\text{corresponding to} \ \varphi_1 \text{ and} \ \varphi_2. \ It \ is \ easy \ to \ see \ that \ the \ orbit \ of \ highest \ weight \ vector 
of \ our \ representation \ is \ the \ intersection \ of \ the \ orbit \ of \ highest \ weight \ vector \ of 
this \ representation \ of \ SL_3 \times SL_3 \ with \ the \ hyperplane \ of \ matrices \ with \ trace \ zero.
Hence \ X \ is \ a \ nonsingular \ hyperplane \ section \ of \ the \ Segre \ variety \ \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \ or, 
equivalently, \ X \ is \ the \ projectivization \ of \ tangent \ bundle \ of \ \mathbb{P}^2 \ naturally \ embedded 
in \ \mathbb{P}^7. \ The \ algebra \ of \ invariants \ has \ the \ form \ I_G[V] = K[Q, \det], \ where \ Q \ is \ the 
quadratic \ form \ defined \ by \ trace. \ In \ this \ case \ \Lambda + M = 0, \ Z \ is \ a \ five-dimensional 
variety, \ SX \ is \ a \ cubic \ hypersurface \ in \ \mathbb{P}^7, \ and \ SingSX = X. \ Here \ representation 
G \to \text{Aut}V \ is \ naturally \ isomorphic \ to \ the \ contragredient \ representation \ G \to 
\text{Aut}V^*, \ and \ the \ dual \ variety \ X^* \subset \mathbb{P}^7^* \ is \ defined \ by \ equation \ 4Q^3 - 27 \det^2 = 0. 
Furthermore, \ X^* \supset \pi(\mathbb{P}^2 \times \mathbb{P}^2), \ where \ \pi: \mathbb{P}^8 \dashrightarrow \mathbb{P}^7, \ \mathbb{P}^8 = \mathbb{P}(M_3) \ is \ the \ projection 
with \ center \ at \ the \ point \ corresponding \ to \ the \ unit \ matrix \ and \ the \ subvariety \ \mathbb{P}^2 \times 
\mathbb{P}^2 \subset \mathbb{P}^8 \ corresponds \ to \ matrices \ of \ rank \ one, \ Z \cap \pi(\mathbb{P}^2 \times \mathbb{P}^2) = X, \ and \ SingX^* = 
Z \cup \pi(\mathbb{P}^2 \times \mathbb{P}^2).

A_3) \ G = SL_6, \ \Lambda = \varphi_2 \ (or \ \Lambda = \varphi_4), \ n = 8, \ N = 14, \ X = G(5,1) \ is \ the 
Grassmann \ variety \ of \ lines \ in \ \mathbb{P}^5. \ The \ space \ V \ is \ identified \ with \ the \ vector \ space 
of \ skew-symmetric \ matrices \ of \ order \ 6 \times 6, \ and \ the \ algebra \ of \ invariants \ has \ the \ form 
I_G[V] = K[\text{Pf}], \ where \ Pf \ denotes \ the \ Pfaffian \ of \ skew-symmetric \ matrix. \ The 
cone \ of \ null-forms \ consists \ of \ three \ orbits, \ viz. \ \{0\}, \ the \ set \ of \ nonzero \ decomposable 
bivectors \ corresponding \ to \ X, \ and \ the \ set \ of \ bivectors \ of \ rank \ four. \ Theorem \ \ref{thm:nonsingular} 
shows \ that \ SX \subset Z \ and \ therefore \ SX = Z. \ Thus \ SX \ is \ a \ cubic \ hypersurface \ in \ \mathbb{P}^{14}, \ X \ is \ a \ Severi \ variety, \ SingSX = X, \ and \ from \ the \ structure \ of \ orbits \ of 
the \ representation \ of \ G \ in \ V \ and \ the \ contragredient \ representation \ of \ G \ in \ V^* \ it \ immediately \ follows 
that \ there \ exist \ natural \ G-isomorphisms \ (SX)^* \simeq X, \ X^* \simeq SX. \ Moreover, \ X^* \ is \ a \ normal \ variety, \ and \ for \ \alpha \in \text{Sing}X^* \simeq X \ the \ hyperplane \ L_\alpha \ is \ tangent \ to \ X \ along \ a 
nonsingular \ quadric \ Q^1 \subset X.

A_4) \ G = SL_7, \ \Lambda = \varphi_3 \ (or \ \Lambda = \varphi_5), \ n = 10, \ N = 20, \ X = G(6,1) \ is \ the 
Grassmann \ variety \ of \ lines \ in \ \mathbb{P}^6. \ This \ example \ is \ similar \ to \ the \ previous \ one. \ Here 
\mathfrak{g} = V \text{ and } V \text{ consists of four orbits, viz. } \{0\}, \text{ the set of bivectors of rank two} 
corresponding to } X, \text{ the set of bivectors of rank four, and the set of bivectors of}
rank six which is dense in $V$. The variety $SX$ corresponds to the cone of bivectors of rank less than or equal to four, $\dim SX = 2n - 3 = 17$. $\Sing SX = X$, $(SX)^* \simeq X$, $X^* \simeq SX$, the variety $X^*$ is normal, and for $\alpha \in \Sing X^* \simeq X$ the hyperplane $L_\alpha$ is tangent to $X$ along a six-dimensional Grassmannian $G(4,1)$ while for $\alpha \in X^* \setminus \Sing X^*$ this hyperplane is tangent to $X$ along a projective plane.

C) $G = \Sp_6$, $\Lambda = \varphi_2$, $n = 7$, $N = 13$. We consider $\Sp_6$ as subgroup of $SL_6$ and restrict the representation of $SL_6$ in the vector space of skew-symmetric $6 \times 6$-matrices described in $A_3$ on $\Sp_6$. The resulting representation of $\Sp_6$ is a direct sum of our representation and a trivial representation in the one-dimensional space spanned by the skew-symmetric matrix

\[
\begin{pmatrix}
0 & 1_3 \\
-1_3 & 0
\end{pmatrix}
\]

It is easy to see that this case is similar to $A_2$, viz. $X$ is a hyperplane section of the variety $G(5,1)$ from example $A_3$). The algebra of invariants is free and is generated by two elements $Q$ and $Pf$ of degrees two and three respectively ($Q$ and $Pf$ are coefficients of the ‘characteristic pfaffian polynomial’). In this case $\Lambda + M = 0$, $Z$ is an eleven-dimensional variety, $SX$ is a cubic hypersurface in $\mathbb{P}^{13}$ and $\Sing SX = X$. Representation $G \to \Aut V$ is naturally isomorphic to the contragredient representation $G \to \Aut V^*$, and the dual variety $X^* \subset (\mathbb{P}^{13})^*$ is defined by equation $4Q^3 - 27Pf^2 = 0$. Furthermore, $X^* \supset \pi(G(5,1))$, where $\pi: \mathbb{P}^{14} \to \mathbb{P}^{13}$ is the projection with center at the point corresponding to the matrix

\[
\begin{pmatrix}
0 & 1_3 \\
-1_3 & 0
\end{pmatrix}
\]

the subvariety $G(5,1) \subset \mathbb{P}^{14}$ corresponds to decomposable bivectors,

\[
Z \cap \pi(G(5,1)) = X,
\]

$\Sing X^* = Z \cup \pi(G(5,1))$.

E) $G = E_6$, $\Lambda = \varphi_1$ (or $\Lambda = \varphi_6$), $n = 16$, $N = 26$. The space $V$ is identified with the 27-dimensional exceptional Jordan algebra $3_3$ of Hermitean $3 \times 3$-matrices over the Cayley algebra (multiplication in the commutative, but not associative algebra $3_3$ is defined as follows: $\psi \psi = \frac{1}{2}(\psi \psi + \psi \psi)$, where $\psi, \omega \in 3_3$ and $\psi \omega$ and $\omega \psi$ are the ordinary products of matrices; cf. [43]). Trace defines a quadratic form $Q(v) = \Tr(v \psi(v)) = \Tr(\psi(v))^2$ and determinant a cubic form $\det v$ on the space $V$ (cf. [23]). It is known (cf. [13; 23; 44]) that the group $E_6$ is identified with the group of linear transformations of $V$ preserving $\det$. Thus $I_G[V] = K[\det]$. The cone $\mathcal{N}$ is a union of three orbits, viz. $\{0\}$, the punctured cone over $X$ (consisting of the so-called ‘primitive idempotents’; cf. [23]), and the complement of this cone. From Theorem 1.4 it follows that $SX = Z$ is a cubic hypersurface in $\mathbb{P}^{20}$, so that $X$ is a Severi variety, $\Sing SX = X$, and from the structure of orbits of the representation of $G$ in $V$ and the contragredient representation of $G$ in $V^*$ it follows that there exist natural $G$-isomorphisms $(SX)^* \simeq X$, $X^* \simeq SX$. Furthermore, $X^*$ is a normal variety, and for $\alpha \in \Sing X^* \simeq X$ the hyperplane $L_\alpha$ is tangent to $X$ along a nonsingular quadric $Q_8 \subset X$. The above interpretation of representation $E_6 \to \Aut V$ is essentially due to Chevalley, Schafer, Freudenthal, Springer, and Jacobson (cf. [13; 23; 44; 76]). However the representation itself was studied already in É. Cartan’s dissertation published in 1894 (cf. [12], ch. VIII, § 8, n° 5). Cartan has also written out defining equations for $SX$ and $X$. The fact that $X$ is a Severi variety was discovered by Chevalley [13] and independently by Lazarsfeld [56] who used Kempf’s results [47].
F) $G = F_4$, $\Lambda = \varphi_4$, $n = 15$, $N = 25$. The group $F_4$ is identified with a subgroup of the group $E_6$ acting on the space $3_3$ as described in E). More precisely, $F_4$ is the subgroup of $E_6$ consisting of linear transformations preserving the unit element $e \in 3_3$, so that $F_4$ coincides with the group of automorphisms $\text{Aut} 3_3$ of the algebra $3_3$. The representation of $F_4$ in $3_3$ splits into a direct sum of two representations, viz. the trivial representation in the one-dimensional space spanned by $e$ and an irreducible representation in the subspace $V = e^c \subset 3_3$ (the orthogonal complement with respect to the form $Q$). It is clear that $V$ coincides with the hyperplane of matrices with trace zero in $3_3$ and $X$ and $SX$ are identified with hyperplane sections of the corresponding varieties from E). The algebra of invariants has the form $I_G[V] = K[Q, \text{det}]$. Thus the case F) is similar to the cases A$_2$) and C) (in particular, as in these cases, all elements $x \in e^c = V$ satisfy characteristic equation $x^3 - Q(x) \cdot x - \text{det} x \cdot e = 0$ (cf. [23]), $\Lambda + M = 0$, $Z$ is a variety of dimension 23, $SX$ is a cubic hypersurface in $\mathbb{P}^{25}$, $\text{Sing} SX = X$). The representation $G \to Aut V$ is naturally isomorphic to the contragredient representation $G \to Aut V^*$, and the dual variety $X^* \subset \mathbb{P}^{25*}$ is defined by equation $4Q^3 - 27\text{det}^2 = 0$. Moreover, $X^* \cap \pi(E)$, where $\pi: \mathbb{P}^{26*} \to \mathbb{P}^{25}$ is the projection with center at the point corresponding to $K \cdot e$ and $E$ is the variety described in E),

$$Z \cap \pi(E) = X, \quad \text{Sing} X^* = Z \cap \pi(E).$$

The 25-dimensional representation of the group $F_4$ was also described by È. Cartan (cf. [12, ch. VIII, §8, n°8]), but it is not clear how to deduce the connection between the simplest nontrivial representations of the groups $F_4$ and $E_6$ from his description.

2.6. Remark. As we have already observed, the varieties described in A$_2$), C), and F) are hyperplane sections of the Severi varieties $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, $G(5, 1) \subset \mathbb{P}^{14}$, and $E^{16} \subset \mathbb{P}^{20}$ respectively. The (nonsingular) hyperplane sections of the fourth Severi variety, viz. the Veronese surface from example A$_1$), are curves in $\mathbb{P}^4$, and they no longer satisfy the condition $N \leq 2n + 1$. However they admit a similar description.

A$_0$) $G = SL_2$, $\Lambda = 4\varphi_1$, $n = 1$, $N = 4$, $X = v_4(\mathbb{P}^1)$ is a linearly normal curve in $\mathbb{P}^4$. The space $V$ is identified with the vector space of symmetric $3 \times 3$-matrices with trace zero. The group $G$ is identified with the subgroup of orthogonal matrices $SO_3 \subset SL_3$ whose representation in the six-dimensional vector space of symmetric matrices splits into a direct sum of two representations, viz. the trivial representation in the subspace $K \cdot \left( \begin{array}{ll} 1 & 0 \\ 0 & 1 \end{array} \right)$ and our representation in the subspace $V = K \cdot \left( \begin{array}{lll} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$ (scalar product with respect to the bilinear form defined by trace). Thus $X$ is the hyperplane section of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ (cf. A$_1$)) corresponding to the hyperplane of matrices with trace zero. The algebra of invariants has the form $I_G[V] = K[Q, \text{det}]$, where $Q$ is the quadratic form defined by trace. In this case $\Lambda + M = 0$, $Z$ is a surface, $SX$ is a cubic hyper-surface in $\mathbb{P}^4$, and $\text{Sing} SX = X$. The representation $G \to Aut V$ is naturally isomorphic to the contragredient representation $G \to Aut V^*$, and the dual variety…
$X^* \subset \mathbb{P}^4$ is defined by equation $4Q^3 - 27 \det^2 = 0$. Moreover, $X^* \supset \pi(v_2(\mathbb{P}^2))$, where $\pi: \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is the projection with center at the point corresponding to the unit matrix, the surface $v_2(\mathbb{P}^2)$ corresponds to symmetric matrices with trace zero, $Z \cap \pi(v_2(\mathbb{P}^2)) = X$, $\operatorname{Sing} X^* = Z \cup \pi(v_2(\mathbb{P}^2))$. Thus example $A_0$ is quite similar to examples $A_2$, $C$, and $F$.

2.7. Remark. In all cases except $A_0$, $A_2$, $C$, and $F$ we gave a complete description of orbits of the representation of $G$ in $V$. Now we describe the orbits in the remaining cases. Put $V_{ab} = (Q, d)^{-1}(a, b)$, where $a, b \in K$, $d = \det$ in cases $A_0$, $A_2$, $C$, and $F$) and $d = \operatorname{Pf}$ in case $C$) (so that in particular $V_{00} = \Re$), and let

$$D(a, b) = 4a^3 - 27b^2$$

be the discriminant of characteristic polynomial in cases $A_0$, $A_2$, and $F$) and the discriminant of characteristic pfaffian polynomial in case $C$). Then $V_{ab}$ is an orbit of dimension $2^{k_G - 1} \cdot 3$ if $D(a, b) \neq 0$, and $V_{ab}$ consists of three orbits of dimensions $0$, $2^{k_G}$, and $2^{k_G - 1} \cdot 3$ if $D(a, b) = 0$, $ab \neq 0$.

Summing up, we obtain the following result.

2.8. Theorem. If a variety $X^n \subset \mathbb{P}^N$ corresponds to the orbit of highest weight vector of an irreducible representation of a simple Lie group $G$ and $\dim SX < N \leq 2n + 1$, then $X$ is one of the seven varieties $A_1$–$A_4$), $C$), $E$), $F$). The varieties $A_1$, $A_3$, and $E$) are Severi varieties, and the varieties $A_0$, $A_2$, $C$), and $F$) are hyperplane sections of Severi varieties.

Combining Theorems 2.4 and 2.8 we obtain the following result.

2.9. Theorem. Over an algebraically closed field $K$ of characteristic zero there exist exactly four Severi varieties corresponding to orbits of linear actions of algebraic groups, viz.

1) $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ ($A_1$);
2) $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;
3) $G(5, 1)^8 \subset \mathbb{P}^{14}$ ($A_3$);
4) $E^{16} \subset \mathbb{P}^{26}$ ($E$).

If $X$ is any of these varieties, then the secant variety $SX$ is a cubic hypersurface corresponding to the cone of null-forms $\Re$. Furthermore, $\Re$ consists of two orbits, so that $\operatorname{Sing} SX = X$. There exist natural equivariant isomorphisms $X^* \simeq SX$, $(SX)^* \simeq X$. Nonsingular hyperplane sections of these Severi varieties also correspond to orbits of linear actions of algebraic groups.

Various geometric characteristics of the projective variety corresponding to the orbit of highest weight vector of an irreducible representation of a semisimple Lie group can be computed in terms of the corresponding representation. For example, results of §24 of [8] allow to compute the Hilbert polynomial (and in particular the degree), and results of §§1.4–1.6 of [3] show how to construct triangulation (and in particular allow to compute the Betti numbers). Let $H_X(t)$ be the Hilbert
polynomial, let $b_r(X)$ be the $r$-th Betti number (from [3] it follows that $b_r(X) = 0$
 for all odd $r$), and let $e(X)$ be the Euler characteristic of a variety $X$. One can derive
universal formulae for all these invariants which are valid for all Severi varieties
from Theorem 2.9. However since the Veronese surface is rather special, it is more
convenient to give the corresponding formulae separately for this surface and the
other Severi varieties.

2.10. Proposition. For the Veronese surface we have $H(t) = (t + 1)(2t + 1)$,
deg $(v_2(\mathbb{P}^2)) = 4$, $b_0 = b_2 = b_4 = 1$, $e = 3$. For the homogeneous Severi variety $X^n$,
n > 2 we have

$$H(t) = \prod_{k=1}^{\frac{n-1}{2}} \left( \frac{t}{k} + 1 \right)^{n_k},$$

where $n_k = 1$ for $k < \frac{n}{4}$ and $k > \frac{n}{2}$, $n_k = 2$ for $\frac{n}{4} \leq k \leq \frac{n}{2}$,

$$\deg X^n = \frac{3n}{4} \cdots \frac{n}{2} = \frac{(n-3)!}{(\frac{n}{2})! (\frac{3n}{4} - 1)!}$$

$(\deg X^4 = 6$, $\deg X^8 = 14$, $\deg X^{16} = 78)$, $b_{2k} = b_{2n-2k} = n_k$ for $1 \leq k < \frac{n}{2}$,
$b_0 = b_{2n} = 1$, $b_n = 3$, and the Euler characteristic is equal to the number of
quadratic generators of the ideal of $X$, i.e. $e(X) = \frac{3n}{2} + 3 = N + 1$.

2.11. Let $X^n \subset \mathbb{P}^N = \mathbb{P}(V)$ be the variety corresponding to the orbit of highest
weight vector $v_\Lambda \in V$, with respect to the action of $G$ on $V$, let $x_\Lambda$ and $x_M$ be the
points of $X$ corresponding respectively to the highest and lowest weight vectors,
and let $z$ be a general point of the line $\langle x_\Lambda, x_M \rangle$. Denote by $S_z$ the stabilizer of the point $z$.
Then

$$\dim (S_z \cdot x_\Lambda) = \dim (S_z/P_\Lambda \cap S_z) = 2n + 1 - \dim Gz = 2n + 1 - s,$$

where $s = \dim SX$ (cf. 1.3). In particular, for the Severi varieties described in
Theorem 2.9 $S_z \cdot x_\Lambda$ is a $n/2$-dimensional quadric. Shifting this quadric by means
of the group $G$ we obtain an $n$-dimensional family of quadrics on the Severi variety
$X^n$.

The family of quadrics on Severi varieties can be also obtained in the following
way. Besides the parabolic subgroup $P_\Lambda$, the Borel subgroup $B = B_\Lambda$ is also
contained in the parabolic subgroup $P_\Lambda$ corresponding to highest weight vector
of the contragredient representation (the corresponding highest weight is equal to
$-M$). Since for Severi varieties $\Lambda + M \neq 0$, it is clear that $P_{-M} \neq P_\Lambda$, but

$$C/P_{-M} \simeq G/P_\Lambda,$$

which yields another interpretation of the isomorphism $(SX)^* \simeq X$. This situation
was essentially described by É. Cartan (cf. [12, ch. VIII, §8, n° 11]). Consider the
orbit of the point $x_\Lambda$ with respect to the action of the subgroup $P_{-M}$. Denote by
$H_{-M}$ the semisimple group corresponding to $P_{-M}$ (the Coxeter graph for $H_{-M}$ is
obtained from the Coxeter graph for $G$ by deleting the vertices at which the weight
M is distinct from zero). It is clear that

$$P_{-M} \cdot x_\Lambda = H_{-M} \cdot x_\Lambda,$$
and the action of $H_{-M}$ corresponds to the restriction of the weight $\Lambda$ on the maximal torus of $H_{-M}$. For the Severi varieties 1–4) from Theorem 2.9 this representation of $H_{-M}$ has the following form:

1) \[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \\
(SL_2, 2\varphi_1);
\]
2) \[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \\
(SL_2 \times SL_2, \varphi_1 \oplus \varphi_1);
\]
3) \[
\begin{array}{c}
\circ \\
\circ
\end{array} \\
(SL_4 \times SL_2, \varphi_2 \oplus 0);
\]
4) \[
\begin{array}{c}
\circ \\
\circ
\end{array} \\
(S\pi_{10}, \varphi_1).
\]

In all these cases $Q_\Lambda = H_{-M} \cdot x_\Lambda$ is a nonsingular quadric of dimension $\frac{1}{2}n$. Shifting $Q_\Lambda$ by means of the group $G$ we obtain the desired family of quadrics.

2.5 and 2.11 yield the following proposition.

2.12. Proposition. On each of the Severi varieties $X^n \subset \mathbb{P}^N$ described in Theorem 2.9 there exists an $n$-dimensional family of $\frac{1}{2}n$-dimensional quadrics. The quadrics on $X^n$ are parametrized by the variety $(SX)^* \simeq X$ (the hyperplane $L_\alpha$ corresponding to a point $\alpha \in (SX)^*$ is tangent to $X$ along a quadric $Q_\alpha$), and the quadrics passing through a point $x \in X$ are parametrized by an $\frac{1}{2}$-dimensional quadric $Q^x \subset (SX)^*$.

2.13. Remark. Let $X^n$ be a Severi variety from 2.9. Arguing as in [47, pp. 234–235] we obtain a resolution of singularities $G \times P_{-M} (Q_\Lambda)$ of the variety $SX$. This resolution is a fiber bundle with fiber $(Q_\Lambda) = \mathbb{P}^{n+1}$ over the variety $G/P_{-M} \simeq X$. The projective fiber bundle $P_\Lambda \times P_{\Lambda \cap P_{-M}} (Q_\Lambda)$ of rank $\frac{n}{2} + 1$ over $P_\Lambda/P_\Lambda \cap P_{-M} \simeq Q_{-M}$ yields a resolution of singularities of the cone $P_\Lambda \cdot (Q_\Lambda) = S(x_\Lambda, X)$, and the fiber bundle $P_\Lambda \times P_{\Lambda \cap P_{-M}} Q_\Lambda$ over the $\frac{n}{2}$-dimensional quadric $Q_{-M}$ whose fiber is an $\frac{n}{2}$-dimensional quadric $Q_\Lambda$ is birationally mapped onto $X$.

We turn to the study of varieties $X^n \subset \mathbb{P}^N$ for which $SX = \mathbb{P}^N$. Classification of such varieties was given in Theorem 1.4 and Corollary 1.7. Apart from projective spaces, the only varieties with small secant varieties (i.e., varieties $X^n$ for which $\dim SX \leq 2n$ or, which is the same, $SX = TX$) are $Q^n \subset \mathbb{P}^{n+1}$, $G(4,1)^6 \subset \mathbb{P}^9$, $S^{10} \subset \mathbb{P}^{35}$ and $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ ($n \geq 3$). From this and the structure of orbits of the representation of $G$ in $V$ and the contragredient representation of $G$ in $V^*$ we derive the following result.

2.14. Proposition. Let $X^n \subset \mathbb{P}^N$, $\mathbb{P}^N = \mathbb{P}(V)$, $N > n$ be a variety corresponding to a linear action of the group $G$ on a vector space $V$ for which $SX = TX = \mathbb{P}^N$. Then there are the following possibilities:

1) $\{X^n \subset \mathbb{P}^N\} = \{Q^n \subset \mathbb{P}^{n+1}\}$ (a nonsingular quadric);
2) $\{X^n \subset \mathbb{P}^N\} = \{\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}\}$ (a Segre variety);
3) $\{X^n \subset \mathbb{P}^N\} = \{G(4,1)^6 \subset \mathbb{P}^9\}$ (the Grassmann variety);
4) $\{X^n \subset \mathbb{P}^N\} = \{S^{10} \subset \mathbb{P}^{35}\}$ (the spinor variety).
In all these cases $X$ is a self-dual variety, i.e. $X^* \simeq X$.

In [33] R. Hartshorne conjectured that for $n > \frac{3}{2} N$ each nonsingular variety $X^n \subset \mathbb{P}^N$ is a complete intersection. In this connection it is natural to introduce the following definition.

2.15. **Definition.** A nonsingular (not necessarily homogeneous) variety $X^n \subset \mathbb{P}^N$ is called a Hartshorne variety if $n = \frac{3}{2} N$ and $X$ is not a complete intersection.

2.16. **Corollary.** The Grassmann variety $G(4,1)^6 \subset \mathbb{P}^9$ and the spinor variety $S^{10} \subset \mathbb{P}^{15}$ are the only Hartshorne varieties corresponding to orbits of linear algebraic groups.

We describe the corresponding representations in more detail.

AH) $G = SL_5$, $\Lambda = \varphi_2$ (or $\Lambda = \varphi_3$), $n = 6$, $N = 9$, $X = G(4,1)$, $I_G[V] = K$, $\mathfrak{g} = V$, and there are three orbits: $\{0\}$, the punctured cone of nonzero decomposable 2-vectors, and the set of indecomposable 2-vectors (of rank 4).

DH) $G = Spin_{10}$, $\Lambda = \varphi_5$ (or $\Lambda = \varphi_4$), $n = 10$, $N = 15$, $X = S^{10}$, $I_G[V] = K$, $\mathfrak{g} = V$, and there are three orbits: $\{0\}$, the punctured cone over $X$ (the set of nonzero ‘pure’ spinors), and the complement of (the closure of) this cone. The variety $S^{10}$ parametrizes the four-dimensional linear subspaces from one family on the eight-dimensional quadric. Furthermore, $S^{10}$ corresponds to the orbit of highest weight vector of the spinor representation of the group $B_4$ (Spin$_9$) with highest weight $\Lambda = \varphi_4$; here $I_G[V] = K[Q]$, where $Q$ is a nondegenerate quadratic form, and in $\mathfrak{g}$ there are three orbits similar to the three orbits of the spinor representation of $D_5$.

Using [3] and [8], we obtain the following result analogous to Proposition 2.10.

2.17. **Proposition.** For a Hartshorne variety $X^n$ corresponding to linear algebraic group the Hilbert polynomial has the form

$$H(t) = \prod_{k=1}^{3n-2} \left( \frac{t}{k} + 1 \right)^{n_k},$$

where $n_k = 1$ for $k < \frac{n+2}{4}$ and $k > \frac{n}{2}$, $n_k = 2$ for $\frac{n+2}{4} \leq k \leq \frac{n}{2}$,

$$\deg X^n = \frac{3n+2}{4} \cdots \frac{n}{2} = \frac{(n-2)!}{(n-4)!} \left( \frac{n}{4} \right)! \left( \frac{3n-2}{4} \right)!.$$  

(deg $X^6 = 5$, deg $X^{10} = 12$). The Betti numbers of the Hartshorne variety $X^n$ are given by the formula $b_r = 0$ for $r \equiv 1 \pmod{2}$, $b_0 = b_{2n} = 1$, $b_{2k} = 2$ for $1 \leq k \leq n-1$, and the Euler characteristic $e$ is equal to $N + 1 = \frac{3}{2} n + 1$. The variety $G(4,1)$ is defined in $\mathbb{P}^9$ by five quadratic equations, and $S^{10}$ is defined in $\mathbb{P}^{15}$ by ten quadratic equations.

2.18. While the $n$-dimensional Severi variety carries a family of $n^2$-dimensional quadrics, the $n$-dimensional Hartshorne variety carries a natural family of ($\frac{n}{2} -$
1)-dimensional linear subspaces. This family can be easily constructed using the equality
\[ \text{codim}_{\mathbb{P}^N} X^* = \text{codim}_{\mathbb{P}^N} X = \frac{n}{2} \]
(to each point of $X^*$ we associate the linear subspace along which the corresponding hyperplane is tangent to $X$), but we shall show how to construct it using representation theory. In the notations of 2.11, the representation of $H_{-M}$ corresponding to the restriction of $\Lambda$ on the maximal torus of $H_{-M}$ has the form
\[ \mathbb{C}^3 \times \mathbb{C}^2 \]
for $G = \text{SL}_6$, $X = G(4,1)$; \[ \mathbb{C}^3 \times \mathbb{C}^2 \]
for $G = \text{Spin}_{10}$, $X = S^{10}$.

It is clear that the orbit of $x_\Lambda$ under the action of $H_{-M}$ is a linear subspace $P_\Lambda$ of dimension $\frac{n}{2} - 1$ shifting which by means of the group $G$ we obtain the desired family of linear subspaces.

Apart from the above family of linear subspaces of dimension $\frac{n}{2} - 1$ the $n$-dimensional homogeneous Hartshorne variety carries also a family of $(n/2 + 1)$-dimensional quadrics. To construct this family we consider the orbit of $x_\Lambda$ with respect to the action of the maximal subgroup $P_e \subset G$ for which representation of the corresponding semisimple group has the form
\[ \mathbb{C}^3 \times \mathbb{C}^2 \]
for $G = \text{SL}_6$, $X = G(4,1)$; \[ \mathbb{C}^3 \times \mathbb{C}^2 \]
for $G = \text{Spin}_{10}$, $X = S^{10}$.

It is clear that the orbit of $x_\Lambda$ under this action is an $(n/2 + 1)$-dimensional quadric $Q_e$. Shifting this quadric by means of the group $G$ we obtain the desired family.

2.19. Proposition. On the Hartshorne variety $X^n \subset \mathbb{P}^N$ described in Corollary 2.16 there is a family of $(n/2 - 1)$-dimensional linear subspaces parametrized by the variety $X^* \simeq X$. There is also a family of $(n/2 + 1)$-dimensional quadrics on $X$ (parametrized by $\mathbb{P}^4$ for $n = 6$ and by $Q^8$ for $n = 10$) such that arbitrary two points of $X$ can be joined by a quadric from this family.

2.20. Remark. Let $X^n \subset \mathbb{P}^N$ be the Hartshorne variety from 2.16. Arguing as in [47], we see that the variety $P_\Lambda \times P_{\Lambda \cap P_{-M}} \mathbb{P}_\Lambda$ is a projective fiber bundle of rank $\frac{n}{2} - 1$ over $\mathbb{P}^{\frac{n}{2}-1}$ and the map $P_\Lambda \times P_{\Lambda \cap P_{-M}} \mathbb{P}_\Lambda \to P_\Lambda \mathbb{P}_\Lambda$ is birational for $X = G(4,1)$ and is a bundle with fiber $\mathbb{P}^4$ for $X = S^{10}$. Here $P_\Lambda \mathbb{P}_\Lambda = T_{X,x_\Lambda} \cap X$ is a cone with vertex $x_\Lambda$ whose base is $\mathbb{P}^1 \times \mathbb{P}^2$ if $X = G(4,1)$ and $G(4,1)$ if $X = S^{10}$ (cf. also 3.1 and 3.6). The map $G \times P_e (Q_e) \to \mathbb{P}^N$ is birational, $G/P_e \simeq \mathbb{P}^4$ for $X = G(4,1)$ and $G/P_e \simeq Q^8$ for $X = S^{10}$. $P_e \cdot Q_e = P_e \cdot P_e \cdot x_\Lambda = X$, and the bundle $P_\Lambda \times P_{\Lambda \cap P_e} Q_e$ with fiber $Q_e^{n/2+1}$ over $P_\Lambda / P_\Lambda \cap P_e \simeq \mathbb{P}^{\frac{n}{2}-1}$ is birationally mapped onto $X$. 

3. HV-varieties as birational images of projective spaces

As we already mentioned (cf. 1.1 and 1.3), from a general theorem of Rosenlicht [72] it follows that HV-varieties are rational. However for HV-varieties one can give an explicit geometric construction of birational isomorphism with projective space.

3.1. Let \( X^n = Gx_\Lambda \subset \mathbb{P}^N \) be an HV-variety, where \( \mathbb{P}^N = \mathbb{P}(V) \), \( G \) is a semisimple group, \( G \to \text{Aut} \, V \) is an irreducible representation, and \( x_\Lambda \in \mathbb{P}^N \) is the point corresponding to highest weight vector \( v_\Lambda \in V \) (here \( \Lambda \) is the highest weight). We denote by \( P_\Lambda \) the stabilizer of the point \( x_\Lambda \) (so that \( X \simeq G/P_\Lambda \)), and let \( H_\Lambda \) be the semisimple group corresponding to the parabolic subgroup \( P_\Lambda \).

The representation of \( H_\Lambda \) in \( V \) obtained by restricting the representation of \( G \) is reducible: it is clear that

\[
H_\Lambda \cdot (Kv_\Lambda), \quad H_\Lambda \cdot (gv_\Lambda) = gv_\Lambda,
\]

where \( gv_\Lambda \) is the tangent space to \( Gv_\Lambda \) at the point \( v_\Lambda \). Thus the \( H_\Lambda \)-module \( V \) can be represented in the form

\[
V = Kv_\Lambda \oplus \mathfrak{t}V \oplus ^nV, \quad (3.1.1)
\]

where \( Kv_\Lambda \oplus \mathfrak{t}V = gv_\Lambda \) is the tangent and \( ^nV \) is the ‘normal’ subspace to \( Gv_\Lambda \subset V \) at the point \( v_\Lambda \in Gv_\Lambda \).

It is clear that the subset \( Gv_\Lambda \cap (\mathfrak{t}V \oplus ^nV) \) is stable with respect to the action of \( H_\Lambda \), and the stabilizer of an arbitrary point of \( Gv_\Lambda \cap (\mathfrak{t}V \oplus ^n) \) contains a maximal unipotent subgroup of \( H_\Lambda \). The corresponding projective variety \( H = X \cap \mathbb{P}(\mathfrak{t}V \oplus ^nV) \) is the singular hyperplane section of \( X \) corresponding to the hyperplane \( \mathbb{P}(\mathfrak{t}V \oplus ^nV) \subset \mathbb{P}(V) \) which is tangent to \( X \) along the subvariety

\[
Y = \text{Sing} \, H = X \cap \mathbb{P}(^nV). \quad (3.1.2)
\]

3.2. Next we consider the rational projection

\[
\pi_X = \pi \big|_X : X^n \to \mathbb{P}^n, \quad \mathbb{P}^n = \mathbb{P}(Kv_\Lambda \oplus \mathfrak{t}V)
\]

with center in the \((N - n - 1)\)-dimensional linear subspace \( \mathbb{P}(^nV) \subset \mathbb{P}^N \). Let

\[
\mathbb{P}^{n-1} = \mathbb{P}(\mathfrak{t}V) \subset \mathbb{P}^n = \mathbb{P}(Kv_\Lambda \oplus \mathfrak{t}V),
\]

and let \( A \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \) be the projective variety corresponding to \( Gv_\Lambda \cap \mathfrak{t}V \). We observe that, since \( X \) is defined in \( \mathbb{P}^N \) by quadratic equations (cf. [57]), \( T_{X,x_\Lambda} X \cap X \) is a cone with vertex \( x_\Lambda \). It is clear that \( A \subset \mathbb{P}^{n-1} \) is the base of this cone. Under the projection \( \pi \), the hyperplanes passing through \( \mathbb{P}(^nV) \) are mapped onto hyperplanes in \( \mathbb{P}^n \). Furthermore,

\[
\pi(\mathbb{P}(\mathfrak{t}V \oplus ^nV)) = \mathbb{P}(\mathfrak{t}V) = \mathbb{P}^{n-1} \subset \mathbb{P}^n,
\]

and

\[
\pi(H) = A \subset \mathbb{P}^{n-1}.
\]

The projection \( Gv_\Lambda \to Kv_\Lambda \oplus \mathfrak{t}V \) is a map of \( H_\Lambda \)-spaces, and its fibers over points lying in the same orbit are isomorphic to each other. It is not hard to see that all ramification points of \( \pi \) lie in \( H \), so that \( \pi|_{X \setminus H} \) is an isomorphism, \( \pi \) is a birational isomorphism, and the variety \( X \) is rational.
3. H(V)-VARIETIES AS BIRATIONAL IMAGES OF PROJECTIVE SPACES

Summing up the discussion in 3.1 and 3.2, we obtain the following result.

3.3. Theorem. Let $X^a \subset \mathbb{P}^N$ be a projective $H(V)$-variety. Then there exist a linear subspace $M \subset \mathbb{P}^N$, $\dim M = N - n - 1$ and a hyperplane $L \supset M$ such that the projection $\pi : X \dashrightarrow \mathbb{P}^n$ with center in $M$ is a birational isomorphism. More precisely, if $H = L \cap X$, then $\pi|_{X \setminus H}$ is an isomorphism and $\{\pi(H) \subset \pi(L)\} = \{A \subset \mathbb{P}^{n-1} = \pi(L) \subset \mathbb{P}^n\}$.

3.4. For applications in Chapters IV and VI we need to know the structure of representation $R$ of the group $H_\Lambda$ in $V$ for some specific representations of the group $G$. We proceed with listing the results of our computations (in what follows 0 denotes the trivial representation, $R(\psi)$ is the irreducible representation corresponding to weight $\psi$; in the case when $H_\Lambda$ is a simple group we denote by $\varphi_i$ the $i$-th fundamental weight in the notations of Bourbaki [9]; if the group $H_\Lambda$ is not simple, then $H_\Lambda$ is a product of two simple groups, $\varphi_i$ denotes the $i$-th fundamental weight of one of them, and $\varphi'_i$ the $i$-th fundamental weight of the other one; the first summand in the formulae below corresponds to the line $Kv_\Lambda$, the second to the subspace $^1V$ and the third to the subspace $^nV$).

1) $G = SL_{n+1}$, $\Lambda = 2\varphi_1$, $X = v_2(\mathbb{P}^n)$, $H_\Lambda = SL_n$, $R = 0 \oplus R(\varphi_1) \oplus R(2\varphi_1)$;  
2) $G = SL_{n+1} \times SL_{b+1}$, $a, b \geq 1$, $\Lambda = \varphi_1 + \varphi'_1$, $X = \mathbb{P}^a \times \mathbb{P}^b$, $H_\Lambda = SL_a \times SL_b$, $R = 0 \oplus(R(\varphi_1) + R(\varphi'_1)) \oplus R(\varphi_1 + \varphi'_1)$;  
3) $G = SL_{n+1}$, $m \geq 3$, $\Lambda = \varphi_2$, $X = G(m, 1)$, $H_\Lambda = SL_2 \times SL_{m-1}$, $R = 0 \oplus R(\varphi_1 + \varphi'_1) \oplus R(\varphi'_1)$;  
4) $G = E_6$, $\Lambda = \varphi_1$, $X = E^{16}$, $H_\Lambda = \text{Spin}_{10}$, $R = 0 \oplus R(\varphi_5) \oplus R(\varphi_1)$;  
5) $G = \text{Spin}_{10}$, $\Lambda = \varphi_5$, $X = S^{10}$, $H_\Lambda = SL_5$, $R = 0 \oplus R(\varphi_3) \oplus R(\varphi_1)$.

3.5. If the representation of $H_\Lambda$ in $^nV$ is irreducible, then its lowest weight vector coincides with $v_{n,M}$, where $v_{n,M}$ is the lowest weight vector of the representation of $G$ in $V$ corresponding to the lowest weight $M$. Hence the variety $Y$ corresponding to the orbit of highest weight vector of the representation of $H_\Lambda$ in $^nV$ can be also defined by the following formulae:

$$Y = P_{-\Lambda}x_M = w_0P_{-M}w_0x_M = w_0P_{-M}x_\Lambda,$$

where $P_{-\Lambda}$ is the stationary subgroup of the lowest weight vector of the contragredient representation of $G$ in $V^*$. This allows to compute $Y$ in cases 1)–5) from 3.4:

1) $Y = v_2(\mathbb{P}^{n-1})$, $\mathbb{P}(^nV) = \langle Y \rangle = \mathbb{P}^{\frac{n(n-1)}{2}}$;  
2) $Y = \mathbb{P}^{n-1} \times \mathbb{P}^{b-1}$, $\mathbb{P}(^nV) = \langle Y \rangle = \mathbb{P}^{ab-1}$;  
3) $Y = G(m-2, 1)$, $\mathbb{P}(^nV) = \langle Y \rangle = \mathbb{P}^{\frac{m(m-3)}{2}}$;  
4) $Y = Q^8$, $\mathbb{P}(^nV) = \langle Y \rangle = \mathbb{P}^9$;  
5) $Y = P_{-M}P_{\Lambda} = \mathbb{P}^4$.

3.6. From 1.3 it follows that the representation of $H_\Lambda$ in $^1V$ is irreducible, and it is clear that $A$ is an $H_\Lambda$-variety. Using the explicit form of representation of $H_\Lambda$ in $^1V$ computed in 3.4 it is easy to describe $A$ in cases 1)–5):
1) $A = \emptyset$;
2) $A = \mathbb{P}^{n-1} \prod \mathbb{P}^{b-1}$;
3) $A = \mathbb{P}^1 \times \mathbb{P}^{m-2}$;
4) $A = S^{10}$;
5) $A = G(4,1)$.

In case 1) $\pi$ is an isomorphism, and in cases 2)-5) $\pi|_{X \setminus H}$ is an isomorphism and $H$ is a rational bundle with fiber:

2) $\mathbb{P}^b$ over $\mathbb{P}^{n-1}$, $\mathbb{P}^a$ over $\mathbb{P}^{b-1}$;
3) $\mathbb{P}^{m-2}$;
4) $\mathbb{P}^5$;
5) $\mathbb{P}^3$.

3.7. Using tables in the end of [94] it is easy to verify that in cases 1)-5) from 3.4 we have the following formulæ ($S^2$ denotes the second symmetric power):

1) $S^2(0 \oplus R(\varphi_1)) \simeq 0 \oplus R(\varphi_1) \oplus R(2\varphi_1)$;
2) $S^2(0 \oplus [R(\varphi_1) \oplus R(\varphi'_1)]) \simeq 0 \oplus [R(\varphi_1) \oplus R(\varphi'_1)] \oplus R(\varphi_1 + \varphi'_1) \oplus [R(2\varphi_1) \oplus R(2\varphi'_1)]$;
3) $S^2(0 \oplus R(\varphi_1 + \varphi'_1)) \simeq 0 \oplus R(\varphi_1 + \varphi'_1) \oplus R(\varphi_2) \oplus R(2\varphi_1 + \varphi'_1)$;
4) $S^2(0 \oplus R(\varphi_3)) \simeq 0 \oplus R(\varphi_3) \oplus R(\varphi_1) \oplus R(2\varphi_3)$;
5) $S^2(0 \oplus R(\varphi_3)) \simeq 0 \oplus R(\varphi_3) \oplus R(\varphi_1) \oplus R(2\varphi_3)$.

It is easy to see that in these formulæ the first term corresponds to the summand $K_{vA}$, the second term to the summand $^4V$, and the third one to the summand $^nV$ in decomposition (3.1.1). From this and 3.2 it follows that the linear system of quadrics in $T_{X,xA}$ containing the subvariety $A \subset \mathbb{P}^{n-1} \subset T_{X,xA}$ defines a rational map

$\sigma : \mathbb{P}^n \to \mathbb{P}(K_{vA} \oplus ^4V \oplus ^nV)$,

which is inverse to $\pi$. From formulæ 1)-5) it also follows that all the above varieties are rational projections of the Veronese varieties $v_2(\mathbb{P}^n)$, viz.

1) $X = v_2(\mathbb{P}^n)$;
2) the Segre variety $\mathbb{P}^a \times \mathbb{P}^b$ is obtained from $v_2(\mathbb{P}^{a+b})$ by projecting it from $\langle v_2(\mathbb{P}^{a-1}), v_2(\mathbb{P}^{b-1}) \rangle$, $\mathbb{P}^{a-1} \cap \mathbb{P}^{b-1} = \emptyset$;
3) the Grassmann variety $G(m, 1)$ is obtained from $v_2(\mathbb{P}^{2m-2})$ by projecting it from $\langle v_2(\mathbb{P}^1) \times v_2(\mathbb{P}^{m-2}) \rangle$;
4) $E^{16}$ is obtained from $v_2(\mathbb{P}^{16})$ by projecting it from $\langle v_2(S^{10}) \rangle = \mathbb{P}^{125}$;
5) $S^{10}$ is obtained from $v_2(\mathbb{P}^{10})$ by projecting it from $\langle v_2(G(4,1)) \rangle = \mathbb{P}^{49}$.

Summing up the discussion in 3.4–3.7, we obtain the following result.
3.8. Theorem. 1) Let $X = v_2(\mathbb{P}^n) \subset \mathbb{P}^{n(n+3)/2}$. The projection $\pi : X \to \mathbb{P}^n$ with center in $\langle v_2(\mathbb{P}^{n-1}) \rangle$ is an isomorphism inverse to the Veronese map $\sigma = v_2 : \mathbb{P}^n \to X$.

2) Let $X = \mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{a+b}$ be a direct application of Theorem 3.3 shows that for all $\pi$ which will be discussed in Chapters IV and VI and analyzed the structure of maps $\pi^{-1}$; the Grassmann variety $G(4, 1)$ can be birationally projected onto $\mathbb{P}^{26}$. Moreover, if $H = \mathbb{P}^a \times \mathbb{P}^{b-1} \cap \mathbb{P}^{a-1} \times \mathbb{P}^b$, then $\pi \mid X \setminus H$ is an isomorphism. The inverse map $\sigma : \mathbb{P}^{a+b} \to X$ is defined by the linear system of quadrics passing through $\mathbb{P}^{a-1} \cup \mathbb{P}^{b-1} \subset \mathbb{P}^{a+b}$. The variety $X$ is obtained from the Veronese variety $v_2(\mathbb{P}^{a+b})$ by projecting it from $\langle v_2(\mathbb{P}^{a-1}), v_2(\mathbb{P}^{b-1}) \rangle$.

3) Let $X = G(m, 1) \subset \mathbb{P}^{m(m-1)/2}$. The projection $\pi : X \to \mathbb{P}^{2(m-1)}$ with center in $\langle G(m-2, 1) \rangle$ is a birational isomorphism; moreover, if $H$ is the Schubert divisor corresponding to a subspace $\mathbb{P}^{m-2} \subset \mathbb{P}^n$, then $\pi \mid X \setminus H$ is an isomorphism. The inverse map $\sigma : \mathbb{P}^{2(m-1)} \to X$ is defined by the linear system of quadrics containing the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{m-2} \subset \mathbb{P}^{2(m-3)} \subset \mathbb{P}^{2(m-1)}$. The variety $X$ is obtained from the Veronese variety $v_2(\mathbb{P}^{2(m-1)})$ by projecting it from $\langle v_2(\mathbb{P}^1) \times v_2(\mathbb{P}^{m-2}) \rangle$.

4) Let $X = E^{16} \subset \mathbb{P}^{26}$. The projection $\pi : X \to \mathbb{P}^{16}$ with center in $\langle Q^8 \rangle = \mathbb{P}^9$, where $Q^8 \subset \mathbb{P}^8$ is a nonsingular quadric, is a birational isomorphism; moreover, if $H$ is the hyperplane section of $X$ such that $\text{Sing} H = Q^8$, then $\pi \mid X \setminus H$ is an isomorphism. The inverse map $\sigma : \mathbb{P}^{16} \to X$ is defined by the linear system of quadrics containing the spinor variety $S^{10} \subset \mathbb{P}^{15} \subset \mathbb{P}^{16}$ parametrizing four-dimensional linear subspaces from one family on the eight-dimensional quadric. The variety $X$ is obtained from the Veronese variety $v_2(\mathbb{P}^{16}) \subset \mathbb{P}^{132}$ by projecting it from $\langle v_2(S^{10}) \rangle = \mathbb{P}^{125}$.

5) Let $X = S^{10} \subset \mathbb{P}^{15}$. The projection $\pi : X \to \mathbb{P}^{10}$ with center in a linear subspace $\mathbb{P}^4 \subset X$ is a birational isomorphism; moreover, if $H$ is a singular hyperplane section, then $\pi \mid X \setminus H$ is an isomorphism. The inverse map $\sigma : \mathbb{P}^{10} \to X$ is defined by the linear system of quadrics containing the Grassmann variety $G(4, 1) \subset \mathbb{P}^9 \subset \mathbb{P}^{10}$. The variety $X$ is obtained from the Veronese variety $v_2(\mathbb{P}^{10}) \subset \mathbb{P}^{65}$ by projecting it from $\langle v_2(G(4, 1)) \rangle = \mathbb{P}^{49}$.

3.9. Remark. In Theorem 3.8 we described projections of those $HV$-varieties which will be discussed in Chapters IV and VI and analyzed the structure of maps $\pi \mid H$ and $\sigma = \pi^{-1}$. Some other $HV$-varieties for which $H$ may contain more than two orbits of the group $H_A$ and $\sigma$ has a more complex structure also present geometric interest. A direct application of Theorem 3.3 shows that for all $d > 1$ projection of the Veronese variety $v_d(\mathbb{P}^n)$ with center in a subspace $\langle v_d(\mathbb{P}^{n-1}) \rangle$ is an isomorphism inverse to $v_d$; the Grassmann variety $G(m, k)$ can be birationally projected onto $\mathbb{P}^{(k+1)(m-k)}$, and the fundamental subset $A = \pi(H)$ of the map $\sigma$ coincides with the Segre variety

$$\mathbb{P}^k \times \mathbb{P}^{m-k-1} \subset \mathbb{P}^{(k+1)(m-k)-1} \subset \mathbb{P}^{(k+1)(m-k)};$$

the spinor variety $S_k$ parametrizing the $k$-dimensional linear subspaces from one family on a nonsingular $2k$-dimensional quadric $Q^{2k} \subset \mathbb{P}^{2k+1}$ ($S_k$ corresponds to
the orbit of highest weight vector of the spinor representation of the group $D_{k+1} = \text{Spin}_{2k+2}$ can be birationally projected onto $\mathbb{P}^{k(k+1)/2}$, and the fundamental subset $A = \pi(H)$ of the map $\sigma$ coincides with the Grassmann variety $G(k, 1) \subset \mathbb{P}^{k(k+1)/2} - 1 \subset \mathbb{P}^{k(k+1)}$, etc. Nonsingular hyperplane sections of Severi varieties (examples $A_0$, $A_2$, $C$, and $F$) from §2) can also be interpreted in this way (for such a variety $A$ is a hyperplane section of the variety $A$ for the corresponding Severi variety).

3.10. Remark. In the case of Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$ and Grassmannians existence of a birational projection $\pi: X \dashrightarrow \mathbb{P}^n$ was proved by different methods in the classical papers [79] and [81].
1. Reduction to nonsingular case

1.1. The goal of this chapter is to give classification of extremal varieties with small secant varieties, i.e. varieties for which the inequality in Theorem 2.8 of Chapter II turns into equality. In other words, we classify nondegenerate varieties

\[ X^n \subset \mathbb{P}^N, \quad n = \frac{2N + b}{3} - 1, \quad b = \text{dim} (\text{Sing} X) \]  

(1.1.1)

which can be J-isomorphically projected to \( \mathbb{P}^{N-1} \). By Proposition 1.5 of Chapter II, the last condition holds if and only if \( SX \neq \mathbb{P}^N \) (in view of Remark 2.10 in Chapter II, for \( n > 1 \) this condition can be replaced by the condition \( T'X \neq \mathbb{P}^N \) which, according to Proposition 1.5 of Chapter II, ensures the existence of a J-unramified projection of \( X \) to \( \mathbb{P}^{N-1} \). From Theorem 2.8 of Chapter II it follows that under these assumptions

\[ \text{dim} SX = N - 1 = \frac{3n - b + 1}{2}. \]  

(1.1.2)

Moreover, from Theorem 2.3 of Chapter V and Theorems 1.4 and 4.7 of the present chapter it follows that if

\[ X' \subset \mathbb{P}^{N'}, \quad SX' \neq \mathbb{P}^{N'}, \quad \text{dim} X' = n, \quad \text{dim} (\text{Sing} X') = b, \quad \text{dim} SX' = \frac{3n - b + 1}{2} \]  

(1.1.3)

is a nondegenerate variety, then \( N' = N = \frac{3(n + 1) - b}{2} \) (a priori one can only claim that any variety \( X' \) satisfying (1.1.3) can be J-isomorphically projected onto a variety \( X \) satisfying (1.1.1)).

Throughout this chapter we consider varieties defined over an algebraically closed field \( K \), char \( K = 0 \). We recall the following definition (cf. Definition 2.3 in Chapter III).

1.2. Definition. A nondegenerate nonsingular variety \( X^n \subset \mathbb{P}^N, \ n = \frac{2}{3}(N - 2) \) is called Severi variety if \( X \) can be isomorphically projected to \( \mathbb{P}^{N-1} \).

In view of Proposition 1.5 b), d) and Corollary 1.7 from Chapter II, a nonsingular nondegenerate variety \( X^n \subset \mathbb{P}^N, \ n = \frac{2}{3}(N - 2) \) is a Severi variety if any of the following equivalent conditions holds:

a) \( SX \neq \mathbb{P}^N \);

b) there exists an unramified projection of \( X \) to a projective space of smaller dimension;

c) \( TX \neq \mathbb{P}^N \).

1.3. Remark. Severi varieties are named after Francesco Severi who gave their classification in the case \( n = 2 \) (cf. [82] and also [62; 15]). More historical details are given in Remark 4.11. We recall that in Chapter III (cf. Theorem 2.9) we gave four examples of Severi varieties in dimensions 2, 4, 8, and 16, viz. the Veronese surface \( v_2(\mathbb{P}^2) \subset \mathbb{P}^5 \), the Segre variety \( \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \), the Grassmann variety \( G(5, 1)^8 \subset \mathbb{P}^{14} \), and the variety \( E^{16} \subset \mathbb{P}^{26} \) corresponding to the orbit of highest weight vector of the simplest nontrivial representation of the group \( E_6 \). In Theorem 4.7 we show that these are the only Severi varieties.
1.4. Theorem. Let $X^n \subset \mathbb{P}^N$, $SX \neq \mathbb{P}^N$ be a nondegenerate variety satisfying condition (1.1.1) (so that $X$ has maximal possible dimension for given $N$ and $b$). Then $X$ is a projective cone with vertex $\mathbb{P}^b = \text{Sing} X$ whose base is a Severi variety $X_0^{n-b-1} \subset \mathbb{P}^{N-b-1}$. Conversely, let $X_0^n \subset \mathbb{P}^{N_0}$ be a Severi variety, let $b \geq 0$ be an integer, and let $X^n \subset \mathbb{P}^N$, $n = n_0 + b + 1$, $N = N_0 + b + 1$ be the projective cone over $X_0$ with vertex $\mathbb{P}^b$. Then $SX \neq \mathbb{P}^N$, $\text{Sing} X = \mathbb{P}^b$, and $n = \frac{2N+3b}{3} - 1$.

Proof. We already observed that $SX$ is a hypersurface (cf. (1.1.2)). Let $z$ be a general point of $SX$, and let $Y_z = p_1(\varphi^{-1}(z))$ (cf. Chapter II, (2.8.1)). Since for our variety $X$ the inequality in Theorem 2.8 of Chapter II turns into equality, from the proof of this theorem it immediately follows that $\dim (Y_z \cap \text{Sing} X) = b$, i.e. there exists an irreducible component $\Xi$ of the variety $\text{Sing} X$ such that

$$\dim \Xi = b, \quad \Xi \subset \bigcap_{z \in SX} Y_z$$

(cf. Remark 2.13 in Chapter II). From Proposition 1.9 of Chapter II it follows that

$$Y_z \subset T_{SX,z}.$$  

(1.4.2)

Combining (1.4.1) and (1.4.2), we see that

$$\Xi \subset \bigcap_{z \in SX} T_{SX,z}.$$  

(1.4.3)

Since $\text{char} K = 0$, from (1.4.1) and (1.4.3) it follows that $SX$ is a cone with vertex

$$\Xi = \mathbb{P}^b \subset X \subset SX$$  

(1.4.4)

(to verify this without computations it suffices to refer to C. Segre’s reflexivity theorem (cf. e.g. [49; 50])).

Let $x \in \Xi$, let $y, y'$ be a general pair of points of $X$, and let $z$ be a general point of the chord $\langle y, y' \rangle$. Since $SX$ is a cone with vertex $x$, $T_{SX,z} = T_{SX,z'}$ for all points $z' \in \langle x, z \rangle \setminus x$. Therefore from Proposition 1.9 a) of Chapter II it follows that the hyperplane $T_{SX,z}$ is tangent to $X$ at all points of $Y_z \cap \text{Sm} X$, where

$$Y_z = p_1(\varphi^{-1}(\langle x, z \rangle \setminus x)) = \bigcup_{z' \in \langle x, z \rangle \setminus x} Y_{z'}.$$  

Applying the arguments used in the proof of Theorem 2.8 of Chapter II to the subvariety $Y_z$ (or applying Theorem 1.7 of Chapter I to the subvariety $\pi(Y_z) \subset \pi(X) \subset \mathbb{P}^{n-1}$ and the hyperplane $\pi(T_{SX,z}) \subset \mathbb{P}^{N-1}$, where $\pi$ is the projection with center in a general point of $T_{SX,z} \setminus SX$), we see that

$$\dim Y_z = \dim Y_{z'} = \frac{n + b + 1}{2} = 2n - N + 2.$$  

Thus $Y_z$ consists of components of $Y_{z'}$, and we may assume that

$$Y_{z'} \supset Y_z$$  

(1.4.5)
for \( z' \in \langle x, z \rangle \). From (1.4.5) it follows that
\[
C = p_2((p_1 \times \varphi)^{-1}(y' \times \langle x, z \rangle)) \subset X \cap \Pi
\]
is a one-dimensional subvariety of the plane \( \Pi = \langle x, y', z \rangle \). Without loss of generality we may assume that
\[
\langle x, z \rangle \cap C^1 = \langle x, z \rangle \cap X = x,
\]
where \( C^1 \) is a one-dimensional component of \( C \) passing through \( y \). In fact, otherwise \( S_X = S(x, X) \) and therefore \( \dim Y_z = 2n + 1 - \dim SX = 2n + 1 - \dim S(x, X) = n, \)
so that \( Y_z = X \) and from (1.4.2) it follows that \( X \subset T_{SX,z} \), contrary to the assumption that \( X \) is nondegenerate. From (1.4.6) it follows that a general line in \( \Pi \) passing through \( x \) intersects \( C^1 \) only at \( x \), and therefore \( C^1 = \langle x, y \rangle \) (cf. e.g. [64, 5.11]).

Thus for a general and therefore for each point \( y \in X \) we have \( \langle x, y \rangle \subset X \), i.e. \( X \) is a cone with vertex \( x \). Since \( x \) is an arbitrary point of \( \Xi \), from this it follows that \( X \) is a cone with vertex \( \Xi = \mathbb{P}^b \), and since \( X \) is irreducible, \( b < n - 1 \).

Let \( M^{N_0} \subset \mathbb{P}^N \) be a general linear subspace, \( M \cap \text{Sing} X = \emptyset, N_0 = N - b - 1, X_0 = X \cap M, n_0 = \dim X_0 = n - b - 1 > 0 \). By Bertini’s theorem (cf. [28, Vol. I, Chapter I, § 1; 34, Chapter II, 8.18]), \( X_0 \) is irreducible and nonsingular. Furthermore,
\[
SX_0 \subset SX \cap M \neq M
\]
and
\[
n_0 = n - b - 1 = \left( \frac{2N + b}{3} - 1 \right) - b - 1 = \frac{2(N - b - 1) - 4}{3} = \frac{2}{3}(N_0 - 2),
\]
i.e. \( X_0 \) is a Severi variety.

The converse is an immediate consequence of the fact that, as it is easy to see, \( SX \) is the cone over \( SX_0 \) with vertex \( \mathbb{P}^b = \text{Sing} X \), and since \( n_0 = \frac{2}{3}(N_0 - 2) \),
\[
n = n_0 + b + 1 = \frac{2(N_0 + b + 1) + b - 3}{3} = \frac{2N + b}{3} - 1.
\]
\[\square\]
2. Quadrics on Severi varieties

2.1. Proposition. Let $X^n \subset \mathbb{P}^N$, $n = \frac{3}{2}(N - 2)$ be a Severi variety, and let $z$ be a general point of $SX$. Set $L = T_{SX,z}$, $P_L = \{u \in SX \mid T_{SX,u} = L\}$. Then $X \cap P_L = Y_z = p_1(\varphi^{-1}(z))$ is a nonsingular $\frac{3}{2}$-dimensional quadric in the projective space $P_L = \mathbb{P}^{\frac{3}{2} + 1}$.

Proof. According to C. Segre’s reflexivity theorem (cf. [49; 50]), $P_L$ is a linear subspace of $\mathbb{P}^N$.

Let

$$X_L = \{x \in X \mid T_{X,x} \subset L\}.$$ 

Then $X_L$ is a closed subvariety of $X$, $T(X_L, X) \subset L$, and since $X$ is nondegenerate, $S(X_L, X) \not\subset L$. Hence Theorem 1.4 of Chapter I shows that

$$\dim SX = \frac{3n}{2} + 1 \geq \dim S(X_L, X) = \dim X_L + n + 1,$$

so that

$$\dim X_L \geq \frac{n}{2}. \tag{2.1.1}$$

On the other hand, for a general point $u \in P_L$ from Proposition 1.9 of Chapter II it follows that

$$T(Y_u, X) \subset T_{SX,u} \overset{\text{def}}{=} L, \quad Y_u = p_1(\varphi^{-1}(u)),$$

i.e.

$$Y_u \subset X_L. \tag{2.1.2}$$

Since $X$ is a Severi variety, we have

$$\dim Y_z = \dim Y_u = 2n + 1 - \dim SX = 2n + 1 - \left(\frac{3n}{2} + 1\right) = \frac{n}{2}, \tag{2.1.3}$$

Combining (2.1.1), (2.1.2), and (2.1.3), we see that for a general point $u \in P_L$ we have $Y_u = Y_z$. Therefore

$$SY_z = P_L \tag{2.1.4}$$

and by (2.1.3)

$$2 \dim Y_z + 1 - \dim P_L = \dim Y_z, \quad \dim P_L = \dim Y_z + 1 = \frac{n}{2} + 1,$$

so that $P_L = \mathbb{P}^{\frac{3}{2} + 1}$ and $Y_z$ is a hypersurface in $\mathbb{P}^{\frac{3}{2} + 1}$. Hence for a general point $u \in P_L$ there is an inclusion $Y_u \supset X \cap P_L$, and therefore

$$X \cap P_L = Y_z. \tag{2.1.5}$$

From (2.1.4) it follows that $\deg Y_z \geq 2$. From the trisecant lemma (cf. [39, 2.5] and also [34, Chapter IV, §3] and [64, §7 B]) it follows that for a general pair of points $x, y \in X$

$$\langle x, y \rangle \cap X = \{x, y\}. \tag{2.1.6}$$
Since \( z \) is a general point of \( SX \), from (2.1.6) it follows that \( \deg Y_z = 2 \).

It remains to verify that the quadric \( Y_z \) is nonsingular. Suppose that this is not so. Then \( Y_z \) is a quadratic cone with vertex at a point \( y \in Y_z \). Therefore \( SX = S(Y_z, X) \) is a cone with vertex \( y \), and for a general point \( u \in SX \) we have \( y \in P_M, M = TSX, u \). In view of (2.1.5), from this it follows that

\[
y \in \bigcap_{u \in SX} Y_u.
\]

(2.1.7)

From (2.1.2) and (2.1.7) it follows that \( T_{X,y} \subset \bigcap T_{SX,u} \), i.e. \( SX \) is a cone with vertex \( T_{X,y} \) and \( T_{X,y} \subset \bigcap M \).

This contradiction shows that \( Y_z \) is a nonsingular quadric. \( \square \)

2.2. Lemma. Let \( X^n \subset \mathbb{P}^N, n = \frac{3}{2}(N - 2) \) be a Severi variety, and let \( z \) be a general point of \( SX \). Then, in the notations of Section 1 of Chapter I, the morphism \( \varphi_z = \varphi_{Y_z}: S_{Y_z, X} \to S(Y_z, X) = SX, Y_z = p_1(\varphi^{-1}(z)) \) is birational.

Proof. From Proposition 1.9 a) of Chapter II it follows that

\[
T(Y_z, X) \subset T_{SX, z}^{N-1},
\]

and since

\[
\langle S(Y_z, X) \rangle = \langle X \rangle = \mathbb{P}^N,
\]

Theorem 1.4 of Chapter I shows that

\[
\dim S(Y_z, X) = \dim S_{Y_z, X} = \dim SX = N = 1 = \frac{3n}{2} + 1.
\]

Hence \( S(Y_z, X) = SX \) and the morphism \( \varphi_z \) is generically finite, i.e. for a general point \( u \in SX \) we have \( \langle Y_z \cap Y_u \rangle < \infty \). To prove Lemma 2.2 it suffices to verify that for a general point \( u \in SX \) the quadrics \( Y_z \) and \( Y_u \) (cf. Proposition 2.1) transversely intersect at a unique point \( (Y_z \cap Y_u) \neq \varnothing \) since \( u \in S(Y_z, X) = SX \).

Let \( P_z \) (resp. \( P_u \)) be the \((\frac{3n}{2} + 1)\)-dimensional linear subspace spanned by the quadric \( Y_z \) (resp. \( Y_u \)). Suppose that \( Y_z \cap Y_u \ni x, y \), and let \( l = \langle x, y \rangle \) (in the case when \( y = x \), i.e. \( Y_z \) and \( Y_u \) are tangent at \( x \), \( l \) is their common tangent line). Then

\[
l \subset P_z \cap P_u \subset \text{Sing} (SX)
\]

since by Proposition 2.1 the tangent space to \( SX \) at an arbitrary point of \( P_z \cap P_u \) contains both \( TSX, z \) and \( TSX, u \neq TSX, z \). Varying \( u \in SX \), we see that a general point \( y \in Y_z \) lies on a line \( l_y \subset P_z \cap \text{Sing} (X \setminus X) \), from which it follows that \( z \in P_z \subset \text{Sing} SX \) contrary to the choice of \( z \). This contradiction completes the proof of Lemma 2.2. \( \square \)
2.3. Lemma. Let $X^n \subset \mathbb{P}^N$, $n = \frac{2}{3}(N - 2)$ be a Severi variety. Then the quasiprojective variety $SX \setminus X$ has nonsingular normalization.

Proof. Let $v \in \text{Sing}(SX \setminus X)$, and let $Y'_v$ be an irreducible component of $Y_v = p_1(\varphi^{-1}(v))$. Then either
\[ S(Y'_v, X) \neq SX \] (2.3.1)
and for a general point $z \in SX$
\[ Y_z \cap Y'_v = \emptyset \] (2.3.2)
or
\[ S(Y'_v, X) = SX. \] (2.3.3)
We claim that in the case (2.3.3)
\[ \dim Y'_v = \frac{n}{2}. \] (2.3.4)
Suppose that this is not so and
\[ \dim Y'_v > 2n + 1 - \dim SX = \frac{n}{2}. \] (2.3.5)
In view of (2.3.3) and Theorem 1.4 of Chapter I, from (2.3.5) it follows that
\[ T(Y'_v, X) = SX. \] (2.3.6)
Remark 1.10 in Chapter II shows that if $v \in (y, x)$, $y \in Y'_v$, $x \in X$, $x \neq y$, then $T'_v, S(x, X) \supset T_{X, y}$, where $T'_v, S(x, X)$ is the tangent cone to $S(x, X)$ at the point $v$ (we use the notations of §1 of Chapter I). Therefore
\[ T'_v, SX \supset T'_v, S(x, X) \supset T_{X, y}. \]
On the other hand, if $v \in T_{X, y}$, then it is clear that $T'_v, SX \supset T_{X, y}$. Thus
\[ T(Y'_v, X) \subset T'_v, SX. \] (2.3.7)
Combining (2.3.6) and (2.3.7) we see that $SX \subset T'_v, SX$, i.e. $SX$ is a cone with vertex $v$. Hence, in the notations of Lemma 2.2,
\[ v \in \bigcap_{z \in X} P_z. \] (2.3.8)
But in the proof of Lemma 2.2 we verified that for a general pair of points $z, u \in SX$ the variety $P_z \cap P_u = Y_z \cap Y_u$ reduces to a unique point lying in $X$, which contradicts (2.3.8). This contradiction proves (2.3.4).

From (2.3.2) and (2.3.4) it follows that for a general point $z \in SX$ we have $\text{card}(Y_z \cap Y_u) < \infty$, i.e. in the notations of Lemma 2.2 $\varphi^{-1}_v$ is a finite set. Hence Lemma 2.2 and Proposition 2.1 show that in a neighborhood of $v$ the normalization of $SX = S(Y_z, X)$ is isomorphic to $S_{Y_z, X}$ and therefore is nonsingular. Lemma 2.3 is proved. □
2.4. Theorem. Let $X^n \subset \mathbb{P}^N$, $n = \frac{2}{3}(N - 2)$ be a Severi variety. Then

a) $SX$ is a normal hypersurface and $\text{Sing} SX = X$;

b) Each point $z \in SX \setminus X$ is general in the sense of Proposition 2.1, i.e. $Y_z$ is a nonsingular quadric in the projective space $P_z = \mathbb{P}^{\frac{n+1}{2}}$;

c) $\varphi|_{\varphi^{-1}(SX \setminus X)}$ is a smooth morphism, and the tangent spaces at arbitrary points $x, y \in X$ such that the line $(x, y)$ does not lie in $X$ span a hyperplane, i.e. $\dim (T_x X, T_y X) = N - 1 = \frac{3n}{2} + 1$;

d) An arbitrary secant of the variety $X$ either lies on $X$ or intersects $X$ at exactly two points (which may coincide with each other);

e) $SX$ is a cubic and $\text{mult}_x SX = 2$ for each point $x \in X = \text{Sing} SX$ (here $\text{mult}_x SX$ denotes the multiplicity of $x$ on $SX$);

f) For an arbitrary point $z \in SX \setminus X$ the projection with center in $P_z$ defines a birational map $\pi_z : X \dashrightarrow \mathbb{P}^n$ which is an isomorphism outside $X \cap T_{SX, z}$.

Proof. a) Let

$$\varphi = \nu \circ \tilde{\varphi}, \quad \tilde{\varphi} : SX \rightarrow \overline{SX}, \quad \nu : \overline{SX} \rightarrow SX$$

be the Stein factorization of the morphism $\varphi : SX \rightarrow SX$. From Proposition 2.1 it follows that $\nu : \overline{SX} \rightarrow SX$ is the normalization morphism. Lemma 2.3 shows that

$$\text{Sing} \overline{SX} \subset \nu^{-1}(X). \quad (2.4.1)$$

Suppose that $\text{Sing} SX \neq X$, let $v$ be a general point of $\text{Sing} (SX \setminus X)$, and let $\tilde{v} \in \nu^{-1}(v)$. In view of (2.4.1) and the Serre normality criterion (cf. e.g. [30, Chapter IV, 2, (5.8.6)]),

$$\dim (\text{Sing} (SX \setminus X)) = \dim SX - 1 = \frac{3n}{2}.$$ 

Hence $\dim \tilde{\varphi}^{-1}(v) = \frac{n}{2}$ and $\tilde{\varphi}^*([\tilde{v}]) \equiv [\varphi^{-1}(z)]$, where $z$ is a general point of $SX$, brackets denote the cycle corresponding to subvariety, and $\equiv$ denotes algebraic equivalence. Therefore

$$(p_1)_* (\tilde{\varphi}^*([v])) = (p_1)_* ([\varphi^{-1}(z)]) = [Y_z]. \quad (2.4.2)$$

By Proposition 2.1 and Lemma 2.2, $Y_z$ is a nonsingular quadric and

$$(Y_z^2)_X = 1. \quad (2.4.3)$$

Hence all components of $Y_{\tilde{v}} = p_1(\tilde{\varphi}^{-1}(\tilde{v}))$ are also $\frac{n}{2}$-dimensional quadrics. From (2.4.2) and (2.4.3) it follows that for a general point $v \in \text{Sing} X \setminus X$ the fiber $\varphi^{-1}(v)$ is connected, i.e. the morphism $\nu$ is one-to-one. Since for $x \in X$ the fiber $\varphi^{-1}(x)$ is connected and contains a reduced component (which is isomorphic to $X$), from this it follows that $\nu$ is an isomorphism, i.e. $SX$ is a normal hypersurface. From Lemma 2.3 it follows that $\text{Sing} SX \subset X$. On the other hand, Lemma 1.8 a) from Chapter II shows that $X \subset \text{Sing} SX$. Thus $X = \text{Sing} SX$. Assertion a) is proved.
b) Let \( z \in SX \setminus X \) be an arbitrary point. Then from a) and Proposition 1.9 a) in Chapter II it follows that
\[
T(Y_z, X) \subset T_{SX,z} \neq \mathbb{P}^N.
\]
Since \( S(Y_z, X) \supset X \), \( S(Y_z, X) \neq T(Y_z, X) \), and from Theorem 1.4 of Chapter I it follows that
\[
\dim S(Y_z, X) = \dim Y_z + n + 1, \quad S(Y_z, X) = SX, \quad \dim Y_z = \frac{n}{2}.
\]
(2.4.4)
We already proved in a) that under these conditions \( Y_z \) is a quadric. Moreover, in the proof of Proposition 2.1 it is shown that from (2.4.4) it easily follows that the quadric \( Y_z \) is nonsingular. Assertion b) is proved.

c) The first claim follows from b), and the second claim is a consequence of the first one and the proof of Proposition 1.9 b) in Chapter II.

d) immediately follows from b).

e) Let \( x \) be an arbitrary point of \( X \), and let \( Q_x = p_1(\psi^{-1}(x)) \) (we use the notations of §1 of Chapter I). We observe that \( S(Q_x, X) \neq SX \). In fact, otherwise from Proposition 1.9 a) of Chapter II it would follow that \( x \in \bigcap_{z \in SX} T_{SX,z} \), i.e. \( SX \) is a cone with vertex \( x \), which is impossible in view of b) and the proof of Proposition 2.1.

Let \( z \in SX \setminus S(Q_x, X) \), and let \( P_{z,x}^{2+2} = \langle P_z, x \rangle \). Then
\[
P_{z,x} \cap X = Y_z \cup x, \quad (2.4.5)
\]
and the intersection of subvarieties \( P_{z,x}^{2+2} \) and \( X^n \) at the point \( x \) of the projective space \( P_{\mathbb{P}^n}^{2+2} \) is transverse. In fact, if
\[
x' \in P_{z,x} \cap X, \quad x' \notin Y_z, \quad u \in \langle x, x' \rangle \cap P_z,
\]
then in view of the choice of \( z \)
\[
x \notin T(Y_z, X) \quad (2.4.6)
\]
and from d) it follows that \( u \notin X \), so that b) yields \( Y_u = Y_z \) which is impossible by (2.4.6) since it is clear that \( Y_u \supset x \).

Now let \( u \in P_{z,x} \cap SX \) be an arbitrary point. As we already observed, \( Y_u \cap Y_z \neq \emptyset \), and either
\[
u \in T_{X,y} \cap P_{z,x}, \quad y \in Y_z \quad (2.4.7)
\]
or
\[
u \in \langle y, x' \rangle, \quad y \in Y_z, \quad x' \in X \cap P_{z,x}, \quad (2.4.8)
\]
But
\[
(T_{X,y} \cap P_{z,x}) \subset (T(Y_z, X) \cap P_{z,x}) = P_z^{2+1}, \quad (2.4.9)
\]
and from (2.4.5), (2.4.7), (2.4.8), and (2.4.9) it follows that

$$P_{z,x} \cap SX = P_z \cup S_x Y_z.$$  \hfill (2.4.10)

Assertion e) immediately follows from (2.4.10).

f) immediately follows from the proof of assertion e).

□

2.5. **Remark.** From Theorem 2.4 b) it follows that

$$S(SX^*) \subset X^*.$$  \hfill (2.5.1)

Counting dimensions, we see that each hyperplane in $\mathbb{P}^N$ which is tangent to $X$ at a point $x \in X$ lies in the pencil generated by two hyperplanes which are tangent to $SX$ at some points of $S(x, X)$ (cf. Proposition 3.1 for more details). Therefore

$$S((SX)^*) = X^*.$$  \hfill (2.5.2)

Applying Theorem 2.4 (and specifically assertion e) of this theorem), it is not hard to show that the variety $X = \text{Sing } SX$ is an intersection of quadrics and the linear system of quadrics passing through $X$ defines a birational map

$$\kappa_X : \mathbb{P}^N \dashrightarrow \mathbb{P}^N,$$  \hfill (2.5.3)

under which $SX$ is transformed to $X$ and vice versa (if $F(x_0 : \cdots : x_N) = 0$ is the equation of the cubic hypersurface $SX$, then $\kappa_X$ is defined by the formula

$$\kappa_X(x) = \left( \frac{\partial F}{\partial x_0}(x) : \cdots : \frac{\partial F}{\partial x_N}(x) \right).$$

Since by definition $\kappa_X(SX) = X^*$, from this it follows that

$$X^* \simeq SX, \quad (SX)^* \simeq X.$$  \hfill (2.5.4)
3. Dimension of Severi varieties

3.1. Proposition. Let \( X^n \subset \mathbb{P}^N \), \( N = \frac{2}{3}(N - 2) \) be a Severi variety, let \( x \in X \) be an arbitrary point, and let \( Y^x = \{ (T_{SX,z})^* \mid z \in S(x, X) \setminus X \} \) (here \((T_{SX,z})^* \in \mathbb{P}^{N*} \) is the point corresponding to the hyperplane \( T_{SX,z} \)). Then \( Y^x \) is a nonsingular quadric in the \((\frac{n}{2} + 1)\)-dimensional projective space \((T_{X,x})^* \subset \mathbb{P}^{N*} \). Let \( X \)

\( \text{Proof.} \) It is clear that \( Y^x \) is an irreducible variety. Since \( \dim(S(x, X) = n + 1 \) and for \( z \in S(x, X) \setminus X \) the hyperplane \( T_{SX,z} \) is tangent to \( SX \) along the \((\frac{n}{2} + 1)\)-dimensional linear subspace

\[ P_z = SY_z \subset S(x, X), \]

we conclude that

\[ \dim Y^x = (n + 1) - \left( \frac{n}{2} + 1 \right) = \frac{n}{2}. \]

From Theorem 2.4 it follows that \( Y^x \) is a (closed) irreducible hypersurface in \((T_{X,x})^* \). Since the variety \((SX)^* \subset \mathbb{P}^{N*} \) is not a hypersurface (cf. e.g. (2.5.1)), from the trisecant lemma (cf. \([39, 2.5] \) and also \([34, \text{Chapter IV, §3}] \) and \([64, \text{§7B}] \)) it follows that \( \deg Y^x = 2 \).

It remains to show that the quadric \( Y^x \) is nonsingular. In fact, otherwise the quadric \( Y^x \) would be a cone, so that there would exist a point \( z \in S(x, X) \setminus X \) such that

\[ \dim(Y_z \cap Y_{u}) > 0 \ \forall u \in S(x, X) \setminus X, \]

which contradicts Theorem 2.4f). \( \square \)

3.2. Proposition. Let \( X^n \subset \mathbb{P}^N \), \( n = \frac{2}{3}(N - 2) \) be a Severi variety, and let \( z_1, z_2 \in SX \setminus X \). Then either \( z_1 \in P_{z_2}, z_2 \in P_{z_1}, Y_{z_1} = Y_{z_2}, P_{z_1} = P_{z_2} \) or \( Y_{z_1} \cap Y_{z_2} \) is a linear subspace (we use the notations from the proof of Lemma 2.2).

\( \text{Proof.} \) As we have already observed,

\[ S(Y_{z_i}, X) = SX, \quad i = 1, 2, \]

and therefore \( Y_{z_1} \cap Y_{z_2} \neq \emptyset \). Suppose that \( Y_{z_1} \cap Y_{z_2} \) is not a linear subspace. Then from Theorem 2.4 it follows that

\[ S(Y_{z_1} \cap Y_{z_2}) = P_{z_1} \cap P_{z_2} \subset X. \]

But for an arbitrary point \( z \in S(Y_{z_1} \cap Y_{z_2}) \setminus X \) from Theorem 2.4b) it follows that

\[ Y_{z_1} = Y_{z_2}. \]

\( \square \)

Let \( X^n \subset \mathbb{P}^N \), \( N = \frac{2}{3}(n - 2) \) be a Severi variety, let \( x \) be an arbitrary point of \( X \), and let

\[ Y_1 = Y_{z_1}, \quad Y_2 = Y_{z_2}, \quad z_1, z_2 \in S(x, X) \setminus X \]

be two quadrics for which \( (Y_1 \cdot Y_2) = x \) (cf. Lemma 2.2). Put

\[ C_i = Y_i \cap T_{X,x} = Y_i \cap T_{Y_i,x}, \quad i = 1, 2. \]

Then \( C_i \) is an \((\frac{n}{2} - 1)\)-dimensional cone with vertex \( x \) in the \((\frac{n}{2} - 2)\)-dimensional projective space \( T_{Y_i,x} \) whose base is a nonsingular \((\frac{n}{2} - 2)\)-dimensional quadric \((i = 1, 2) \). It is clear that for \( n > 2 \), \( S(C_1, C_2) \subset X \) (here, as in \( §1 \) of Chapter I, \( S(C_1, C_2) \) is the join of cones \( C_1 \) and \( C_2 \)).
3.3. Proposition. a) \( \dim S(C_1, C_2) = n - 2; \)

b) Let \( n > 2, z \in S(C_1, C_2) \setminus X, Y_z \neq Y_i \) (\( i = 1, 2 \)). Then \( Y_z \cap Y_i \) is a linear subspace of dimension \( \left\lfloor \frac{n}{2} \right\rfloor \);

c) For \( n > 2 \) we have \( n \equiv 0 \pmod{4} \);

d) For \( n > 4 \) we have \( n \equiv 0 \pmod{8} \).

Proof. a) Let \( Q_i \) be the base of the cone \( C_i \) (i.e. \( Q_i \) is the intersection of \( C_i \) with a general hyperplane in \( T_{Y_i, x}, i = 1, 2 \)). Then

\[
S(C_1, C_2) = S(x, S(Q_1, Q_2))
\]

and

\[
\langle Q_1 \rangle \cap \langle Q_2 \rangle = \emptyset.
\]

Hence

\[
\dim S(C_1, C_2) = \dim S(Q_1, Q_2) + 1 = 2(\frac{n}{2} - 2) + 2 = n - 2.
\]

Assertion a) is proved.

b), c). Let \( z \) be a general point of \( S(C_1, C_2) \setminus X \). By Proposition 3.2,

\[
Y_z \cap Y_i = \mathbb{P}^{\alpha_i}, \quad i = 1, 2.
\]

Since \( C_1 \) and \( C_2 \) are cones with vertex \( x, \alpha_i > 0 \) (\( i = 1, 2 \)) and

\[
\mathbb{P}^{\alpha_i} \ni x, \quad i = 1, 2 \quad \mathbb{P}^{\alpha_1} \cap \mathbb{P}^{\alpha_2} = C_1 \cap C_2 = Y_1 \cap Y_2 = x, \quad z \in S(\mathbb{P}^{\alpha_1}, \mathbb{P}^{\alpha_2}).
\]

Furthermore, if \( z' \in S(C_1, C_2) \setminus X \), then \( Y_z' = Y_z \) if and only if

\[
z' \in S(\mathbb{P}^{\alpha_1}, \mathbb{P}^{\alpha_2}) \setminus \mathbb{P}^{\alpha_1 + \alpha_2} \setminus X.
\] (3.3.1)

By a) and (3.3.1), varying \( z \in S(C_1, C_2) \setminus X \) we obtain an \( (n - 2) - (\alpha_1 + \alpha_2) \)-dimensional family of quadrics passing through \( x \) and intersecting \( Y_1 \) and \( Y_2 \) along linear subspaces of positive dimension.

We already know (cf. Proposition 3.1) that there is an \( \frac{n}{2} \)-dimensional family of quadrics \( Y_u \) passing through \( x \) and parametrized by a quadric \( Y^x \). Furthermore, the \( (\frac{n}{2} - 1) \)-dimensional subfamily of quadrics \( Y_u \) intersecting \( Y_1 \) along a positive-dimensional linear subspace is parametrized by the subcone with vertex \( (T_{X,S', z})^* \) in \( Y^x \). From this it follows that the dimension of the family of quadrics \( Y_u \) passing through \( x \) and intersecting \( Y_1 \) and \( Y_2 \) along positive-dimensional linear subspaces is equal to \( \frac{n}{2} - 2 \) (the base \( Y_{12} \) of this family is the intersection of two \( (\frac{n}{2} - 1) \)-dimensional subcones in \( Y^x \) with vertices \( (T_{X,S, z}, z')^* \) and \( (T_{X,S, z}, z')^* \), so that \( Y_{12} \) is an \( (\frac{n}{2} - 2) \)-dimensional quadric). Thus \( n - 2 - (\alpha_1 + \alpha_2) = \frac{n}{2} - 2 \), i.e.

\[
\alpha_1 + \alpha_2 = \frac{n}{2} - 2.
\] (3.3.2)

On the other hand, it is well known (cf. e.g. [37, Chapter XIII, §4; 28, Chapter VI, §1]) that the maximal dimension of linear subspace lying on a nonsingular \( \frac{n}{2} \)-dimensional quadric \( Y_i \) (\( i = 1, 2 \)) is equal to \( \left\lfloor \frac{n}{4} \right\rfloor \). Hence

\[
\alpha_i \leq \left\lfloor \frac{n}{4} \right\rfloor, \quad i = 1, 2.
\] (3.3.3)
Combining (3.3.2) and (3.3.3), we see that
\[ \alpha_1 = \alpha_2 = \left[ \frac{n}{4} \right] = \frac{n}{4} \]
which simultaneously proves b) and c) (under specialization of \( z \) the dimension of \( Y_z \cap Y_i \) could only jump).

d) Let \( z \in S(C_1, C_2) \setminus X \). From b), c), and Proposition 3.1 it follows that the set of quadrics \( Y_u \) passing through \( x \) and intersecting \( Y_z \) along a linear subspace of dimension \( \frac{n}{4} \) is parametrized by the cone with vertex \((T_{S_X,z})^* \) in \( Y^z \) whose base is a nonsingular \( \left( \frac{n}{2} - 2 \right) \)-dimensional quadric. For \( n > 4 \) this cone is irreducible, and therefore all \( \frac{n}{4} \)-dimensional linear subspaces of the form \( Y_z \cap Y_u \) belong to one and the same family of linear subspaces on \( Y_z \). It is well known (cf. [37, Chapter XIII, § 4; 28, Chapter VI, § 1]) that the dimension of intersection of two \( \frac{n}{4} \)-dimensional linear subspaces from one family on \( Y^x \) has the same parity as \( \frac{n}{4} \) (we recall that on the nonsingular even-dimensional quadric \( Y_{n/2}^z \) there exist two irreducible families of \( \frac{n}{4} \)-dimensional linear subspaces). On the other hand, in the notations used in the proof of assertions b) and c)
\[ \mathbb{P}^{\alpha_1} \cap \mathbb{P}^{\alpha_2} = Y_1 \cap Y_2 = x. \]
Hence for \( n > 4 \)
\[ 0 \equiv \frac{n}{4} \pmod{2}. \]
This completes the proof of assertion d).

\[ \square \]

3.4. Remark. Assertions c) and d) of Proposition 3.3 were independently proved by Fujita and Roberts (cf. Propositions 5.2 and 5.4 in [25]) who used the techniques of computations with Chern classes. Their approach was developed by Roberts (unpublished) and Tango [89] (cf. Remark 4.11 below).

3.5. Corollary. If in the conditions of Proposition 3.2 \( Y_{z_1} \neq Y_{z_2} \), then \( Y_{z_1} \cap Y_{z_2} \) is either a point or a linear subspace of dimension \( \frac{n}{4} \).

In the proof of Proposition 3.3 we showed that varying \( z \) in \( S(C_1, C_2) \setminus X \) we obtain a family of \( \frac{n}{4} \)-dimensional linear subspaces on the \( \frac{n}{2} \)-dimensional quadric \( Y_z \). The base of this family is a nonsingular \( \left( \frac{n}{2} - 2 \right) \)-dimensional quadric. Hence for \( n > 4 \) all linear subspaces of the form \( Y_z \cap Y_1, z \in S(C_1, C_2) \setminus X \) belong to one and the same irreducible family of \( \frac{n}{4} \)-dimensional linear subspaces on \( Y_1 \) passing through \( x \) (cf. the proof of Proposition 3.3 d)). We denote this family by \( \mathcal{F} \) and the other family by \( \mathcal{F}' \).

3.6. Lemma. Let \( n > 4 \), and let \( P_0 \) be an arbitrary linear subspace on \( Y_1 \) passing through \( x \) and belonging to the family \( \mathcal{F} \). Then for some \( z \in S(C_1, C_2) \setminus X \) we have \( Y_z \cap Y_1 = P_0 \).

Proof. We argue by induction. Let \( z \in S(P_0, C_2) \setminus X \). Then \( Y_z \cap P_0 \) is a linear subspace of positive dimension. It is clear that it suffices to prove the following assertion. Let
\[ z \in S(C_1, C_2) \setminus X, \quad Y_z \cap P_0 = \mathbb{P}^{\alpha} \ni x, \quad 0 < \alpha < \frac{n}{4} \]
IV. SEVERI VARIETIES

(since \( \alpha \equiv \frac{n}{4} \mod 2 \), Proposition 3.3.d) shows that \( \alpha \) is even). Then there exists a point \( u \in S(C_1, C_2) \setminus X \) such that \( Y_u \cap P_0 \supseteq \mathbb{P}^\alpha \).

Let \( y \in P_0 \setminus \mathbb{P}^\alpha \). Then there exists a point \( y' \in Y_z \cap Y_2 \) such that

\[
(y, y') \not\subseteq X, \quad (y', \mathbb{P}^\alpha) \subseteq Y_z.
\]

In fact, the variety

\[
X_{z,y} = \{ y' \in Y_z \mid (y, y') \subset X \}
\]

is a linear subspace since otherwise there would exist a point

\[
z' \in (y', y'') \setminus X, \quad y', y'' \in X_{z,y},
\]

and it is clear that

\[
(y, y') \subset Y_{z'}, \quad (y, y'') \subset Y_{z'},
\]

so that

\[
y \in Y_{z'} = Y_z
\]

(cf. Theorem 2.4b) contrary to the choice of \( y \). Since \( (y, \mathbb{P}^\alpha) \subset P_0 \subset X \), we see that \( \mathbb{P}^\alpha \subset X_{z,y} \) and therefore

\[
X_{z,y} \neq Y_z \cap Y_2.
\]

The linear subspace \( \mathbb{P}^\alpha \) is contained in two \( \frac{1}{2} \left( \frac{n}{4} - \alpha \right) \left( \frac{n}{4} - \alpha - 1 \right) \)-dimensional families of \( \frac{n}{4} \)-dimensional linear subspaces on \( Y_z \) belonging to \( \mathcal{F} \) and \( \mathcal{F}' \) respectively (cf. [37, Chapter XIII, § 4; 28, Chapter VI, § 1]). Since \( \frac{n}{4} \) and \( \alpha \) are even,

\[
\frac{1}{2} \left( \frac{n}{4} - \alpha \right) \left( \frac{n}{4} - \alpha - 1 \right) > 0.
\]

Hence there exists a linear subspace on \( Y_z \) which passes through \( \mathbb{P}^\alpha \) and intersects \( Y_2 \) at a point \( y' \in Y_2 \setminus X_{z,y} \). This point \( y' \) satisfies all the above conditions.

Let

\[
u \in (y, y') \setminus X \subset S(C_1, C_2) \setminus X,
\]

and let \( a \in \mathbb{P}^\alpha \) be an arbitrary point. By construction

\[
(y, a) \subset P_0 \subset X, \quad (y', a) \subset Y_z \subset X,
\]

and therefore

\[
(y, a) \subset Y_u, \quad (y', a) \subset Y_u.
\]

Thus

\[
\mathbb{P}^\alpha \subseteq (y, \mathbb{P}^\alpha) \subset Y_u.
\]

\( \square \)

3.7. Remark. We can also take as \( P_0 \) the \( \frac{n}{4} \)-dimensional linear subspace passing through \( x \) and belonging to the family \( \mathcal{F}' \) and repeat the arguments used in the proof of Lemma 3.6 up to the last step when \( \frac{n}{4} - \alpha = 1 \). In this case \( \mathbb{P}^\alpha \) is contained in exactly two linear subspaces, viz. the subspace \( Y_z \cap Y_1 \) from the family \( \mathcal{F} \) and a subspace \( P_{\alpha'} \) from the family \( \mathcal{F}' \). Since at that step the process of constructing
u must terminate, we see that if \( y \in P_0 \setminus P_0^* \), then \( X_{x,y} = P_0^* \) and the \( \left( \frac{n}{4} + 1 \right) \)-dimensional linear subspace \( \langle P_0, P_0^* \rangle \) lies in \( X \). Thus each \( \frac{n}{4} \)-dimensional linear subspace from the family \( \mathcal{F} \) on \( Y_1 \) is cut by an \( \left( \frac{n}{4} + 1 \right) \)-dimensional linear subspace lying on \( X \).

3.8. Remark. It is clear that for \( n = 4 \) each of the two lines making up \( C_1 \) lies on one of the quadrics \( Y_z, z \in S(C_1, C_2) \setminus X \). These lines lie on certain planes in \( X \).

In the proof of Proposition 3.3 b) we already observed that the quadrics

\[ Y_z, \quad z \in S(C_1, C_2) \setminus X \]

form a family parametrized by a nonsingular \( \left( \frac{n}{2} - 2 \right) \)-dimensional quadric \( Y_{12} \), where \( Y_{12} \) is the intersection in \( Y^z \) of the \( \left( \frac{n}{2} - 1 \right) \)-dimensional quadratic cones with vertices \((T_{SX,z_1})^*\) and \((T_{SX,z_2})^*\). On the other hand, the \( \frac{n}{2} \)-dimensional linear subspaces on the quadric \( Y_1 \) passing through the point \( x \) and belonging to the family \( \mathcal{F} \) are parametrized by the spinor variety \( S^x \) corresponding to the orbit of highest weight vector of the spinor representation of the group \( \text{Spin}_{\frac{n}{2}}(D_{n/4}) \), where \( \dim S^x = \frac{1}{2} \cdot \frac{n}{4} \cdot \left( \frac{n}{4} - 1 \right) \) (cf. [11; 35; 87; 74]). The correspondence \( Y_z \to Y_z \cap Y_1 \) induces a morphism \( \rho: Y_{12} \to S^x \).

Lemma 3.6 can now be restated as follows.

3.9. Corollary. For \( n > 4 \) the morphism \( \rho \) is surjective.

3.10. Theorem. Let \( X^n \subset \mathbb{P}^N, n = \frac{2}{5}(N - 2) \) be a Severi variety. Then \( n = 2, 4, 8, \) or \( 16 \).

Proof. From Lemma 3.6 and Corollary 3.9 it follows that for \( n > 4 \)

\[
\frac{n}{2} - 2 = \dim Y_{12} \geq \dim S^x = \frac{1}{2} \cdot \frac{n}{4} \cdot \left( \frac{n}{4} - 1 \right) \quad (3.10.1)
\]

(from Remark 3.8 it follows that for \( n = 4 \) the inequality (3.10.1) turns into equality). Thus for \( n > 2 \) we obtain the following inequality:

\[
4 \leq n \leq 16. \quad (3.10.2)
\]

From the definition of Severi varieties (cf. 1.2) it is clear that \( n \) is even, and so Theorem 3.10 follows from (3.10.2) and Proposition 3.3 d). \( \square \)

3.11. Remark. If \( n = 16 \), then \( S^x \) is a six-dimensional quadric and \( \rho: Y_{12} \to S^x \) is an isomorphism. For \( n = 8 \) we have \( S^x = \mathbb{P}^1 \), and \( \rho \) is the projection of two-dimensional quadric onto one of its generatrices. In fact, for \( n = 8 \) each quadric

\[
Y_z, \quad z \in S(P_0, C_2) \setminus X \quad (3.11.1)
\]

intersects \( Y_1 \) along \( P_0 \), and thus each plane from the family \( \mathcal{F} \) on \( Y_1 \) is cut by a pencil of quadrics from the family (3.11.1). Similarly, from Remark 3.7 it follows that for \( n = 8 \) each plane from the family \( \mathcal{F}' \) on \( Y_1 \) is cut by a pencil of three-dimensional linear subspaces (or, which is the same, by a four-dimensional linear subspace) on \( X \).

3.12. Remark. Tango [89] proved that if there exists a Severi variety \( X^n, n > 16 \), then \( n = 2^m \) (\( m \geq 7 \)) or \( n = 3 \cdot 2^m \) (\( m \geq 5 \)).
4. Classification theorems

Let \( x \in S_X \setminus X \) be an arbitrary point. In this section we study the projection \( \pi_x : X \rightarrow \mathbb{P}^n \) with center at the linear subspace \( P_x = SY_x \) in more detail. We already know (cf. Theorem 2.4 f)) that \( \pi_x \) is an isomorphism outside the hyperplane section \( H_x = X \cap T_{SX,x} \).

### 4.1. Lemma

For \( n > 2 \) the fibers of the map \( \pi_x \vert_{H_x} \) are \( (\frac{n}{4} + 1) \)-dimensional linear subspaces intersecting \( Y_x \) along \( \frac{n}{4} \)-dimensional linear subspaces. For \( n > 4 \) these subspaces belong to the family \( F' \), and for \( n = 4 \) the intersections contain lines from both families.

**Proof.** Let \( u = \langle x, x' \rangle \cap P \in SX \setminus X \) by Theorem 2.4 d) and therefore \( x \in Y_u = Y_x \) contrary to the choice of \( x \) (cf. the proof of assertion e) of Theorem 2.4). As in the proof of Lemma 3.6, from this it follows that \( \pi^{-1}_x(\pi(x)) = (\mathbb{P}^n \cap X) \setminus Y_x \) is a linear subspace. The intersection of this linear subspace with the quadric \( Y_x \) coincides with the linear subspace \( X_{x,x} \subset Y_x \) introduced in the proof of Lemma 3.6. The assertion of Lemma 4.1 now follows from Remarks 3.7 and 3.8. \( \square \)

Thus for \( n > 2 \) \( B_x = \pi_x(H_x) \) is an \( [(n-1) - (\frac{n}{4} + 1)] = (\frac{3n}{4} - 2) \)-dimensional subvariety in the hyperplane in \( \mathbb{P}^n \) corresponding to the hyperplane \( T_{SX,z} \subset \mathbb{P}^N \).

### 4.2. Lemma

a) If \( n = 4 \), then \( B_x \) is a union of two skew lines in \( \mathbb{P}^3 \subset \mathbb{P}^4 \).

b) If \( n = 8 \), then \( B_x \) is the Segre variety \( \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7 \subset \mathbb{P}^8 \).

c) If \( n = 16 \), then \( B_x \subset \mathbb{P}^{15} \subset \mathbb{P}^{16} \) is the spinor variety corresponding to the orbit of highest weight vector of the spinor representation of the group \( \text{Spin}_{10}(D_{5}) \) (cf. § 2 of Chapter III).

**Proof.** Let \( S_x \) be the variety parametrizing the \( \frac{n}{4} \)-dimensional linear subspaces on \( Y_x \) belonging to the family \( F' \). Then \( S_x \) corresponds to the orbit of highest weight vector of the spinor representation of the group \( \text{Spin}_{\frac{n}{4}+2}(D_{\frac{n}{4}+1}) \) and

\[
\dim S_x = \frac{1}{2} \cdot \frac{n}{4} \cdot \left( \frac{n}{4} + 1 \right)
\]

(cf. [11; 35; 87; 37, Chapter XIII, § 4]). By Lemma 4.1, the correspondence

\[
a \sim \pi^{-1}_x(a) \cap Y_x
\]

induces a morphism

\[
B_x \rightarrow S_x.
\] (4.2.1)
From Remark 3.7 it follows that this morphism is surjective.

In case c) the varieties $B_z$ and $S_z$ have the same dimension ($\dim B_z = \dim S_z = 10$) and the map (4.2.1) is an isomorphism. In fact, if

$$x, x' \in H_z \setminus Y_z, \quad P_{z,x} \neq P_{z,x'}, \quad P_{z,x} \cap Y_z = P_{z,x'} \cap Y_z,$$

then for each $u \in S(P_{z,x}, P_{z,x'}) \setminus X$

$$Y_u \supset P_{z,x}, P_{z,x'},$$

which is impossible since $Y_u$ is a nonsingular $\frac{n}{2}$-dimensional quadric and $\dim P_{z,x} = \frac{n}{2} + 1$. Therefore $S(P_{z,x}, P_{z,x'}) \subset X$ and the fiber of the map (4.2.1) over the point corresponding to $P_{z,x} \cap Y_z$ is a linear subspace of positive dimension which is also impossible since otherwise the variety $A_z^{10} \subset P^{15}$ would contain an exceptional divisor contrary to a theorem of Barth (cf. [6; 33]). Thus in case c) the map (4.2.1) is an isomorphism, and from the fact that

$$\text{Pic} S_z \simeq \mathbb{Z}$$

(cf. [11; 35; 87]; by virtue of Theorem 5 from [95] this also follows from the results of §2 of Chapter III, and in the case when $\dim S_z = 10$ one can apply a Barth type theorem [54; 65; 60]) it follows that the embedding

$$A_z^{10} \hookrightarrow P^{15} \hookrightarrow P^{16}$$

corresponds to the spinor representation.

In case b)

$$\dim B_z = 4, \quad S_z = \mathbb{P}^3.$$

From Remark 3.11 it follows that the fibers of the morphism (4.2.1) are projective lines and the preimage of an arbitrary projective line from $S_z$ (corresponding to a point from $Y_z$) is a nonsingular two-dimensional quadric. Furthermore, to each point of $S_z$ there corresponds a four-dimensional linear subspace on $X$ mapping to a line on $B_z$, and thus we obtain a map $S_z \times \mathbb{P}^1 \to B_z$. Thus in case b) $B_z$ is projectively isomorphic to the Segre embedding of the variety $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7 \subset \mathbb{P}^8$.

In case a)

$$\dim B_z = 1, \quad S_z = \mathbb{P}^1.$$

By Remark 3.8, each line on the quadric $Y_z$ is cut by a plane, and we obtain a surjection $B_z \to S_z \coprod S_z$. Arguing as in case c), we see that $B_z = \mathbb{P}^1 \coprod \mathbb{P}^1$ and $H_z$ consists of two irreducible components intersecting along $Y_z$.

\[\Box\]

4.3. Remark. Since

$$T_{SX,z} \cap SX = T(Y_z, X),$$

we see that

$$H_z = T_{SX,z} \cap X = \bigcup_{y \in Y_z} T_{X,y} \cap X.$$
Furthermore,
\[ T_{X,y} \cap X = \bigcup_{u \in \pi^{-1} X \setminus X} C_u = \bigcup_{y \in \mathbb{P}^1} \mathbb{P}^1, \quad C_u = Y_u \cap T_{Y_u,y}. \]

Hence \( T_{X,y} \cap X \) is a cone with vertex \( y \), and from Proposition 3.1 and Corollary 3.5 it follows that
\[ \dim (T_{X,y} \cap X) = \frac{n}{2} + \left( \frac{n}{2} - 1 \right) - \frac{n}{4} = \frac{3n}{4} - 1. \]

Under the mapping \( \pi_z \) each of the cones \( T_{X,y} \cap X \) is projected onto its base which is isomorphic to the variety \( B_z \).

4.4. Remark. For \( n = 2 \)
\[ H_z = 2Y_z, \quad T_{X,y} \cap X = y \]
(by Corollary 1.15 from Chapter I, the hyperplane section \( H_z \) is reduced for \( n > 2 \) and is normal for \( n > 4 \)). It is not hard to show that in this classical case the map \( \pi_z \) is an isomorphism, so that \( B_z = \emptyset \) (cf. e.g. \([82; 62]\)).

Next we describe the rational map inverse to the map \( \pi_z : X \to \mathbb{P}^n \). This map \( \sigma_z \) is defined by \( \mathbb{P}^n + 3 \) forms
\[ G_0, \ldots, G_{\frac{n}{2}+2}, \quad \deg G_i = d, \quad i = 0, \ldots, \frac{3n}{2} + 2 \]
vanishing on a subvariety \( B_z \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \). It is easy to see that \( B_z \) is defined in \( \mathbb{P}^{n-1} \) (and in \( \mathbb{P}^n \)) by quadratic equations. This immediately follows from Lemma 4.2, but can also be proved without any computations. It suffices to observe that, according to Remark 4.3, \( B_z \) is the base of the cone \( T_{X,y} \cap X \) \((y \in Y_z)\).

Hence \( B_z = \mathbb{P}^{n-1} \cap X \), where \( \mathbb{P}^{n-1} \) is a hyperplane in \( T_{X,y} \) not passing through \( y \). In Remark 2.5 we observed that \( X \) is defined by quadratic equations. Restricting these equations on \( \mathbb{P}^{n-1} \) we obtain equations for \( B_z \). It is clear that the image of the restriction of the map \( \pi_X \) from Remark 2.5 (cf. (2.5.3)) on \( T_{X,y} \) (or \( \mathbb{P}^{n-1} \subset T_{X,y} \)) coincides with the \( \frac{n}{2} \)-dimensional quadric \( Y^2 \) from Proposition 3.1.

Hence the number of linearly independent quadratic equations defining \( B_z \) in \( \mathbb{P}^{n-1} \) is equal to \( \frac{n}{2} + 2 \). Adding \( n + 1 \) quadratic equations defining \( \mathbb{P}^{n-1} \) in \( \mathbb{P}^n \), we see that the subvariety \( B_z \subset \mathbb{P}^n \) is defined by \( \frac{3n}{2} + 3 \) linearly independent quadratic equations. From this it follows that for \( n > 2 \) we have \( d = 2 \). The case \( n = 2 \) \((B_z = \emptyset)\) is dealt with in a similar way (cf. Remark 4.4); this case was first studied by Severi (cf. \([82; 62; 15]\)). Summing up, we obtain the following result.

4.5. Theorem. If \( X^n \subset \mathbb{P}^N \), \( n = \frac{3}{2}(N - 2) \) is a Severi variety, then \( n = 2, 4, 8, \) or 16 and \( X \) is the image of \( \mathbb{P}^n \) under the rational map \( \sigma : \mathbb{P}^n \to \mathbb{P}^N \) defined by the linear system of quadrics passing through a subvariety \( A \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \), where

- a) for \( n = 2 \) \( A = \emptyset \);
- b) for \( n = 4 \) \( A = \mathbb{P}^1 \coprod \mathbb{P}^1 \) is a union of two skew lines;
- c) for \( n = 8 \) \( A = \mathbb{P}^1 \times \mathbb{P}^3 \) is the Segre variety in \( \mathbb{P}^7 \);
- d) for \( n = 16 \) \( A = S^{10} \) is the spinor variety parametrizing one of the two families of four-dimensional linear subspaces on the nonsingular quadric in \( \mathbb{P}^9 \) (cf. §2 of Chapter III).
In other words, the Severi variety $X^n$ is obtained from the Veronese variety $v_2(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{P}^n$ by projecting it from the linear span $\langle v_2(A) \rangle$ of the image of the subvariety $A \subset \mathbb{P}^n$ under the Veronese embedding $v_2$.

4.6. Remark. From Remark 4.3 and the arguments given before the statement of Theorem 4.5 it immediately follows that the linear system of quadrics cut in a general linear subspace $\mathbb{P}^{n-1} \subset T_{X,x}$ by the linear system of quadrics passing through $X$ and defining a rational map

$$\mathbb{P}^{n-1} \dashrightarrow Q^2 \subset \mathbb{P}^{n+1}$$

(where $Q^2 = Y^x$ is a nonsingular quadric) is the second fundamental form in the sense of [29] and the subvariety $A \subset \mathbb{P}^{n-1}$ is the fundamental subset of this form.

Theorem 4.5 shows that in each of the dimensions 2, 4, 8, 16 there exists at most one Severi variety. To complete classification of Severi varieties it remains to verify that the necessary conditions formulated in Theorem 4.5 are also sufficient, i.e. the varieties $X^n$ described in Theorem 4.5 are nonsingular and can be isomorphically projected to $\mathbb{P}^{n+1}$.

However in Chapter III we already constructed four examples of Severi varieties (the first three of them, viz. the Veronese, Segre, and Grassmann varieties, are classical; cf. Remark 1.3, [33; 38]). Moreover, using methods from representation theory, in § 3 of Chapter III we studied the maps $\pi_z$ and $\sigma_z$ in these examples and described geometric properties and computed invariants of the corresponding varieties. Thus Theorem 4.5 yields the following basic result.

4.7. Theorem. Over an algebraically closed field of characteristic zero each Severi variety is projectively equivalent to one of the following four projective varieties:

- a) $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ (Veronese surface);
- b) $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ (Segre variety);
- c) $G(5,1)^8 \subset \mathbb{P}^{14}$ (Grassmann variety);
- d) $E_{16} \subset \mathbb{P}^{26}$ (Cartan variety);

All these varieties are homogeneous, rational, and are defined by quadratic equations. Furthermore, the variety $X$ corresponds to the orbit of highest weight of an irreducible representation of a semisimple group $G$ in a vector space $V$ with highest weight $\Lambda$, where:

- a) $G = SL_3$, $\Lambda = 2\varphi_1$;
- b) $G = SL_3 \times SL_3$, $\Lambda = \varphi_1 \oplus \varphi_1$;
- c) $G = SL_6$, $\Lambda = \varphi_2$;
- d) $G = E_6$, $\Lambda = \varphi_1$

(here $\varphi_i$ is the $i$-th fundamental weight).

The variety $SX$ corresponds to the cone of ‘null-forms’ $\mathcal{R} = \{ v \in V \mid F(v) = 0 \}$, where $F$ is the cubic form generating the algebra of $G$-invariant polynomials on $V$.

There remains one question: how to ‘explain’ the fact that dimension of Severi varieties assumes only these four values (or at least that they are powers of two)? One approach suggested by Roberts (unpublished) and Tango [89] is based on a
study of arithmetic properties of Chern characters (Roberts used well known number theoretic results on denominators of Bernoulli numbers). Using this approach Tango showed that if there exists a Severi variety $X^n$ of dimension $n > 16$, then $n = 2^m (m \geq 7)$ or $n = 3 \cdot 2^m (m \geq 5)$. More intriguing is the ‘explanation’ based on the following result (independently discovered by Roberts).

4.8. Theorem. Let $\mathfrak{A}$ be a composition algebra over the field $K$, and let $\mathfrak{J}$ be the Jordan algebra of Hermitean $3 \times 3$-matrices over $\mathfrak{A}$ (a matrix $A$ is called Hermitean if $A^t = A$, where $t$ denotes transposition and the bar denotes the involution in $\mathfrak{A}$), so that $\dim_K \mathfrak{J} = 3(\dim_K \mathfrak{A} + 1)$ (cf. [10; 44; 76]). Let $X^n \subset \mathbb{P}(\mathfrak{J}) = \mathbb{P}^N$ be the projective variety corresponding to the cone $\{ A \in \mathfrak{J} \mid \operatorname{rk} A \leq 1 \}$. Then $X$ is a Severi variety and $SX$ is the hypersurface corresponding to the cone $\{ A \in \mathfrak{J} \mid \det A = 0 \}$. Conversely, each Severi variety is obtained in such way.

Proof. By Jacobson’s theorem (cf. [43; 44, Chapter IV, n°3]), there exist exactly four composition algebras—one in each of the dimensions 1, 2, 4, 8, viz. the algebras $\mathfrak{A}_0 = K$, $\mathfrak{A}_1 = K[t]/(t^2 + 1)$, $\mathfrak{A}_2$—the algebra of quaternions over $K$, and $\mathfrak{A}_3$—the Cayley algebra over $K$. For these algebras

$$N_i = \dim \mathbb{P}(\mathfrak{J}_i) = 3 \cdot 2^i + 2, \quad n_i = \dim X_i = 2^{i+1} = 2 \dim \mathfrak{A}_i,$$

where $\mathfrak{J}_i$, and $X_i$ are the Jordan algebra and the projective variety corresponding to the algebra $\mathfrak{A}_i$ ($0 \leq i \leq 3$).

It is clear that the surface $X_0$ coincides with the Veronese surface. The algebra $\mathfrak{J}_1$ is identified with the algebra of $3 \times 3$-matrices over the field $K$, and the variety $X_1$ is identified with the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$ (cf. Theorem 2.4 in Chapter III). Since the field $K$ is algebraically closed, $\mathfrak{A}_2$ is isomorphic to the algebra of $2 \times 2$-matrices over $K$. From this it is easy to deduce that $X_2$ is projectively equivalent to $G(5, 1)$ (cf. Chapter III, 2.5, A$_3$). Finally, from Freudenthal’s results [23] it follows that the variety $X_3$ is isomorphic to $E$ (cf. Chapter III, 2.5, E). $\Box$

It is easy to see that Theorem 4.8 can be restated as follows (judging by [56], a similar result (for complexifications of real division algebras) was proved by T. Banchoff).

4.9. Theorem. $X$ is a Severi variety if and only if $X$ is a ‘Veronese surface’ over one of the algebras $\mathfrak{A}_i$ ($0 \leq i \leq 3$), i.e. $X$ is the image of the ‘projective plane’ $\mathbb{P}^2(\mathfrak{A}_i) = (\mathfrak{A}_i^3 \setminus 0)/\mathfrak{A}_i^*$ (where $\mathfrak{A}_i^*$ is the set of invertible elements of the algebra $\mathfrak{A}_i$) with respect to the map

$$(x_0 : x_1 : x_2) \mapsto (\cdots : x_l \bar{x}_m : \cdots), \quad 0 \leq l \leq m \leq 2.$$

4.10. Remark. We do not know if there exists some intrinsic connection between composition algebras (or some other class of algebras) and Severi varieties or this is an accidental coincidence. In any case, classification of Severi varieties given in Theorems 4.7–4.9 allows to give a new unexpected proof of the well known Jacobson theorem on the structure of composition algebras (cf. e.g. [43; 44, Chapter IV, n°3]). In Chapter VI (cf. Remark 5.10 and Theorem 5.11) we shall see that extremal varieties with small secant varieties also correspond to matrix Jordan algebras (or
4. CLASSIFICATION THEOREMS

Furthermore, all varieties except $E_{16} \subset \mathbb{P}^{26}$ correspond to special Jordan algebras, and the variety $E$ corresponds to the exceptional algebra of Hermitean $3 \times 3$-matrices over the Cayley numbers.

4.11. Remark. In the case of surfaces Theorem 4.7 was first proved by Severi [82] (the proof of Severi is reproduced in [62], and the paper [15] is devoted to finding out which parts of this proof work in the case when $\text{char } K > 0$). Griffiths and Harris who apparently didn’t know about Severi’s paper proved a local version of his result (cf. [29, 6c]). Scorza [77; 78] classified (possibly singular) threefolds and fourfolds with small secant varieties. However these results of Scorza were forgotten, and in 1979 Griffiths and Harris proved that each four-dimensional Severi variety (or a Zariski open subset of such a variety) has the same second fundamental form as the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ (cf. [29, (5.62)]). In the same paper Griffiths and Harris conjectured that up to projective equivalence $\mathbb{P}^2 \times \mathbb{P}^2$ is the only four-dimensional Severi variety. Basing on the author’s results, Fujita and Roberts [25] proved this conjecture, and Fujita [24] gave a modern proof of Scorza’s result for nonsingular threefolds. Tango [89] showed that if there exists a Severi variety $X^n$, $n > 16$, then $n \geq 96$ and either $n = 2^m$ or $n = 3 \cdot 2^m$, where $m$ is a natural number. Theorem 4.7 was first proved in [99] (cf. also [56]).
5. Varieties of codegree three

From Theorem 2.4 and Remark 2.5 it follows that the dual variety of an arbitrary Severi variety is a cubic hypersurface. This property is shared by isomorphic projections of Severi varieties. This observation indicates that in the context of the present chapter it is relevant to give classification of all nonsingular varieties whose dual varieties have degree three.

Classification of varieties of small _degree_ has been a popular topic since A. Weil [104] classified all projective varieties of degree three in 1957 (classification of varieties of degree one and two is trivial). Later Swinnerton-Dyer [86] succeeded in classifying varieties of degree four, and in a series of papers Ionescu classified smooth projective varieties up to degree eight. Several papers are devoted to low-dimensional varieties of small degree and to varieties whose degree is not too big with respect to codimension. Due to efforts of Hartshorne, Barth, Van de Ven, and Ran it was found out that if the dimension of a nonsingular variety is sufficiently large with respect to its degree, then the variety is a complete intersection.

However here we are more interested in _class_ which is another _classical_ invariant of projective varieties whose role in enumerative geometry is not less than that of degree. Traditionally, the class of a nonsingular variety $X^n \subset \mathbb{P}^N$ is defined as the number of singular divisors in a generic pencil of hyperplane sections of $X$. Thus if the dual variety $X^*$ is a hypersurface, then the class of $X$ is equal to the degree of $X^*$. Varieties for which $\text{codim} \, X^* > 1$ have class zero, but for us it is more convenient to use the notion of _codegree_.

5.1. Definition. The number $d^* = \deg X^*$ is called the codegree of $X$ in $\mathbb{P}^N$ and is denoted by $\text{codeg} \, X$.

Thus codegree is equal to class provided that $X^*$ is a hypersurface.

It is clear that the only varieties of codegree one are linear subspaces $\mathbb{P}^n \subset \mathbb{P}^N$ and the only varieties of codegree two are quadrics $Q^n \subset \mathbb{P}^N$.

We notice that if $X^n \subset \mathbb{P}^M \subset \mathbb{P}^N$, then the dual variety of $X$ in $\mathbb{P}^N$ is the cone over the dual variety of $X$ in $\mathbb{P}^M$ with vertex $(\mathbb{P}^M)^* = \mathbb{P}^{N-M-1}$. Hence the codegree of $X$ in $\mathbb{P}^N$ is equal to the codegree of $X$ in $\mathbb{P}^M$, and in classification of varieties of a given codegree it suffices to consider the case when $X$ is nondegenerate, i.e. $\langle X \rangle = \mathbb{P}^N$, where $\langle X \rangle$ is the linear span of $X$.

Classification of nonsingular varieties of small codegree is apparently more difficult than that of varieties of small degree, e.g. because in the last case one can proceed by induction on dimension by taking hyperplane sections while in the first case there is no such possibility. Furthermore, the flavor of the problem for codegree is quite different. An important part of the problem is to characterize the structure of singularities of hypersurfaces of a given degree whose dual varieties are nonsingular. While there always exist varieties of a given degree and arbitrary dimension (e.g. hypersurfaces), there are reasons to expect that, if we denote by $n(d)$ the smallest natural number (or $\infty$) such that for each nonsingular variety $X$ with $\text{codeg} \, X = d$ we have $\dim X \leq n(d)$, then $n(d) < \infty$ for $d > 2$ (of course, $n(2) = \infty$). However the number $n(d)$ is not small; in the present section we show that already $n(3) = 16$. 
5. VARIETIES OF CODEGREE THREE

There are many papers devoted to surfaces of small class (codegree) and some papers devoted to threefolds (cf. [53]; a survey and bibliography can be found in [91]), but in general varieties of small codegree remain completely unexplored. In the present section we make the first step and give complete classification of nonsingular nondegenerate varieties of codegree three (it turns out that up to projective equivalence there are exactly ten such varieties).

5.2. Theorem. Let \( X^n \subset \mathbb{P}^N \) be a nonsingular irreducible nondegenerate projective variety of codegree three over an algebraically closed field \( K \) of characteristic zero. Then there are the following possibilities:

0. \( n = 3, X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \) (\( X \) is a Segre variety);

I. \( n = 2, X = \mathbb{F}_1 \subset \mathbb{P}^4 \) (\( \mathbb{F}_1 \) is a bundle with fiber \( \mathbb{P}^1 \) over \( \mathbb{P}^1 \) embedded in \( \mathbb{P}^4 \) so that its fibers and the minimal section \( s \) are projective lines and \( (s^2) = -1 \));

II. \( X \) is a Severi variety. More precisely, in this case there are the following possibilities:

II.1. \( n = 2, X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5 \) (\( X \) is the Veronese surface);

II.2. \( n = 4, X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \) (\( X \) is the Segre variety);

II.3. \( n = 8, X = G(5, 1) \subset \mathbb{P}^{14} \) (\( X \) is the Grassmann variety);

II.4. \( n = 16, X = E \subset \mathbb{P}^{26} \) (\( X \) corresponds to the orbit of highest weight vector of the nontrivial representation of the group \( E_6 \) having the smallest possible dimension);

II'. \( X \) is an isomorphic projection of one of the Severi varieties \( X^n \subset \mathbb{P}^{2n+2} \) described in II to \( \mathbb{P}^{2n+1} \), \( n = 2i, 1 < i < 4 \) (as in II, here we obtain four cases II'.1 – II'.4).

5.3. Remark. In all the above cases \( X \) is the image of \( \mathbb{P}^n \) under the rational map defined by the linear system of quadrics in \( \mathbb{P}^n \) passing through \( B \), where \( B \) has the following form:

0. \( B = \mathbb{P}^0 \coprod \mathbb{P}^1 \);

I. \( B = \mathbb{P}^0 \);

II.1. II'.1. \( B = \emptyset \);

II.2. II'.2. \( B = \mathbb{P}^1 \coprod \mathbb{P}^1 \);

II.3. II'.3. \( B = \mathbb{P}^1 \times \mathbb{P}^3 \);

II.4. II'.4. \( B = S^{10}_1 \), the spinor variety in \( \mathbb{P}^{15} \) corresponding to the orbit of highest weight vector of the spinor representation of the group \( \text{Spin}_{9} \) (or \( \text{Spin}_{10} \)).

In case 0 we have \( X^* \simeq X \simeq \mathbb{P}^1 \times \mathbb{P}^2 \); in case I \( X^* \) is the projection of \( \mathbb{P}^1 \times \mathbb{P}^2 \) from a point of \( \mathbb{P}^5 \setminus \mathbb{P}^1 \times \mathbb{P}^2 \). If \( X \) is a Severi variety, then by (2.5.4) \( X^* \simeq SX \), and in case II' the variety \( X^* \) is obtained from the corresponding Severi variety by intersecting it with a general hyperplane.

According to Theorems 4.8 and 4.9, all Severi varieties can be interpreted as ‘matrices of rank 1’ in the space of Hermitean \( 3 \times 3 \)-matrices (or as ‘Veronese surfaces’) over one of the four standard composition algebras; \( X^* \) is defined by the equation \( \det = 0 \) and therefore has degree three. In case II' \( X^* \) is defined by the same equation in the subspace of matrices with vanishing trace.
Variety \( \Pi \) is a hyperplane section of variety \( \emptyset \); these varieties have degree three. Varieties \( \Pi.1 \) and \( \Pi'.1 \) have degree four, varieties \( \Pi.2 \) and \( \Pi'.2 \) have degree six, varieties \( \Pi.3 \) and \( \Pi'.3 \) have degree 14, and varieties \( \Pi.4 \) and \( \Pi'.4 \) have degree 78 (cf. Chapter III, Proposition 2.10).

The remaining part of this section is devoted to a proof of Theorem 5.2.

5.4. Lemma. In the conditions of the theorem, let \( \Sigma^k = \text{Sing} X^* \). Then \( \mathcal{S}\Sigma \subseteq X^* \).

Proof. For \( \alpha \in \Sigma \) we have \( \text{mult}_\alpha X^* \geq 2 \). Hence if \( \alpha, \beta \in \Sigma, \alpha \neq \beta \), then the line \( \langle \alpha, \beta \rangle \) intersects \( X^* \) with multiplicity at least 4, and therefore this line lies in \( X^* \). Thus \( \mathcal{S}\Sigma \subseteq X^* \). □

5.5. Remark. Since \( \text{codim} X^* \leq \text{deg} X^* - 1 = 2 \), there are two possibilities: \( \text{codim} X^* = 1 \) and \( \text{codim} X^* = 2 \). The second case is easy to investigate since classification of varieties of degree 3 and codimension 2 is fairly simple: all nondegenerate varieties with such invariants are cones over sections of the Segre variety \( \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \) by linear subspaces of \( \mathbb{P}^5 \) (cf. [104]), and since \( X \) is nondegenerate and \( \text{codim} X > 1 \) (because otherwise \( \text{codeg} X = \text{deg} X \cdot (\text{deg} X - 1) \cdot \text{dim} X \neq 3 \), \( X \) is the Segre threefold \( \emptyset \). One can avoid reference to [104] by considering a general hyperplane section \( Y \) of a variety \( X \) with \( \text{codim} X^* = 2 \). It is easy to see that \( Y^* \subseteq \mathbb{P}(N-1)^* \) is obtained by projecting \( X^* \) from a general point of \( \mathbb{P}^N^* \). Since \( Y^* \) is a hypersurface, it suffices to classify varieties \( X \) of codegree 3 for which \( X^* \) is a hypersurface and to find out which of them are smoothly extendible, i.e. are hyperplane sections of nonsingular varieties (cf. Corollaries 5.7 and 5.10).

Thus in what follows we may assume that \( X^* \) is a hypersurface.

5.6. Lemma. Either \( \Sigma = \mathbb{P}^{N-2} \) or the hypersurface \( X^* \) is normal.

Proof. Suppose that \( X^* \) is not normal. Then from the Serre normality criterion it follows that there exists a component \( \Sigma_0 \subset \Sigma \) such that \( \dim \Sigma_0 = \dim X^* - 1 = N - 2 \), and Lemma 5.4 shows that \( \mathcal{S}\Sigma_0 \subseteq X^* \). Let \( \Lambda \) be a general plane in \( \mathbb{P}^N^* \), and put \( X^{**} = X^* \cap \Lambda, \Sigma_0 = \Sigma_0 \cap \Lambda. \) Then \( X^{**} \) is an irreducible plane cubic, \( \Sigma_0^* \) is a union of \( \deg \Sigma_0 \) distinct points and \( \mathcal{S}\Sigma_0^* \subseteq X^{**} \). Hence \( \deg \Sigma_0 = 1 \) and \( \Sigma_0 = \mathbb{P}^{N-2} \). Furthermore, \( \Sigma = \Sigma_0 \) since otherwise from Lemma 5.4 it would follow that \( X^* \) contains the hyperplane spanned by \( \Sigma_0 \) and a point from \( \Sigma \setminus \Sigma_0 \). □

5.7. Corollary. Let \( X^n \subset \mathbb{P}^N \) be a nonsingular projective variety such that \( \text{codeg} X = 3, \text{codim} X^* = 2 \), and let \( Y^{n-1} \subset \mathbb{P}^{N-1} \) be a general hyperplane section of \( X \). Then \( \text{codeg} Y = 3, \text{codim} Y^* = 1 \) and \( \text{Sing} Y^* = \mathbb{P}^{N-3} \).

Proof. It is clear that \( Y^* \) is obtained by projecting \( X^* \subset \mathbb{P}^N^* \) from a general point \( \xi \in \mathbb{P}^N^* \). Since \( SX^* = \mathbb{P}^N^* \) (to prove this it suffices to consider the section of \( X^* \) by a general three-dimensional linear subspace \( \Lambda \subset \mathbb{P}^N^* \)), the finite map \( X^* \to Y^* \) is not an isomorphism. Hence the variety \( Y^* \) cannot be normal, and from Lemma 5.6 it follows that \( \text{Sing} Y^* = \mathbb{P}^{N-3} \). □

Since \( X^* \) is a hypersurface, \( \alpha \in \text{Sm} X^* \) if and only if the hyperplane section \( L_\alpha \cap X \) has a unique nondegenerate quadratic singular point (cf. [16]). For \( x \in X \) we denote by \( \Sigma_x \subset \mathcal{P}_x \) the set of hyperplanes \( \beta \in \mathcal{P}_x \) for which the singularity of the hyperplane section \( L_\beta \cap X \) at the point \( x \) is not a nondegenerate quadratic singular
point (we identify \( P_x \) and \( \Sigma_x \) with their images in \( P^{N*} \) under the morphism \( \pi \)). Then \( \Sigma \supset \left( \bigcup_{x \in X} \Sigma_x \right) \) and the points from \( \Sigma \setminus \left( \bigcup_{x \in X} \Sigma_x \right) \) correspond to hyperplane sections having several nondegenerate quadratic singular points. In the case when \( X^* \) is normal we have \( \Sigma = \bigcup_{x \in X} \Sigma_x \).

Let \( x \in X \) be a point for which \( \Sigma_x \neq P_x \). Taking \( N-n \) points \( \alpha_0, \ldots, \alpha_{N-n-1} \) in general position in \( P_x \) and denoting by \( A_{\alpha} \), \( \alpha \in P_x \), the quadratic term in the Taylor expansion of the equation of the hyperplane section \( L_\alpha \cap X \) in some system of local coordinates in a neighborhood of the point \( x \in X \), we see that \( \Sigma_x \subset P_x \) is defined by the equation \( \det \left[ t_0A_{\alpha_0} + \cdots + t_{N-n-1}A_{\alpha_{N-n-1}} \right] = 0 \), and so for \( N \geq n + 2 \) \( \Sigma_x \) is a hypersurface of degree at most \( n \) in \( P_x = \mathbb{P}^{N-n-1} \). Since \( \Sigma \subset \mathbb{P}^{N*} \) is defined by quadratic equations (by vanishing of the partial derivatives of the equation of \( X^* \)), for all \( x \) for which \( \Sigma_x \) is distinct from \( P_x \) the subvariety \( \Sigma_x \subset P_x \) is a hypersurface of degree at most two.

For \( N \geq n + 2 \) we will distinguish between the following two main cases:

I. For all \( x \in X \) for which \( P_x \notin \Sigma \) the subvariety \( \Sigma_x \) is a hyperplane in \( P_x \);

II. For a general \( x \in X \), \( \Sigma_x \) is a quadric in \( P_x \).

For the sake of completeness this list can be supplemented by the following case which occurs if and only if \( \dim X^* < N - 1 \):

0. \( \Sigma_x = P_x \) for all \( x \in X \).

5.8. Lemma. Under the above assumptions, case I occurs iff \( \Sigma \) is a linear subspace of \( \mathbb{P}^N \), and case II occurs iff \( \Sigma \subset \mathbb{P}^N = X^* \).

Proof. If for a general point \( x \in X \) the subvariety \( \Sigma_x \) is a quadric, then \( S\Sigma_x = P_x \) and therefore \( S\Sigma = X^* \).

If \( \Sigma \) is a linear subspace, then \( \Sigma_x \subset \Sigma \cap P_x \), and if \( P_x \notin \Sigma \), then \( \Sigma_x \) is a hyperplane in \( P_x \). It remains to show that in case I \( \Sigma \) is a linear subspace. Suppose that this is not so. By Lemma 5.6 we may assume that \( X^* \) is normal so that \( \Sigma = \bigcup_{x \in X} \Sigma_x \) and for each \( x \in X \) \( \Sigma_x = \Sigma \cap P_x \) is a linear subspace of \( P_x \) (of codimension 0 or 1). Let \( \alpha, \beta \) be a generic pair of points of \( \Sigma \), and let \( l = \langle \alpha, \beta \rangle \notin \Sigma \). Consider the family of planes \( \Pi_t \subset \mathbb{P}^N \) passing through \( l \) (here \( t \) runs through an \( (N-2) \)-dimensional linear space of parameters). Then \( \Pi_t \cap X^* = C_t \), where \( C_t \) is a curve of degree three, \( C_t = l + Q_t \). By definition, \( C_t \) is singular at the points \( \alpha \) and \( \beta \), so that for all \( t \) \( Q_t \ni \alpha, \beta \). Since \( t \notin \Sigma = \text{Sing} X^* \), for a general point \( \gamma \in l \) the tangent space \( T_{X^*, \gamma} \) is a hyperplane in \( \mathbb{P}^{N*} \). Furthermore, the hyperplane \( T_{X^*, \gamma} \supset l \) is non-constant when \( \gamma \) runs through \( l \setminus \Sigma \) since otherwise we would have \( l \subset P_y \), where \( y = p(\pi^{-1}(\gamma)) = p(\pi^{-1}(l \setminus \Sigma)) \), and since \( \alpha, \beta \in \Sigma_y \) and \( \Sigma_y \) is a linear subspace of \( P_y \), \( l \subset \Sigma_y \subset \Sigma \) contrary to the choice of \( l \). Thus \( \bigcup_{\gamma \in l \setminus \Sigma} T_{X^*, \gamma} = \mathbb{P}^{N*} \) and for a general plane \( \Pi_t \supset l \) there exists a point \( \gamma \in l \setminus \Sigma \) such that \( \Pi_t \subset T_{X^*, \gamma} \), i.e. \( C_t \) is singular at the point \( \gamma \) and \( (Q_t \cap l) \ni \alpha, \beta, \gamma \). Since \( Q_t \) is a conic, from this it follows that \( Q_t \supset l \) so that \( \Pi_t \subset T_{X^*, \gamma} \) for all \( \gamma \in l \) which is clearly impossible for generic \( t \). This contradiction shows that \( l \subset \Sigma \), i.e. \( S\Sigma = \Sigma \) and \( \Sigma \) is a linear subspace of \( \mathbb{P}^{N*} \). ❑
We begin with investigating case I.

5.9. Lemma. In case I, \( X = \mathbb{P}_1 \) is a rational scroll of degree three (the image of \( \mathbb{P}^2 \) under the rational map defined by the linear system of conics passing through a fixed point of \( \mathbb{P}^2 \)).

Proof. If \( X^o \) is a hypersurface of degree \( m \), then \( \deg X^* = m(m-1)^o \) and so \( \text{cod} X \geq 2 \). From Lemma 5.8 it follows that for each point \( x \in X \) the subvariety \( \Sigma \cap P_x \) is a linear subspace of \( P_x \). Let \( x \) be a general point of \( X \). Then \( \Sigma_x = P_x \cap \Sigma = \mathbb{P}^{N-n-2} \) is a hyperplane in \( P_x \). Let \( S_X = \bigcup \Sigma_x \subset P_X \), where \( x \) runs through the set of general points of \( X \), so that \( S_X \) is a divisor in \( P_X \). Then \( \pi(S_X) = \Sigma^c \subset \Sigma \), and for a general point \( x \in X \) we have \( \Sigma_x = P_x \cap \Sigma = P_x \cap \Sigma' = \Sigma_x' \).

Let \( H \) be a hyperplane section of \( X^* \) passing through \( \Sigma \). Then \( \pi(H) = S_X + P_D \), where \( D \) is an effective divisor in \( X \) and \( P_D = p^{-1}(D) \) is a divisor in \( P_X \). For an arbitrary point \( \alpha \in \Sigma \) we denote by \( Y_\alpha \) both the preimage of \( \alpha \) in \( P_X \) and the image of this preimage in \( X \). By the projection formula, for \( \alpha \in \Sigma \setminus \Sigma' \) we have

\[
0 = (Y_\alpha \cdot \pi^*(H))_{P_X} = (Y_\alpha \cdot P_D)_{P_X} = (Y_\alpha \cdot D)_{X}.
\]

We notice that if \( k = \dim \Sigma < N - 2 \), then \( \dim Y_\alpha > 0 \) since in this case the hypersurface \( X^* \) is normal. Furthermore, if \( N \leq 2n - 2 \), then \( \text{Pic} X = \mathbb{Z} \) (cf. [54]), so that \( D \) is a positive multiple of the hyperplane section of \( X \) and the equality \((Y_\alpha \cdot D)_{X} = 0 \) is impossible.

We claim that for \( k \leq N - 5 \) \( X \) can be isomorphically projected to \( \mathbb{P}^{2n-2} \). In fact, for a generic pair of points \( x, y \in X \) we have

\[
\dim P_x \cap P_y \geq \dim \Sigma_x \cap \Sigma_y' \geq \dim \Sigma_x + \dim \Sigma_y' - \dim \Sigma \geq 2(N-n-2) - k,
\]

so that

\[
\dim \langle T_x, T_y \rangle \leq N - 2(N-n-2) + k - 1 = 2n + 3 - (N-k).
\]

By Terracini’s lemma (cf. Theorem 1.13 in Chapter II), \( X \) can be isomorphically projected to \( \mathbb{P}^{\dim\langle T_x, T_y \rangle} \). Hence for \( k \leq N - 5 \) \( X \) can be isomorphically projected to \( \mathbb{P}^{2n-2} \). We have already shown that for \( N \leq 2n - 2 \) there are no varieties of codegree three. Since codegree is stable with respect to general projections, such varieties also do not exist for \( 0 \leq k \leq N - 5 \).

It remains to consider the case when \( k \geq N - 4 \). We observe that \( \Sigma' \neq \Sigma \) since otherwise from the Bertini theorem it would follow that \( X = p(S_X) \subset L \), where \( L = (\Sigma^c)^* = \mathbb{P}^{N-k-1} \) contrary to the assumption that \( X \) is nondegenerate. It is clear that \( \pi^{-1}(\Sigma) = S_X^{N-k} \cup P_E \), where \( E \) is a subvariety of \( X \), \( P_E = p^{-1}(E) \).

Suppose that \( k = N - 3 \). Then \( \dim P_E - \dim \Sigma = \dim E - n + 2 \geq 1 \), i.e. \( \dim E = n - 1 \). When \( \alpha \) runs through the set of general points of \( \Sigma \), the varieties \( Y_\alpha \) sweep out a dense subset in \( E \), and by Bertini’s theorem \( E^{n-1} \subset L = (\Sigma)^* = \mathbb{P}^{2} \), so that \( n \leq 3 \). For \( \beta \in \Sigma' \) we have \( \dim Y_\beta \geq \dim S_X - (k - 1) = 2 \). Hence \( n = 3 \) and for \( \beta \in \Sigma' \) the hyperplane section \( \langle \beta \rangle \cdot X \) is not reduced. In particular, for a general point \( x \in X \) a general hyperplane section from \( \Sigma_x \) is not reduced at \( x \),
and from the Bertini theorem it follows that the linear subspace \((\Sigma_x)^* = \mathbb{P}^4\) is tangent to \(X\) along a surface in contradiction with the theorem on tangencies (cf. Corollary 1.8 in Chapter I). Thus the case \(k = N - 3\) is impossible.

Suppose now that \(k = N - 4\). As we already observed, in this case \(X\) can be isomorphically projected to \(\mathbb{P}^{2n-1}\), and without loss of generality we may assume that \(N = 2n - 1\). For \(\beta \in \Sigma'\) we have \(\dim Y_{\beta} \geq \dim S_X - (k - 1) = 3\), and from the theorem on tangencies it follows that \(n > 4\).

Since \(\dim \mathcal{P}_E - \dim \Sigma = \dim E - n + 3 \geq 1\), we have \(\dim E \geq n - 2\), and arguing as in the case \(k = N - 3\) we see that \(E \subset L = (\Sigma)^* = \mathbb{P}^3\). Hence the case \(\dim E = n - 1\) is impossible, and for \(\dim E = n - 2\) we have \(n = 5, N = 9\) and \(E = L = \mathbb{P}^3\). Moreover, from the above it follows that in the last case \(\Sigma'\) is a hypersurface in \(\Sigma = \mathbb{P}^3\) and \(\dim Y_{\beta} = 3\) for a general point \(\beta \in \Sigma'\).

Let \(x\) be a general point of \(X\), and let \(Y_x = \bigcup_{\beta} \bar{Y}_{\beta}\), where \(\beta\) runs through the set of general points of \(\Sigma_x\). By the theorem on tangencies, \(\dim Y_x \geq 4\). On the other hand, if \(\dim Y_x = 5\), then \(Y_x = X\) so that for a general point \(y \in X\) there exists a hyperplane \((\beta)^*\) which is tangent to \(X\) at \(x\) and \(y\). From the Terracini lemma it follows that \(X\) can be isomorphically projected to \(\mathbb{P}^3\) which was already shown to be impossible \((8 = 2n - 2)\). Thus \(\dim Y_x = 4\), and Bertini’s theorem yields the inclusion \(Y_x \subset ((\Sigma_x)^*)^6 = (T_{\Sigma_X}^5 \setminus L^3)\). Let \(y\) be a general point of \(Y_x\).

A dimension count shows that \(y\) lies on a one-dimensional family of \(Y_{\beta}\). Hence \(\Sigma_x \cap \Sigma_y = \mathcal{P}_x \cap \mathcal{P}_y = \mathbb{P}^1\), and a dimension count shows that, varying \(y \in Y_x\), we thus obtain a general line in the plane \(\Sigma_x\). From the theorem on tangencies it follows that \(\dim \bigcup_{\gamma} \bar{Y}_{\gamma} = 4\), where \(\gamma\) runs through the set of general points of the line \(\Sigma_x \cap \Sigma_y\). Thus \(\bigcup_{\gamma} \bar{Y}_{\gamma}\) coincides with both \(Y_x\) and \(Y_y\), so that \(Y_y = Y_x\) and

\[Y_x \subset \bigcap_y (T_{X,y}, L) = \bigcup_y \bar{Y}_{y}^{*},\]

where \(y\) runs through the set of general points of \(Y_x\).

Since for a general point \(y \in Y_x\) we have \(\dim \Sigma_x \cap \Sigma_y = 1\), \(\dim (\Sigma_x, \Sigma_y)^* = 5\) and \(Y_x^* \subset (\Sigma_x, \Sigma_y)^* = \mathbb{P}_x^5 \supset L^3\). Let \(x'\) be another general point of \(X\). Then \(Y_{x'} \subset \mathbb{P}_x^5 \supset L^3\) and \(\dim (\mathbb{P}_{x'}, \mathbb{P}_x^5) \leq 10 - 3 = 7\). Since \(Y_x \cap Y_{x'}\) is nonempty (these subvarieties intersect with each other on \(L\)), we have \(\dim Y_x \cap Y_{x'} \geq 8 - 5 = 3\). Thus the linear subspace \((\mathbb{P}_{x}, \mathbb{P}_{x'}^{5})\) is tangent to \(X\) along the subvariety \(Y_x \cap Y_{x'}\), which contradicts the theorem on tangencies since

\[\dim Y_x \cap Y_{x'} \geq 3 \geq 2 \geq \dim (\mathbb{P}_{x'}^{5}, \mathbb{P}_x^5) = \dim X.\]

Thus the case \(k = N - 4\) is also impossible.

It remains to consider the case \(k = N - 2\). Since \(\Sigma = \pi(\mathcal{P}_E)\), we have \(\dim \mathcal{P}_E = \dim E + (N - n - 1) > \dim \Sigma = N - 2\), and so \(\dim E = n - 1\). Arguing as in the case \(k = N - 3\), we see that \(E^{n-1} \subset L = (\Sigma)^* = \mathbb{P}^1\) and therefore \(n \leq 2\). Since for \(\beta \in \Sigma'\) we have \(\dim Y_{\beta} \geq (N - 2) - (k - 1) = 1\), \(X\) is a surface and \(E = L\) is a line on \(X\). Now the lemma follows from Proposition 3 from [96].
5.10. Corollary. Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety such that $\text{codeg} X = 3$, $\text{codim} X^* = 2$. Then $n = 3$, $N = 5$ and $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ is a Segre variety.

Proof. From Corollary 5.7 it follows that a general hyperplane section $Y$ of the variety $X$ satisfies the conditions of Lemma 5.9, and so $Y = \mathbb{F}_1$. The corollary now follows from well known results on extension of projective varieties (in this particular case it is easy to verify directly that the standard morphisms $\mathbb{F}_1 \to \mathbb{P}^1$ and $\mathbb{F}_1 \to \mathbb{P}^2$ defined by the linear systems $|F|$ and $|s + F|$ respectively extend to $X$ and define an isomorphism $X \cong \mathbb{P}^1 \times \mathbb{P}^2$). $\square$

A different proof of the corollary is given in Remark 5.5.

5.11. Remark. A close analysis of our proof of Lemma 5.9 shows that we actually used only the fact that $\Sigma = \text{Sing} X$ has a unique nondegenerate quadratic point. Thus our method allows to give classification of all varieties having this property (the list of such varieties includes all rational scrolls $\mathbb{F}_e (e \geq 0)$ of degree $e + 2$ embedded in $\mathbb{P}^N (N \leq e + 3)$ by means of a very ample linear subsystem of the linear system $|s + (e + 1)F|$, where $s \simeq \mathbb{P}^1$, $(s^2)_X = -e$ is the minimal section and $F \simeq \mathbb{P}^1$ is a fiber). Similarly, using our techniques it should be possible to classify all varieties for which $S \Sigma \subset X^*$ (cf. Lemma 5.4).

To prove the theorem it remains to consider case II. We begin with giving a lower bound for the dimension of $\Sigma$ which holds under very general assumptions.

5.12. Lemma. Let $X^n \subset \mathbb{P}^N$ be a nonsingular projective algebraic variety such that $\dim X^* = N - 1$, and let $\Sigma = \text{Sing} X^*$. Then either $X$ is a quadric and $\Sigma = \emptyset$ or $\dim \Sigma \geq n - 1$.

Proof. Since $X^*$ is a hypersurface, $\alpha \in X^* \setminus \Sigma$ if and only if the hyperplane section $(\alpha)^* \cap X$ has a unique nondegenerate quadratic point. If $X$ is a hypersurface of degree $d$, then the Gauss map $\pi: X \to X^*$ is finite, and it is clear that for $d > 2 \dim \Sigma = n - 1$ (for $d = 2$ $X$ is a quadric, $X^* \simeq X$, $\Sigma = \emptyset$).

Suppose now that $n \leq N - 2$. We have already shown that in this case there exists an irreducible subvariety $S_X \subset \mathcal{P}_X$ such that $\dim S_X = \dim \mathcal{P}_X - 1 = N - 2$, $p(S_X) = X$ and $\pi(S_X) \subset \Sigma$. Thus for an arbitrary point $\alpha \in \pi(S_X)$ we have $\dim Y_\alpha \geq \dim S_X - \dim \pi(S_X) \geq N - \dim \Sigma - 2$. On the other hand, by the theorem on tangencies $\dim Y_\alpha \leq N - n - 1$, so that $N - \dim \Sigma - 2 \leq N - n - 1$ and $\dim \Sigma \geq n - 1$. $\square$

In case II one can give an upper bound for the dimension of singular locus which is almost equal to the lower bound.

5.13. Lemma. In case II $\dim \Sigma \leq n$.

Proof. From Lemma 5.8 it follows that in case II $S \Sigma = X^*$. Hence from the Terracini lemma it follows that for a general point $\alpha \in X^*$ we have $T_{X^*, \alpha} \supset T(\Sigma_x, \Sigma)$, where $x = p(\pi^{-1}(\alpha))$, $T(\Sigma_x, \Sigma) = \bigcup_{\beta \in \Sigma_x} T_{\Sigma, \beta}$. From Theorem 1.4 in Chapter I it follows that

$$\dim S(\Sigma_x, \Sigma) = \dim \Sigma_x + \dim \Sigma + 1 = \dim \Sigma + N - n - 1.$$
On the other hand, \( S(\Sigma, X) \subset SS = X^* \) so that \( \dim \Sigma + N - n - 1 \leq N - 1 \), i.e. \( \dim \Sigma \leq n \). □

5.14. Lemma. In case II there exists a nonsingular irreducible component \( \Sigma_0 \subset \Sigma \) such that for generic point \( x \in X \) we have \( \Sigma_0 \cap P_x = \Sigma_x \) (and therefore \( p(\pi^{-1}(\Sigma_0)) = X \), \( \dim \pi^{-1}(\Sigma_0) = N - 2 \), \( SS_0 = X^* \) and \( \Sigma_0 \) is nondegenerate) and one of the following conditions holds:

\[ \text{II}. \quad \dim \Sigma_0 = n, N = \frac{3n}{2} + 2; \]
\[ \text{II'}. \quad \dim \Sigma_0 = n - 1, N = \frac{3n}{2} + 1 \]
(in particular, \( n \) is always even).

Proof. Let \( \Sigma_0 \subset \Sigma \) be a component for which \( p(\pi^{-1}(\Sigma_0)) = X \), \( \dim \pi^{-1}(\Sigma_0) = N - 2 \) (\( \Sigma_0 = \pi(SX) \), where \( SX \subset PX \) is the subvariety considered in the proof of Lemma 5.9). For a general point \( x \in X \), \( \Sigma_x \) is a quadric, \( \Sigma_0 \cap P_x \subset \Sigma_x \) and \( \dim (\Sigma_0 \cap P_x) = N - n - 2 = \dim \Sigma_x \), so that either \( \Sigma_0 \cap P_x = \Sigma_x \) or \( \Sigma_x \) is a union of two hyperplanes in \( P_x \) and \( \Sigma_0 \cap P_x \) is one of these hyperplanes. In the first case \( SS_0 = X^* \), and in the second case there exists another component \( \Sigma_0' \subset \Sigma \) such that \( p(\pi^{-1}(\Sigma_0')) = X \), \( \dim \pi^{-1}(\Sigma_0') = N - 2 \) and \( (\Sigma_0 \cup \Sigma_0') \supset \Sigma_x \). Then \( \Sigma' \cap P_x \) is the other component of \( \Sigma_x \), and, applying to \( \Sigma_0 \) and \( \Sigma_0' \) the argument given in the proof of Lemma 5.8, we see that \( \Sigma_0 \) and \( \Sigma_0' \) are linear subspaces of \( \mathbb{P}^{N^*} \). Furthermore, \( S(\Sigma_0, \Sigma_0') \subset X^* \) and so \( S(\Sigma_0, \Sigma_0') = X^* \) because \( S(\Sigma_0, \Sigma_0') \supset \bigcup_x \Sigma_x \times P_x = X^* \), where \( x \) runs through the set of generic points of \( X \). Hence in this case \( X^* \) is a hyperplane which is clearly impossible. Thus \( SS_0 = X^* \) and in particular \( \Sigma_0 \) is a nondegenerate variety.

From the Terracini lemma it follows that a general point of \( X^* = SS_0 \) lies on an exactly \( (\dim \Sigma_x) \)-dimensional family of chords of \( \Sigma_0 \), and therefore \( (2 \dim \Sigma_0 + 1) - \dim X^* = \dim \Sigma_x \), i.e. \( \dim \Sigma_0 = N - \frac{n}{2} - 2 \). Hence if \( \dim \Sigma_0 = n \), then \( N = \frac{3n}{2} + 2 \), and if \( \dim \Sigma_0 = n - 1 \), then \( N = \frac{3n}{2} + 1 \).

It remains to verify that \( \Sigma_0 \) is nonsingular. Let \( \alpha \in \Sigma_0 \) be an arbitrary point. We observe that \( \Sigma_0 \) is not a cone with vertex \( \alpha \) since otherwise the variety \( X^* = SS_0 \) would also be a cone with vertex \( \alpha \) and \( X \) would lie in the hyperplane \( (\alpha)^* \) contrary to the assumption that \( X \) is nondegenerate. Hence the cone \( S_\alpha \Sigma_0 = S(\alpha, \Sigma_0) \) has dimension \( \dim \Sigma_0 + 1 \), and for a general point \( \beta \in \Sigma_0 \) and an arbitrary point \( \gamma \in \langle \alpha, \beta \rangle \setminus \Sigma \) we have \( p(\pi^{-1}(\gamma)) \supset T_{X, \gamma} = T_{S_\alpha \Sigma_0, \gamma} \). By the Terracini lemma (cf. Proposition 1.10 a) in Chapter II), \( p(\pi^{-1}(\gamma)) \) \( \supset \supset T_{S_\alpha \Sigma_0, \alpha} \).

Let \( S_\alpha^0 \Sigma_0 \) be a dense open subset of general points of the cone \( S_\alpha \Sigma_0 \). We put \( R_\alpha = p(\pi^{-1}(S_\alpha^0 \Sigma_0)) \subset X \). Then

\[
\dim \pi^{-1}(S_\alpha^0 \Sigma_0)) = \dim S_\alpha^0 \Sigma_0 = \dim \Sigma_0 + 1,
\]
\[
\dim R_\alpha \geq \dim \Sigma_0 + 1 - (N - n - 1) = \frac{n}{2}
\]
and, as we have just shown, \( \langle R_\alpha \rangle \supset \supset T_{S_\alpha \Sigma_0, \alpha} \), i.e. \( \dim T_{S_\alpha \Sigma_0, \alpha} \leq N - 1 - \dim \langle R_\alpha \rangle \). Since \( \dim \Sigma_0 = N - \frac{n}{2} - 2 \leq \dim T_{S_\alpha \Sigma_0, \alpha} \), from this it follows that \( \dim \langle R_\alpha \rangle \leq \frac{n}{2} + 1 \), where equality holds if and only if \( \dim T_{S_\alpha \Sigma_0, \alpha} = \dim \Sigma_0 \), i.e. \( \alpha \) is a nonsingular point of \( \Sigma_0 \).
IV. SEVERI VARIETIES

We observe that $R_\alpha$ is not a linear subspace of $\mathbb{P}^N$. In fact, when $\beta$ runs through the set of general points of $\Sigma_0$ and $\gamma$ runs through the set of general points of $(\alpha, \beta)$, $x = p(\pi^{-1}(\gamma))$ by definition runs through the set of general points of $R_\alpha$, and from the Terracini lemma it follows that $(x)^* \supset T_{\Sigma_0, \beta}$. If $R_\alpha$ were a linear subspace of $\mathbb{P}^N$, from the Bertini theorem it would follow that $\Sigma_0 \subset (R_\alpha)^* = (R_\alpha)^*$ which contradicts the nondegeneracy of $\Sigma_0$.

Thus $R_\alpha \neq (R_\alpha)$ and since $\dim R_\alpha \geq \frac{n}{2}$ and $\dim (R_\alpha) \leq \frac{n}{2} + 1$ we see that $R_\alpha$ is an irreducible hypersurface in $(\alpha)^* = \mathbb{P}^{\frac{n}{2} + 1}$, $\pi^{-1}(S_0^\alpha \Sigma_0) = P_{R_\alpha}$ and $\alpha \in \text{Sm} \Sigma_0$. Since this is true for an arbitrary point $\alpha \in \Sigma_0$, $\Sigma_0$ is a nonsingular variety. \hfill $\square$

5.15. Lemma. In case II $X^n$ is a Severi variety, and in case II' $X^n$ is an isomorphic projection of a Severi variety $\tilde{X}^n$ to $\mathbb{P}^{\frac{n}{2} + 1}$ (we use the notations of Lemma 5.14).

Proof. In case II $\Sigma_0$ is a Severi variety by definition (cf. Definition 1.2). From the description of the structure of Severi varieties given in Remark 2.5 it follows that $X = (S \Sigma_0)^*$ is also a Severi variety. Furthermore, $S X = \Sigma_0^n$, $\Sigma = \Sigma_0$. This proves Lemma 5.15 in case II.

It remains to consider case II'. As in the proof of Lemma 5.9, let $S = \bigcup_{x} S_x \subset P_x$, where $x$ runs through the set of general points of $X$. From Lemma 5.14 it follows that for a general point $x \in X$ we have $\Sigma_0 \cap P_x = \Sigma_0$, so that $\pi(S) = \Sigma_0$. Since $\dim S = n + (N - n - 2) = N - 2$ and $\dim \Sigma_0 = n - 1$, for each point $\alpha \in \Sigma_0$ we have
\[
\dim Y_\alpha \geq (N - 2) - (n - 1) = N - n - 1
\]
(here as above $Y_\alpha$ denotes the varieties $\pi^{-1}(\alpha)$ and $p(\pi^{-1}(\alpha))$ which are naturally isomorphic to each other). But according to the theorem on tangencies $\dim Y_\alpha \leq N - n - 1$. Hence $\dim Y_\alpha = n - n - 1 = \frac{n}{2}$ for all $\alpha \in \Sigma_0$.

Let $\alpha \in \Sigma_0$, $y \in Y_\alpha$ be general points. Then the hyperplane $(y)^*$ is tangent to $X^*$ along $P_y$, and from Lemma 5.14 and the Terracini lemma it follows that $(y)^*$ is tangent to $\Sigma_0$ along the quadric $\Sigma_0 \cap P_y = \Sigma_0 \supseteq \alpha$. In particular, the linear subspace $(Y_\alpha)^* \subset P^N$ is tangent to $\Sigma_0$ at the point $\alpha$, and therefore $(Y_\alpha)^* \subset \left( T_{\Sigma_0, \alpha}^* \right)^{\frac{n}{2} + 1}$. If $\beta \in \Sigma_y$ is another point for which $\langle \alpha, \beta \rangle \not\subset \Sigma_y$, then it is clear that $Y_\alpha \cap Y_\beta = y \in R_\alpha$ and the intersection is transverse (so that in particular $(Y_\alpha^2)_X = 1$). For a general point $\alpha \in \Sigma_0$ we have $Y_\alpha \subset R_\alpha$, where $R_\alpha$ is the variety introduced in the proof of Lemma 5.14. Since $\dim Y_\alpha = \frac{n}{2}$ and $R_\alpha$ is an irreducible nonlinear hypersurface in $(R_\alpha) = \left( T_{\Sigma_0, \alpha}^* \right)^{\frac{n}{2} + 1}$, we conclude that $Y_\alpha = R_\alpha$.

Let $x$ be a general point of $X$. Then the hyperplane $(x)^*$ is tangent to $\Sigma_0$ along the subvariety $\Sigma_0 \subset \Sigma_0$, i.e. $T(\Sigma_0, \Sigma_0) \subset (x)^*$. From Theorem 1.4 in Chapter I it follows that $\dim S(\Sigma_0, \Sigma_0) = (\frac{n}{2} - 1) + (n - 1) + 1 = \frac{3}{2} n - 1$ and a general point $\xi \in S(\Sigma_0, \Sigma_0)$ lies on a finite number of secants joining points of $\Sigma_0$ with points of $\Sigma_0$. We have $\xi \in P_y = S \Sigma_y$ for a suitable general point $y \in X$, and by the above the intersection $\Sigma_\alpha \cap \Sigma_y = P_x \cap P_y$ reduces to a single point $\beta = \beta(x, y)$. We set $H_x = \bigcup_{\alpha} Y_\alpha^2$, where $\alpha$ runs through the set of general points of $\Sigma_\alpha^{\frac{n}{2} - 1}$.

Then $y \in H_x$ and $y$ lies on a unique variety $Y_\alpha$, $\alpha \in \Sigma_x$, viz. on $Y_\beta$. Hence $\dim H_x = (\frac{n}{2} - 1) + \frac{n}{2} = n - 1$ so that $H_x$ is a divisor on $X$. 

We observe that $H_x$ is not a hyperplane section of $X$. In fact, suppose that there exists a point $ξ ∈ P^N$ such that $H_x ⊂ (ξ)^*$. Then for a general pair of points $α, β ∈ Σ_x$ we have $(ξ)^* ⊃ (Y_α, Y_β)$, and since $Y_α · Y_β = x$, we see that

$$(ξ)^* ⊃ (T_{Y_α,x}, T_{Y_β,x}) = T_{ξ,x},$$

i.e. $ξ ∈ P_x$. Furthermore, since $(ξ)^* ⊃ (Y_α)$, we have $ξ ∈ (Y_α)^* = T_{Σ_α, Σ_α}$ so that $ξ ∈ \bigcap_{α∈Σ_x} T_{Σ_α, Σ_α} = \bigcap_{α∈Σ_x} T_{Σ_α, Σ_α}$, i.e. $Σ_x$ is a cone with vertex $ξ$. But on the other hand the $(\frac{n}{2} - 1)$-dimensional variety $Σ_x$ is a component of the variety of ‘ends’ of secants passing through a general point of $P_x$ (we recall that by Lemma 5.12 $\bigcup_{x∈X} P_x^2 = (X^*)^2 = SΣ_0$), and therefore, for a general point $x ∈ X$, $Σ_x$ is a non-singular quadric. This contradiction shows that $H_x$ cannot be hyperplane section.

We claim that the divisor $H_x$ is linearly equivalent to a hyperplane section. Since $H_x$ itself is not a hyperplane section, this would imply that the variety $X$ is not linearly normal, i.e. $X$ is obtained by projecting a non-singular variety $X^n ⊂ P^{\frac{n(n+1)}{2} + 2}$ from a point outside $SX$. By definition, $X$ is a Severi variety (in this case $Σ_0$ is a hyperplane section of the Severi variety $Sing X ≃ \tilde{X}$).

Thus to prove Lemma 5.15 in case $Π'$ it remains to verify that $H_x$ is linearly equivalent to a hyperplane section. To shorten the proof we may assume that $n > 4$ (to extend the proof given below to the cases $n = 2$ and $n = 4$ one needs some additional arguments). In fact, Proposition 3 from [96] shows that the case $n = 2$ (in a different guise) is actually considered in the main theorem of [96]. The case $n = 4$ (i.e. $dim Σ_0 = 3$) is dealt with in [24]. For $n > 4$ from [54] it follows that Pic $X = Z$. Hence $O_X(H_x) ≃ O_X(m)$, where $m$ is a natural number. We claim that $m = 1$. To prove this it suffices to verify that, for a general point $α ∈ Σ_0$, $H_x ∩ Y_α$ is a hyperplane section of the hypersurface $Y_α^2 ⊂ (T_{Σ_α, Σ_0})^*$. Consider the map $\pi|_{P_{Σ_x} ∩ P_{Y_α}}: P_{Σ_x} ∩ P_{Y_α} → Σ_x$, where $P_{Σ_x} = \pi^{-1}(Σ_x) ⊂ P_X$, $P_{Y_α} = \pi^{-1}(Y_α) ⊂ P_X$.

The fiber of this map over a point $β ∈ Σ_x$ is naturally isomorphic to $Y_α ∩ Y_β$, and if the intersection is not transverse, then $(α, β) ⊂ Σ_0$ and in particular $α ∈ T_{Σ_0, β}$. Hence for $α ∉ (x)^* ⊃ T(Σ_x, Σ_0)$ the map $\pi|_{P_{Σ_x} ∩ P_{Y_α}}$ is an isomorphism, and the variety $P_{Σ_x} ∩ P_{Y_α}$ is naturally isomorphic to the $(\frac{n}{2} - 1)$-dimensional quadric $Σ_x$.

By definition, $p|_{P_{Σ_x} ∩ P_{Y_α}} = H_x ∩ Y_α$ and the fiber over a point $y ∈ H_x ∩ Y_α$ is naturally isomorphic to the linear subspace $Σ_x ∩ Σ_y = P_x ∩ P_y$, so that the morphism $p|_{P_{Σ_x} ∩ P_{Y_α}}$ is birational. Now it is easy to show that the morphism

$$ \left( p|_{P_{Σ_x} ∩ P_{Y_α}} \right)^{-1} \circ (\pi|_{P_{Σ_x} ∩ P_{Y_α}}) : Σ_x → H_x ∩ Y_α$$

is an isomorphism. Since, as it was shown above, $Y_α$ is not a linear subspace, from this it follows that $Y_α$ is a quadric and $H_x ∩ Y_α ≃ Σ_x$ is a hyperplane section of $Y_α$, i.e. $m = 1$ and $H_x ∈ |O_X(1)|$. This completes the proof of Lemma 5.15 and Theorem 5.2.

5.16. Remark. Let $X^n ⊂ P^N$ be a nondegenerate nonsingular variety which can be isomorphically projected to $P^{N−1}$. From Corollary 2.11 in Chapter II and Definition 1.2 it follows that if $n = 2k$ is even, then $s = dim SX > 3k + 1$, and
equality holds if and only if $X$ is a Severi variety. Classification of Severi varieties is given in Theorem 4.7. If $n = 2k - 1$ is odd, then from Corollary 2.11 in Chapter II it follows that $s > 3k$. In this case it also seems interesting to classify extremal varieties for which $s = 3k$. Such classification can be accomplished by using methods of the present chapter, but the corresponding paper is not yet written. Here we only give the expected answer (cf. also Remark 2.7 in Chapter V). Since each nondegenerate curve in $P^N, N > 4$ is an extremal variety, we may assume that $n > 3$. Furthermore, for $N = s + 3, X = v_2(P^3) \subset P^S$ is the Veronese threefold, and for $N = s + 2$ there are three possibilities: $X^3$ is an isomorphic projection of the Veronese threefold $v_2(P^3)$ to $P^8$, $X^3$ is the image of $P^3$ with respect to the rational map defined by the linear system of quadrics passing through a point (i.e. $X$ is obtained from $v_2(P^3)$ by projecting it from a point lying in $v_2(P^3)$; in this case $X$ is isomorphic to the blow up of $P^3$ with center at a point), $X^5 = P^2 \times P^3 \subset P^{11}$ is a Segre variety. Finally (and this is the only result whose proof is not written), for $N = s + 1$ $X$ is either an isomorphic projection of one of the varieties listed above or a hyperplane section of one of the Severi varieties $X^{2k}, k \geq 2$. The proof should be essentially parallel to the proof of classification theorem for Severi varieties given in the present chapter; in particular, one can use the fact that nonsingular hyperplane sections are projectivizations of orbits of highest weight vectors of irreducible representations of algebraic groups (viz. hyperplane sections of $P^2 \times P^2$ correspond to the adjoint representation of $SL_3$, hyperplane sections of the Grassmann variety $G(5, 1)$ of lines in $P^5$ correspond to the second fundamental representation of $Sp_6$, and hyperplane sections of the variety $E_{16} \subset P^{26}$ correspond to the nontrivial representation of lowest possible dimension of $F_4$). In the proof of Lemma 5.15 in case II’ it would be natural to refer to classification of odd-dimensional extremal varieties as we referred to classification of Severi varieties in case II. However, for lack of suitable reference, we argued indirectly by using additional information available in our special case (in particular, in the notations of this remark, we know in advance that the variety $(P^{X^*})^{n+1}$ is nonsingular).
CHAPTER V

LINEAR SYSTEMS OF HYPERPLANE SECTIONS
ON VARIETIES OF SMALL CODIMENSION
1. Higher secant varieties

Let \( X^n \subset \mathbb{P}^N \) be a nondegenerate projective variety over an algebraically closed field \( K \). Put

\[
(S_X^k)^0 = \{x_0, \ldots, x_k; u \in X \times \cdots \times X \times \mathbb{P}^N \mid \dim \langle x_0, \ldots, x_k \rangle = k, \ u \in \langle x_0, \ldots, x_k \rangle \},
\]

and let \( S_X^k \) be the closure of \((S_X^k)^0\) in \( X \times \cdots \times X \times \mathbb{P}^N \). We denote by \( \varphi^k \) (or, if there is no ambiguity, simply by \( \varphi \)) the projection of \( S_X^k \) to \( \mathbb{P}^N \) and by \( p_i^k \) (or simply by \( p_i \)) the projection of \( S_X^k \) onto the \( i \)-th factor of \( X \times \cdots \times X \) \((i = 0, \ldots, k)\).

1.1. Definition. The variety \( S^k X = \varphi(S_X^k) \) is called the variety of \( k \)-secants of the variety \( X \).

Thus \( S^k X \) is the closure of the set of points lying in \( k \)-dimensional linear subspaces spanned by general collections of \( k+1 \) points in \( X \). In particular, \( S^0 X = X \) and \( S^1 X = SX \) is the usual secant variety (cf. §1 of Chapter I).

1.2. It is clear that all \( S^k X \), \( 0 \leq k \leq k_0 \) are irreducible projective varieties and

\[
X \subset S^1 X \subset S^2 X \subset \cdots \subset S^k X \subset \cdots \subset S^{k_0-1} X \subset S^{k_0} X = \mathbb{P}^N,
\]

where

\[
k_0 = \min \{k \mid S^k X = \mathbb{P}^N\}.
\]

The variety \( S^k X \) can also be constructed inductively as follows. Let \( a_0 \leq \cdots \leq a_r \) be a collection of nonnegative integers such that \( a_0 + \cdots + a_r = k - r \), let

\[
S_{S^{a_0} X \ldots, a_r X}^0 = \{(v_0, \ldots, v_r; u) \in S^{a_0} X \times \cdots \times S^{a_r} X \times \mathbb{P}^N \mid \dim \langle v_0, \ldots, v_r \rangle = r, \ u \in \langle v_0, \ldots, v_r \rangle \},
\]

and let \( S_{S^{a_0} X \ldots, a_r X} \) be the closure of \( S_{S^{a_0} X \ldots, a_r X}^0 \) in \( S^{a_0} X \times \cdots \times S^{a_r} X \times \mathbb{P}^N \subset \mathbb{P}^N \times \cdots \times \mathbb{P}^N \). In this case we also denote by \( \varphi \) (or, to avoid ambiguity, by \( \varphi^{a_0 \ldots a_r} \)) the projection map from \( S_{S^{a_0} X \ldots, a_r X} \) to \( \mathbb{P}^N \) and by \( p_i^{a_0 \ldots a_r} \) the projection map from \( S_{S^{a_0} X \ldots, a_r X} \) to \( S^{a_i} X \). It is not hard to see that

\[
S^k X = \varphi^{a_0 \cdots a_r} (S_{S^{a_0} X \ldots, a_r X}).
\]

Definition 1.1 is a special case of (1.2.3) for \( r = k, \ a_0 = \cdots = a_r = 0 \), so that

\[
S_X^k = S_{X, \ldots, X}^{k+1}.
\]

We shall often use another special case of (1.2.3), viz. \( r = 1 \). In this case from (1.2.3) it follows that, in the notations of §1 of Chapter I,

\[
S^k X = S(S^{a_0} X, S^{a_1} X), \quad a_0 + a_1 = k - 1.
\]

In particular, for \( a_0 = 0 \) we obtain the following inductive formula:

\[
S^k X = S(X, S^{k-1} X).
\]
1.3. Proposition. All inclusions in (1.2.1) are strict. In other words, $S^{k-1}X \neq S^kX$ for $1 \leq k \leq k_0$.

Proof. Suppose that $S^{k-1}X = S^kX$. Then from (1.2.5) it follows that for each point $x \in X$ the variety $S^{k-1}X$ is a cone with vertex $x$. But this is possible only if $S^{k-1}X = \mathbb{P}^N$ contrary to (1.2.2). □

1.4. Proposition. Let $v_0 \in S^{a_0}X, \ldots, v_r \in S^{a_r}X$, $\dim \langle v_0, \ldots, v_r \rangle = r$, $u \in \langle v_0, \ldots, v_r \rangle \subset S^kX$, where $k = a_0 + \cdots + a_r + r$, and let $L_u = T_{S^kX,u}$. Then

a) $L_u \supset \langle T_{S^{a_0}X,v_0}, \ldots, T_{S^{a_r}X,v_r} \rangle$;

b) if $\text{char } K = 0$, $v_0, \ldots, v_r$ is a generic collection of points of $X$, and $u$ is a generic point of the linear subspace $\langle v_0, \ldots, v_r \rangle$, then

$L_u = \langle T_{S^{a_0}X,v_0}, \ldots, T_{S^{a_r}X,v_r} \rangle$.

Proof. The proof is by induction on $r$. We use the representation $S^kX = \varphi(S_{S^{a_0}X,S^{a_1}X,\ldots,S^{a_r}X,v_r})$ (cf. (1.2.4)). Then

$u \in \langle v_0, v \rangle, \quad v \in \langle v_1, \ldots, v_r \rangle$,

and it suffices to apply Proposition 1.10 from Chapter II to the subvariety $S^{a_0}X \subset S^{a_1+a_2+\cdots+a_r+r-1}X$ and the points $u, v_0, and v$. □

1.5. From now on we shall assume that the variety $X$ is nonsingular (as in Chapter II, similar results can be proved for singular varieties, but we will not need them). Let $u \in S^kX$. We put

$Y_u = p^k_0((\varphi^k)^{-1}(u))$. (1.5.1)

From Proposition 1.4 a) it follows that the linear subspace $L_u = T_{S^kX,u}$ is tangent to $X$ along the subvariety $Y_u \subset X$. The dimension of $Y_u$ for a generic point $u \in S^kX$ is a projective invariant of the variety $X$; we put

$\delta_k = \dim Y_u$. (1.5.2)

It is easy to compute this invariant in terms of the dimensions of higher secant varieties. To wit, let

$s_k = \dim S^kX$ (in particular, $s_0 = n$, $s_1 = s = \dim SX$). We use the representation $S^kX = \varphi(S_X,S^{k-1}X)$ (cf. (1.2.5)). Then

$Y_u = p^0_{0,k-1}((\varphi^{0,k-1})^{-1}(u))$, and if $1 \leq k \leq k_0$, (so that by Proposition 1.3 $S^{k-1}X \neq S^kX$), then

$\delta_k = \dim Y_u = \dim ((\varphi^{0,k-1})^{-1}(u))$

$= \dim S_X,S^{k-1}X - \dim S^kX = s_{k-1} + n + 1 - s_k$ (1.5.4)
(in particular \( \delta_1 = 2n + 1 - s \); for the sake of brevity we shall denote \( \delta_1 \) simply by \( \delta \)).

From Proposition 1.4 a) it follows that for \( k < k_0 \)

\[
\delta_k < n; \tag{1.5.5}
\]

it is clear that

\[
\delta_{k_0} \leq n, \tag{1.5.6}
\]

and in view of (1.5.4) equality in (1.5.6) holds if and only if

\[
s_{k_0 - 1} = N - 1; \tag{1.5.7}
\]

finally, \( s_k = N \) for \( k \geq k_0 \), and \( \delta_k = n \) for \( k > k_0 \).

Summing up the equalities (1.5.4) for all \( 1 \leq k' \leq k \), we see that for \( k \leq k_0 \)

\[
s_k = (k + 1)(n + 1) - \sum_{i=1}^{k} \delta_i - 1 = \sum_{i=0}^{k} (n - \delta_i + 1) - 1 \tag{1.5.8}
\]

(we recall (cf. (1.5.1), (1.5.2)) that

\[
\delta_0 = \dim Y_x = \dim x = 0, \tag{1.5.9}
\]

where \( x \in X \) is a generic point).

1.6. Example. Let \( n = 1 \), so that \( X \) is a curve. From (1.5.5) it follows that \( \delta_k = 0 \) for \( k < k_0 \). Formula (1.5.8) shows that

\[
s_k = \min \{2k + 1, N\}.
\]

Thus the dimensions of higher secant varieties of a curve do not depend on its properties. Below we shall see that for varieties of higher dimensions this is far from being so.

1.7. Proposition. \( 0 = \delta_0 \leq \delta_1 = \delta \leq \delta_2 \leq \cdots \leq \delta_{k_0 - 1} \leq \delta_{k_0} \leq n - \delta \), and \( \delta_k = n \) for \( k > k_0 \).

Proof. Let \( k \leq k_0 \). Consider the rational maps

\[
\xi: S_{X,X,S^{k-2}X} \rightarrow S_{X,S^{k-2}X}, \quad \xi(x', x, w, u) = (x', w, \langle x', w \rangle \cap \langle x, u \rangle)
\]

and

\[
\eta: S_{X,X,S^{k-2}X} \rightarrow S_{X,S^{k-2}X}, \quad \eta(x', x, w, u) = (x, \langle x', w \rangle \cap \langle x, u \rangle, u).
\]

It is clear that \( \xi \) and \( \eta \) are defined off \( (\varphi^{0,0,k-2})^{-1}(X) \). Furthermore,

\[
\varphi^{0,k-2} \circ \xi = \varphi^{0,0,k-2}, \quad \varphi^{0,k-1} \circ \eta = \varphi^{0,0,k-2}.
\]
Let $x$ be a general point of $X$, let $v$ be a general point of $S^{k-1}X$, and let $u \in \langle x, v \rangle$ be a general point of $S^kX$. Then
\[
\phi^{0,0,k-2}(u) = \eta^{-1}((\phi^{0,0,k-1}(u))) \supset \eta^{-1}(x, v, u),
\]
\[
\xi(\eta^{-1}(x, v, u)) = (\phi^{0,k-2})^{-1}(v),
\]
\[
Y_u = p_0^{0,0,k-2}((\phi^{0,0,k-2})^{-1}(u)) \supset p_0^{0,0,k-2}(\eta^{-1}(x, v, u))
\]
\[
= p_0^{k-2}(\xi(\eta^{-1}(x, v, u))) = p_0^{0,k-2}((\phi^{0,k-2})^{-1}(v)) = Y_v.
\]
Hence
\[
\delta_k = \dim Y_u \geq \dim Y_v = \delta_{k-1},
\]
i.e. the numbers $\delta_k$ form an increasing sequence.

It remains to show that if $S^kX \neq \mathbb{P}^N$, then $\delta_k \leq n - \delta$. Let $u$ be a generic point of $S^kX$, and let $L_u = T_{S^kX,u}$. From Proposition 1.4 a) it follows that in the notations of Section 1 of Chapter I $T(Y_u, X) \subset L_u$. Since $X$ is a nondegenerate variety,
\[
T(Y_u, X) \neq S(Y_u, X) \supset X.
\]
Hence Theorem 1.4 from Chapter I yields:
\[
s = \dim SX \geq \dim S(Y_u, X) = \dim Y_u + \dim X + 1 = \delta_k + n + 1,
\]
i.e.
\[
\delta_k \leq s - n - 1 = n - \delta \quad (1.7.1)
\]
as required. □

The following theorem improves the monotonicity result of Proposition 1.7.

1.8. THEOREM. For $1 \leq k \leq k_0$ we have $\delta_k \geq \delta_{k-1} + \delta$.

PROOF. Consider the commutative diagram of rational maps
\[
(1.8.1)
\]
where for generic points $x \in X, v_{k-2} \in S^{k-2}X, v_0 \in X, u \in \langle x, v_{k-2}, v_0 \rangle \subset S^kX$ the map $\lambda$ is defined by the formula
\[
\lambda(x, v_{k-2}, v_0, u) = (v_{k-1}, v_0, u),
\]
where
\[
v_{k-1} = \langle x, v_{k-2} \rangle \cap \langle v_0, u \rangle
\]
and $\mu$ is defined by the formula
\[
\mu(x, v_{k-2}, v_0, u) = (v_{k-2}, v_1, u),
\]
where
\[ v_1 = \langle x, v_0 \rangle \cap (v_{k-2}, u) \]
(cf. fig. 1).

It is clear that
\[ \varphi^{k-1,0} (\lambda^{-1}(v_{k-2}, v_1, u)) = u. \]

Hence
\[
\begin{align*}
\lambda^{-1} (\lambda^{-1}(v_{k-2}, v_1, u)) & \subset \varphi^{0,k-2,0} (u), \\
p_0^{0,k-2,0} (\lambda^{-1}(\lambda(v_{k-2}, v_1, u))) & \subset p_0^{0,k-2,0} ((\varphi^{0,k-2,0})^{-1}(u)) = Y_u
\end{align*}
\]
and
\[ \delta_k = \dim Y_u \geq \dim p_0^{0,k-2,0} (\lambda^{-1}(\mu^{-1}(v_{k-2}, v_1, u))). \quad (1.8.2) \]

It is clear that
\[ \dim \lambda^{-1}(v_{k-2}, v_1, u) = \dim \mu^{-1}(v_{k-2}, v_1, u) = \dim ((\varphi^{0,0})^{-1}(v_1)) = \delta_1. \quad (1.8.3) \]

On the other hand, for a generic point \((v_{k-1}, v_0, u) \in S_{S^*-1}.X_.X \) we have
\[ \dim \lambda^{-1}(v_{k-1}, v_0, u) = \dim ((\varphi^{0,k-2})^{-1}(v_{k-1})) = \delta_{k-1}. \quad (1.8.4) \]

From (1.8.3) and (1.8.4) it follows that
\[ \dim \lambda^{-1}(\lambda^{-1}(v_{k-2}, v_1, u)) \geq \delta_{k-1} + \delta_1. \quad (1.8.5) \]

From (1.8.2) and (1.8.5) it is clear that in order to prove Theorem 1.8 it suffices to verify that the map \( p_0^{0,k-2,0} \) is finite at a generic point of the variety \( \lambda^{-1}(\mu^{-1}(v_{k-2}, v_1, u)) \). Let \( y \in p_0^{0,k-2,0} (\lambda^{-1}(\mu^{-1}(v_{k-2}, v_1, u))) \) be a generic point. The preimage of \( y \) in \( \lambda^{-1}(\mu^{-1}(v_{k-2}, v_1, u)) \) consists of quadruples of the form \((y, v_{k-2}', v_0, u)\), where
\[ (y, v_{k-2}') \ni v_{k-1}, \quad v_{k-1} = \langle x, v_{k-2} \rangle \cap (v_0, u), \quad \langle x, v_0 \rangle \ni v_1 \]
Hence it remains to show that the subvarieties
\[ Y_{v_k-1} = p_{1}^{k-2.0}((\varphi^{k-2.0})^{-1}(v_{k-1})), \quad u \in \langle y, v_{k-1} \rangle \]
and
\[ Y_{v_1} = p_{1}^{0.0}((\varphi^{0.0})^{-1}) \]
intersect in a finite number of points. This immediately follows from the general observation that if \( v_{k-1} \) is a generic point of \( S^{k-1} X \), then \( \dim S(Y_{v_{k-1}}, X) = \dim Y_{v_{k-1}} + n + 1 \). To prove this equality we argue as in the proof of the last assertion of Proposition 1.7. From Proposition 1.4 a) it follows that \( T(Y_{v_{k-1}}, X) \subset T_{S^{k-1} X, v_{k-1}} \neq \mathbb{P}^N \) is a proper linear subspace of \( \mathbb{P}^N \). Since \( X \) is nondegenerate, we have \( S(Y_{v_{k-1}}, X) \neq T(Y_{v_{k-1}}, X) \), and our claim follows from Theorem 1.4 in Chapter I. □

1.9. Corollary. For \( 0 \leq k \leq k_0 \) we have \( \delta_k \geq k \delta \).

1.10. Remark. In the case when \( \text{char} \ K = 0 \) Theorem 1.8 can be proved using Proposition 1.4 b) for \( k = r, a_0 = \cdots = a_r = 0 \) and general position arguments. A proof of this type was independently discovered by A. Holme and J. Roberts (cf. §5 in [40]).

1.11 Remark. It would be interesting to find out under what conditions Theorem 1.8 can be generalized as announced in [100]: The function \( \delta \) is superadditive on the interval \([0, k_0]\), i.e. if \( k = k_1 + \cdots + k_r \) is a partition of \( k \), then \( \delta_k \geq \delta_{k_1} + \cdots + \delta_{k_r} \). There is a gap in the proof of this statement given in [100], and Adlansvik [103] observed that counterexamples are given by Dale’s surfaces [14] (cf. also [40]). Some approaches to this problem are discussed in [40] and also in [22], where it is shown that superadditivity holds under a rather restrictive assumption which is apparently hard to verify, viz. that the higher secant varieties of \( X \) be almost smooth, i.e. for all \( z \in S^k X \) \( T_{S^k X, z}^z \subset S(z, S^k X) \). However, the problem still remains open. It seems plausible that superadditivity holds for varieties with \( \delta > 0 \) (for Dale’s surfaces \( \delta = 0 \)). Moreover, it might be that quite generally Dale’s surfaces yield the only possible exceptions to superadditivity.
1.12. Theorem. $k_0 \leq \lfloor \frac{n}{3} \rfloor$, i.e. $S^\lfloor \frac{n}{3} \rfloor X = \mathbb{P}^N$ (for $\delta = 0$ the assertion of the theorem is empty).

Proof. From the definition of $k_0$ (cf. (1.2.2)) it follows that for $a \geq k_0$ we have $S^a X = \mathbb{P}^N$, so that to prove Theorem 1.12 it suffices to verify that $\frac{a}{3} \geq k_0$. From Theorem 1.8 it follows that
\[ \delta_{k_0} \geq k_0 \delta, \]  \hspace{1cm} (1.12.1)
and (1.5.6) yields
\[ \delta_{k_0} \leq n. \]  \hspace{1cm} (1.12.2)
Combining (1.12.1) and (1.12.2), we see that $k_0 \delta \leq n$ (one could also use Theorem 1.8 for $k = k_0 - 1$ and the inequality (1.7.1) proved in Proposition 1.7). \[ \Box \]

Theorem 1.12 allows to give a simple proof of the Hartshorne conjecture on linear normality (cf. Corollary 2.11 in Chapter II).

1.13. Corollary. If $S^\delta X \neq \mathbb{P}^N$, then $\delta \leq \frac{n}{2}$ and $n \leq \frac{2}{3}(N - 2)$.

Proof. In fact, for $\delta > \frac{n}{2}$ we have $\lfloor \frac{n}{3} \rfloor = 1$, and from Theorem 1.12 it follows that $S^1 X = \mathbb{P}^N$. Furthermore, if $S^\delta X \neq \mathbb{P}^N$, then
\[ N \geq s + 1 = 2n + 2 - \delta \geq \frac{3n}{2} + 2, \]
i.e. $n \leq \frac{2}{3}(N - 2)$. \[ \Box \]
2. Maximal embeddings of varieties of small codimension

Let $X$ be a nonsingular projective variety over an algebraically closed field $K$. Projections yield a partial order on the set of all nondegenerate embeddings of $X$ in projective spaces. Of special interest are maximal and minimal embeddings with respect to this order.

We denote by $M(n, \delta)$ (resp. $m(n, \delta)$) the maximal (resp. minimal) integer $N$ for which there exists a nonsingular nondegenerate variety $X \subset \mathbb{P}^N$ such that

$$\dim X = n, \quad \delta(X) = \delta$$

(in case there is no finite maximum, we set $M(n, \delta) = \infty$). Ruling out the obvious case $X = \mathbb{P}^n$, we see that the functions $m$ and $M$ are defined on the set of all pairs $(n, \delta) \in \mathbb{Z}^2$ for which $0 \leq \delta \leq n$ and $M(n, 0) = \infty$.

2.1. Example. Suppose that $\delta > \frac{n}{2}$. Then

$$M(n, \delta) = m(n, \delta) = 2n + 1 - \delta.$$ 

In fact, by Corollary 1.13 (or Corollary 2.11 from Chapter II), for $\delta > \frac{n}{2}$

$$SX = \mathbb{P}^N = \langle X \rangle$$

and

$$N = s = 2n + 1 - \delta = M(n, \delta) = m(n, \delta).$$

2.2. Proposition.

(i) $m(n, \delta) = 2n + 1 - \delta$;

(ii) $M(n, \delta - 1) \geq M(n, \delta) + 1$;

(iii) $M(n - 1, \delta - 1) \geq M(n, \delta) - 1$

(we assume that all pairs of integers involved in the statement of the proposition lie in the domain of definition of the functions $m$ and $M$).

Proof. (i) In fact, for each $X$ with $\dim X = n$, $\delta(X) = \delta$ we have

$$m(n, \delta) \geq s_X = 2n + 1 - \delta. \quad (2.2.1)$$

Taking $X$ to be the intersection of $n + 1 - \delta$ generic hypersurfaces

$$H_i \subset \mathbb{P}^{2n+1-\delta}, \quad \deg H_i > 1, \quad i = 1, \ldots, n + 1 - \delta,$$

we see that

$$m(n, \delta) \leq 2n + 1 - \delta. \quad (2.2.2)$$

Combining (2.2.1) and (2.2.2) we see that (i) holds.

(ii) Let $X \subset \mathbb{P}^{M(n, \delta)}$ be a nonsingular nondegenerate variety for which $\dim X = n$, $\delta(X) = \delta$, and let $u \in \mathbb{P}^{M(n, \delta)+1}$ be a generic point. Consider the projective cone $S(u, X) \subset \mathbb{P}^{M(n, \delta)+1}$ with vertex $u$, and let

$$X' = S(u, X)^{n+1} \cap H^{M(n, \delta)},$$
where \( H \subset \mathbb{P}^{M(n, \delta) + 1}, \ \deg H > 1 \)

is a general hypersurface. Then \( X' \) is a nonsingular nondegenerate variety, \( n' = \dim X' = \dim X = n \), and it is easy to see that for \( \delta > 0 \)

\[
S X' = S(u, SX), \quad s' = \dim S X' = \dim SX + 1 = s + 1,
\]

and

\[
\delta' = \delta(X') = 2n' + 1 - s' = 2n - s = \delta - 1,
\]

which proves (ii).

(iii) Let \( X \subset \mathbb{P}^{M(n, \delta)} \) be a nonsingular nondegenerate variety for which \( \dim X = n, \ \delta(X) = \delta \), and let

\[
X' = X \cap H \subset \mathbb{P}^{M(n, \delta) - 1}
\]

be the intersection of \( X \) with a generic hyperplane \( H \subset \mathbb{P}^{M(n, \delta)} \). Then \( X' \) is a nonsingular nondegenerate variety, \( n' = \dim X' = \dim X - 1 = n - 1 \), and it is not hard to see that for \( \delta > 0 \)

\[
S X' = S X \cap H, \quad s' = \dim S X' = \dim SX - 1 = s - 1
\]

and

\[
\delta' = \delta(X') = 2n' + 1 - s' = 2n - s = \delta - 1.
\]

Assertion (iii) and Proposition 2.2 are proved. \( \square \)

2.3. Theorem. \( M(n, \delta) \leq f \left( \left[ \frac{n}{\delta} \right] \right) = \frac{n(n + \delta + 2)}{28} + \frac{1}{2} \left[ \frac{n}{\delta} \right] \left( \delta - \delta \left[ \frac{n}{\delta} \right] - 2 \right) \)

\( \left( \mod \delta \right), \left[ \frac{n}{\delta} \right] \) is the largest integer not exceeding \( \frac{n}{\delta} \), and \( \left\{ \frac{n}{\delta} \right\} = \frac{n}{\delta} - \left[ \frac{n}{\delta} \right] \).

Proof. Let \( X \subset \mathbb{P}^{M(n, \delta)} \) be a nonsingular nondegenerate variety, \( \dim X = n, \ \delta(X) = \delta \). From (1.5.8) and Corollary 1.9 it follows that for \( k \leq k_0 \)

\[
s_k = (k + 1)(n + 1) - \sum_{i=1}^{k} \delta_i - 1 \leq (k + 1)(n + 1) - \sum_{i=1}^{k} i \delta - 1 = f(k). \quad (2.3.1)
\]

The graph of the function \( f(k) \) is a parabola (cf. fig. 3); \( f \) attains maximal value (equal to \( \frac{(2n + \delta + 2)^2}{28} - 1 \)) for \( k = \frac{2n - \delta + 2}{28} = a \), and \( f \) is a monotone increasing function for \( 0 \leq k \leq a \).

FIG. 3
From (1.2.2) and (2.3.1) it follows that

\[ M(n, \delta) = s_{k_0} \leq f(k_0), \quad (2.3.2) \]

and from Theorem 1.12 it follows that \( k_0 \leq \left\lceil \frac{n}{\delta} \right\rceil \). If \( k_0 < \left\lceil \frac{n}{\delta} \right\rceil \), then

\[ k_0 \leq \left\lceil \frac{n}{\delta} \right\rceil - 1 \leq \frac{n}{\delta} - 1 < a \]

and

\[ f(k_0) \leq f \left( \frac{n}{\delta} - 1 \right) = f \left( \frac{n}{\delta} \right) - 1 < f \left( \left\lceil \frac{n}{\delta} \right\rceil \right) \quad (2.3.3) \]

(we observe that from (2.3.1) it follows that \( f \left( \left\lceil \frac{n}{\delta} \right\rceil \right) \) is an integer). From (2.3.2) and (2.3.3) it follows that we always have

\[ M(n, \delta) \leq f \left( \left\lceil \frac{n}{\delta} \right\rceil \right), \]

and to prove Theorem 2.3 it remains to explicitly compute \( f \left( \left\lceil \frac{n}{\delta} \right\rceil \right) \). □

2.4. REMARK. Proposition 2.2 (i) and Theorem 2.3 show that for \( \delta(X) = \delta > 0 \) the dimension \( N \) of the linear span of \( X^n \) \((n \geq \delta - 1)\) lies in the limits depicted in fig. 4.
2.5. Remark. If $\delta > \frac{n}{2}$, then $[\frac{n}{\delta}] = 1$, $f\left(\left[\frac{n}{\delta}\right]\right) = 2n + 1 - \delta$, and Theorem 2.3 shows that

$$M(n, \delta) = m(n, \delta) = 2n + 1 - \delta \leq \frac{3n + 1}{2}$$

(cf. also Corollary 1.13).

2.6. Remark. For $\delta = \frac{n}{2}$ ($n \equiv 0 \pmod{2}$) Theorem 2.3 yields:

$$N \leq f\left(\left[\frac{n}{\delta}\right]\right) = f(2) = \frac{3n}{2} + 2.$$

Here there are two possibilities:

(i) $SX = \mathbb{P}^N, N = s = m\left(n, \frac{n}{2}\right) = \frac{3n}{2} + 1$;

(ii) $SX \neq \mathbb{P}^N, N = s + 1 = M\left(n, \frac{n}{2}\right) = f(2) = \frac{3n}{2} + 2$.

Nonsingular nondegenerate varieties $X^n \subset \mathbb{P}^N$ for which $N > s = \frac{3n}{2} + 1$ are called Severi varieties (cf. Definition 1.2 in Chapter IV). Thus (ii) means that each Severi variety lies in a $(\frac{3n}{2} + 2)$-dimensional projective space. Of course, this also follows from classification of Severi varieties (cf. Theorem 4.7 in Chapter IV).

2.7. Remark. From Remark 2.5 it follows that if

$$SX^n \neq \mathbb{P}^N, \quad n \equiv 1 \pmod{2},$$

then

$$\delta \leq \frac{n - 1}{2}, \quad s = 2n + 1 - \delta \geq \frac{3n + 3}{2}.$$

Consider the extremal case when $\delta = \frac{n - 1}{2}$. By Theorem 2.3, for $n = 3$ ($\delta = 1$) we have

$$N \leq \frac{n(n + 3)}{2} = 9 = s + 3,$$

and for $n > 3$

$$N \leq \frac{3n(n + 1)}{2(n - 1)} + \frac{1}{2} \cdot \frac{2}{n - 1} \left(\frac{n - 1}{2} - \frac{n - 1}{2} \cdot \frac{2}{n - 1} - 2\right) = \frac{3n + 7}{2} = s + 2.$$

Thus if $n \equiv 1 \pmod{2}$, $\delta = \frac{n - 1}{2}$, then there are the following possibilities:

(i) $SX = \mathbb{P}^N, N = s = m\left(n, \frac{n - 1}{2}\right) = \frac{3n + 3}{2}$;

(ii) $N = s + 1 = \frac{3n + 5}{2}$;

(iii) $N = s + 2 = \frac{3n + 7}{2}$ ($= M\left(n, \frac{n - 1}{2}\right)$ if $n > 3$);

(iv) $n = 3, N = s + 3 = M(3, 1) = 9$.

All these cases really occur: as an example of case (ii) one can take a nonsingular hyperplane section of any of the Severi varieties; an example of case (iii) is given by the five-dimensional Segre variety $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^5$, and an example of case (iv) is given by the Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ (cf. [77] and [24], where all threefolds $X^3 \subset \mathbb{P}^N$ with $\delta = 1$ and $SX \neq \mathbb{P}^N$ are classified; cf. also Remark 2.9).
2.8. Definition. Nonsingular nondegenerate varieties \( X^n \subset \mathbb{P}^N \) for which \( \delta(X) = \delta > 0 \) and \( N = M(n, \delta) \) will be called extremal varieties.

2.9. Remark. The bound in Theorem 2.3 is sharp. Examples of extremal varieties \( X^n \subset \mathbb{P}^N \) for which
\[
N = M(n, \delta) = f \left( \left[ \frac{n}{\delta} \right] \right)
\] (2.9.1)
are given by all varieties with \( \delta > \frac{n}{2} \) (cf. Remark 2.5), by the Severi varieties \( \delta = \frac{n}{2} \); cf. Remark 2.6 and Theorem 4.7 in Chapter IV), by the Veronese varieties
\[
v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}, \quad (\delta = 1),
\]
by the Segre varieties
\[
\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{(a+1)(b+1)-1}, \quad |a - b| \leq 1 \quad (\delta = 2),
\]
and by the Grassmann varieties
\[
G(m, 1) \subset \mathbb{P}^{\left( \left[ \frac{m+1}{2} \right] \right)-1}, \quad (\delta = 4).
\]
In Chapter VI we shall show that each variety \( X^n \subset \mathbb{P}^N \) satisfying condition (2.9.1) (i.e. such that the inequality in Theorem 2.3 turns into equality) coincides with one of the varieties listed in this remark (cf. Theorem 5.6 in Chapter VI). We remark that from the proof of Theorem 2.3 it follows that for an extremal variety \( X^n \subset \mathbb{P}^{M(n, \delta)} \), \( \delta = \delta(X) \) \( M(n, \delta) = f \left( \left[ \frac{n}{\delta} \right] \right) \) if and only if \( k_0 = \left[ \frac{n}{\delta} \right] \) and the inequalities (2.3.1) turn into equalities for all \( k \leq k_0 \), i.e. \( \delta_i = \delta \) for \( 0 \leq i \leq k_0 = \left[ \frac{n}{\delta} \right] \) (cf. Proposition 1.2 in Chapter VI).

2.10. Theorem. Let \( X^n \subset \mathbb{P}^r \) be a (not necessarily nondegenerate) nonsingular variety. Then the dimension of the complete linear system of hyperplane sections of \( X \) does not exceed
\[
h^0(X, \mathcal{O}_X(1)) \leq f \left( \left[ \frac{n}{2n+1-r} \right] \right) + 1 \leq \left( \frac{4n-r+3)^2}{8(2n-r+1)} \right).
\]

Proof. Let \( \bar{X} \subset \mathbb{P}^N \) be the image of \( X \) under the embedding defined by the complete linear system \( |\mathcal{O}_X(1)| \), \( N = h^0(X, \mathcal{O}_X(1)) - 1 \). Then
\[
\bar{\delta} = \delta(X) = 2\bar{n} + 1 - \bar{s} = 2n + 1 - s \geq 2n + 1 - r
\]
\[
\bar{n} = \dim \bar{X} = n, \quad \bar{s} = \dim S \bar{X} = \dim SX = s.
\]
By Proposition 2.2 (ii),
\[
N \leq M(n, \bar{\delta}) \leq M(n, 2n + 1 - r).
\] (2.10.1)
From (2.10.1) and Theorem 2.3 it follows that
\[ h^0(X, \mathcal{O}_X(1)) \leq f \left( \left[ \frac{n}{2n + 1 - r} \right] \right) + 1. \] (2.10.2)

On the other hand, it is not hard to see that, for fixed \( n \) and \( r \), the second term in the expression
\[ f \left( \left[ \frac{n}{2n + 1 - r} \right] \right) = \frac{n(3n - r + 3)}{2(2n - r + 1)} + \frac{\epsilon(2n - r - \epsilon - 1)}{2(2n - r + 1)} \]
attains maximal value equal to \( \frac{(2n-r-1)^2}{8(2n-r+1)} \) for \( \epsilon = \frac{2n-r-1}{2} \). Hence
\[ f \left( \left[ \frac{n}{2n + 1 - r} \right] \right) + 1 \leq \frac{n(3n - r + 3)}{2(2n - r + 1)} + \frac{(2n - r - 1)^2}{8(2n - r + 1)} + 1 = \frac{(4n - r + 3)^2}{8(2n - r + 1)}. \] (2.10.3)

Theorem 2.10 immediately follows from (2.10.2) and (2.10.3).

2.11. Definition. Let \( X^n \subset \mathbb{P}^r \) be a nonsingular nondegenerate variety. The variety \( \tilde{X}^n \subset \mathbb{P}^N \), \( N = h^0(X, \mathcal{O}_X(1)) - 1 \) defined as the image of \( X \) under the embedding given by the complete linear system \( |\mathcal{O}_X(1)| \) will be called the linear normalization of \( X \).

2.12. Corollary. Let \( X^n \subset \mathbb{P}^r \), \( r \leq 2n \) be a nonsingular variety. Then \( h^0(X, \mathcal{O}_X(1)) \leq \frac{n+2}{2} \). In other words, if \( \text{codim}_P \ X \leq \dim X \) and \( \tilde{X} \) is the linear span of \( X \), then \( \dim \langle \tilde{X} \rangle \leq \frac{n(n+3)}{2} \).

Proof. Since in the statement of Theorem 2.10 \( X \) is not supposed to be nondegenerate, we may assume that \( r = 2n \). In this case Corollary 2.12 immediately follows from Theorem 2.10 because
\[ f(n) = \frac{n(n+3)}{2} = \left( \frac{n+2}{2} \right) - 1. \]

2.13. Remark. In the case when \( X \) is a complex manifold and
\[ r = s(X) = \dim SX = \dim TX = 2n \]
Corollary 2.12 was proved in [28, 6.16] by completely different (analytic) methods.

2.14. Remark. If for a nonsingular variety \( X^n \subset \mathbb{P}^{2n} \) we have \( h^0(X, \mathcal{O}_X(1)) = \binom{n+2}{2} \), then \( X \simeq \mathbb{P}^n \) and the embedding \( \mathbb{P}^n \to \mathbb{P}^{2n} \) is defined by a generic collection of \( 2n + 1 \) quadratic forms (cf. Remark 2.9 and Corollary 2.9 in Chapter VI).

2.15. Remark. It is clear that for \( r > 2n \) the dimension \( h^0(X, \mathcal{O}_X(1)) \) can assume arbitrarily large values; it suffices to take a nonsingular linearly normal variety \( \tilde{X} \subset \mathbb{P}^N \), \( N \geq r \) (dim \( \tilde{X} = n \), \( h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(1)) = N + 1 \)) and to project it isomorphically to \( \mathbb{P}^r \); the image of \( \tilde{X} \) under this projection is a nonsingular variety \( X \subset \mathbb{P}^n \) for which \( \dim X = n \) and \( h^0(X, \mathcal{O}_X(1)) = N + 1 \). This is only a restatement of the fact that \( M(n, 0) = \infty \).

It is convenient to interpret Theorem 2.10 using the notion of abnormality index.
2.16. **Definition.** Let $X^n \subset \mathbb{P}^r$ be a nonsingular nondegenerate variety, and let $\tilde{X} \subset \mathbb{P}^N$ be the linear normalization of $X$. The number $\lambda(X) = N - r = h^0(X, \mathcal{O}_X(1)) - r - 1$ is called the (linear) abnormality index of the projective variety $X$.

It is clear that linearly normal varieties have abnormality index zero; if $\delta > 0$, then curves have abnormality index zero, the abnormality index of surfaces does not exceed one, and the abnormality index of threefolds lies in the interval between zero and three (cf. Remarks 2.5–2.7).

Theorem 2.10 can be restated as follows.

2.17. **Corollary.** For a nonsingular nondegenerate variety $X^n$ with $s(X) \leq s$ the following inequality holds:

\[
\lambda(X) \leq \frac{(2s - 3n)(s - n - 1)}{2(2n - s + 1)} + \frac{\epsilon(2n - s - \epsilon - 1)}{2(2n - s + 1)} = \\
\frac{(2s - 3n)(s - n - 1)}{2(2n - s + 1)} + \left( n - \frac{s + 1}{2} \right) \left\{ \frac{n}{2n - s + 1} \right\}^2 - \left( n - \frac{s - 1}{2} \right) \left\{ \frac{n}{2n - s + 1} \right\}^2,
\]

where $\epsilon = n \pmod{2n - s + 1}$. In particular, if $s(X) \leq 2n$ (or, which is equivalent, $SX = TX$ cf. Theorem 1.4 in Chapter I), then $\lambda(X) \leq \frac{n(n-1)}{2}$.
CHAPTER VI

SCORZA VARIETIES
1. Properties of Scorza varieties

1.1. Definition. Let \( X^n \subset \mathbb{P}^N \) be a nonsingular nondegenerate variety. We call \( X \) a Scorza variety if, in the notations of Chapter V,

(i) \( N > m(n, \delta) \), where \( \delta = \delta(X) = 2n + 1 - s \);

(ii) \( N = M(n, \delta) < \infty \), i.e. \( \delta > 0 \) and \( X \) is an extremal variety in the sense of Definition 2.8 from Chapter V;

(iii) \( M(n, \delta) = f(\left[ \frac{n}{\delta} \right]) = \frac{n(n + \delta + 2) + \varepsilon(\delta - \varepsilon - 2)}{25} \), where \( f \) is the function defined by formula (2.3.1) from Chapter V (i.e. \( f(k) = (k + 1)(n + 1) - \frac{k(k + 1)}{2}\delta - 1 \)) and \( \varepsilon = n \pmod{\delta} \).

By Proposition 2.2 from Chapter V, condition (i) is equivalent to condition

\((i')\) \( SX \neq \mathbb{P}^N \).

From Theorem 2.3 in Chapter V it follows that

\((i) \& (iii) \Leftrightarrow (iv): \)

(iv) \( N = f\left(\left[ \frac{n}{\delta} \right]\right) \).

Finally, from the definition of \( f \) it follows that for \( 1 \leq \delta \leq \frac{n}{2} \)

\((iv) \Rightarrow (i')\)

(for \( \delta > \frac{n}{2} \) we have \( f\left(\left[ \frac{n}{\delta} \right]\right) = f(1) = m(n, \delta) = M(n, \delta) \); cf. Remark 2.5 in Chapter V).

Thus \( X \) is a Scorza variety if and only if \( n \geq 2\delta > 0 \), \( N = f\left(\left[ \frac{n}{\delta} \right]\right) \).

Scorza varieties are named in memory of the Italian mathematician Guido Scorza who obtained pioneer results in the study of linear normalizations of varieties of small codimension (cf. [77; 78]).

The goal of the present chapter is to give classification of Scorza varieties. Throughout this chapter we consider varieties defined over an algebraically closed field \( K \), char \( K = 0 \).

1.2. Proposition. Let \( X^n \subset \mathbb{P}^N \) be a nonsingular nondegenerate variety, and let \( \delta(X) = \delta \leq \frac{n}{2} \). Consider the following conditions:

(a) \( X \) is a Scorza variety;

(b) \( k_0 = \left[ \frac{n}{\delta} \right] \) (where \( k_0 = \min \{ k | S^k X = \mathbb{P}^N \} \));

(c) \( \delta_i = i\delta \) for \( 0 \leq i \leq \left[ \frac{n}{\delta} \right] \) (where \( \delta_i \) is defined by formula (1.5.2) in Chapter V).

Then (a) \( \Leftrightarrow (b) \& (c) \). Furthermore, (b) \( \Rightarrow (c) \) for \( n \equiv 0 \pmod{\delta} \) and (c) \( \Rightarrow (b) \) for \( n \not\equiv 0 \pmod{\delta} \).

Proof. The equivalence (a) \( \Leftrightarrow (b) \& (c) \) immediately follows from the proof of Theorem 2.3 in Chapter V (cf. Remark 2.9 in Chapter V).

Suppose that \( n \equiv 0 \pmod{\delta} \) and that condition (b) holds. By Theorem 1.8 in Chapter V,

\[ i\delta \leq \delta_i \leq \delta_{k_0} - (k_0 - i)\delta \leq n - (k_0 - i)\delta = i\delta \quad 0 \leq i \leq \frac{n}{\delta}, \]
i.e. $\delta_i = i\delta$ for $0 \leq i \leq \frac{n}{\delta}$. Hence for $n \equiv 0 \pmod{\delta}$ we have $(b) \Rightarrow (c)$ and $(a) \Leftrightarrow (b)$.

Suppose now that $n \not\equiv 0 \pmod{\delta}$ and that condition $(c)$ holds. Then $\delta_{k_0+1} = n > \left[\frac{n}{\delta}\right]$. Hence $k_0 + 1 > \left[\frac{n}{\delta}\right]$ and $k_0 \geq \left[\frac{n}{\delta}\right]$. On the other hand, by Theorem 1.12 in Chapter V we have $k_0 \leq \left[\frac{n}{\delta}\right]$, i.e. for $n \not\equiv 0 \pmod{\delta}$ we have $(c) \Rightarrow (b)$ and $(a) \Leftrightarrow (c)$.  

1.3. Remark. If $n \not\equiv 0 \pmod{\delta}$, then in general $(b) \not\Rightarrow (a)$. An example is given by the projection of Segre variety $P^m \times P^{m+1} \subset P^{m^2+3m+1}$ from a generic point $u \in P^{m^2+3m+1}$. We denote this variety by $X$. Then

$$n = 2m + 1, \quad N^2 = m^2 + 3m, \quad \delta = 2, \quad \delta_i = 2i, \quad 0 \leq i < m, \quad k_0 = m,$$

but

$$\delta_m = 2m + 1, \quad N = f\left(\left[\frac{n}{2}\right]\right) - 1.$$

If $n \equiv 0 \pmod{\delta}$, then in general $(c) \not\Rightarrow (a)$. An example is given by an arbitrary variety $X^\circ \subset P^{3n+1}$. In this case

$$\delta = \frac{n}{2}, \quad \delta_2 = n,$$

but Remark 2.6 from Chapter V shows that

$$k_0 = 1, \quad N = f(2) - 1.$$

1.4. Theorem. Let $X^n \subset P^N, N = f\left(\left[\frac{n}{2}\right]\right), \delta = \delta(X)$ be a Scorza variety, and let $u$ be a general point of $S^kX$, $2 \leq k \leq k_0 - 1$ (by Proposition 1.2, $k_0 = \left[\frac{n}{\delta}\right]$). Then in the notations of Chapter V (cf. Chapter V, (1.5.1), (1.5.2), and (2.3.1)), $Y_u \subset P_u$ is a Scorza variety, where

$$P_u = \left\{ u' \in S^kX \mid T_{S^kX,u'} = T_{S^kX,u} \right\} = S^kY_u,$$

and we have

$$\dim Y_u = k\delta, \quad k_0(Y_u) = k, \quad \delta_i(Y_u) = i\delta, \quad 0 \leq i \leq k, \quad \dim P_u = f(k).$$

If $k = k_0 \geq 2$, then $Y_u$ is a Scorza variety of dimension $k_0\delta = n - \varepsilon$, $\delta = \delta(Y_u)$ in the projective space $P_u = (Y_u) = S^{k_0}Y_u$, $\dim P_u = f(k_0)$.  

Proof. From Proposition 1.2 (c) and usual general position arguments it follows that for a generic point $u \in S^kX$, $0 \leq k \leq k_0$ the variety $Y_u$ is nonsingular and has dimension $k\delta$. Arguing by descending induction on $k$, we see that it suffices to prove Theorem 1.4 in the cases $k = k_0 - 1$ ($k_0 \geq 3$) and $k = k_0$.

First we consider the case

$$k = k_0 - 1, \quad \dim Y_u = (k_0 - 1)\delta.$$
Since for \( k_0 \geq 3 \) we have \( (k_0 - 1)\delta \geq 2\delta \), from Proposition 1.2 ((b) \( \Rightarrow \) (a) for \( n \equiv 0 \) (mod \( \delta \))) it follows that to show that \( Y_u \) is a Scorza variety it suffices to verify that

\[
\delta(Y_u) = \delta, \quad k_0(Y_u) = \left\lfloor \frac{\dim Y_u}{\delta} \right\rfloor = k_0 - 1. \tag{1.4.1}
\]

Let

\[
L_u = T_{S^{k_0-1}X,u}, \quad \dim L_u = s_{k_0-1} = N - 1 - (n - \delta_{k_0}(X)) = N - \varepsilon - 1,
\]

where \( \varepsilon = n - \delta_{k_0}(X) = n - k_0\delta \) (cf. formula (1.5.4) in Chapter V). Put

\[
Y_{L_u} = \{ x \in X \mid Tx \subset L_u \}.
\]

By Proposition 1.4 a) from Chapter V

\[
Y_u \subset Y_{L_u} \tag{1.4.2}
\]

Projecting \( X \) to \( \mathbb{P}^s \) \((s = 2n + 1 - \delta)\) from a general \((N - s - 1)\)-dimensional linear subspace of \( L_u \) and applying the theorem on tangencies (cf. Corollary 1.8 in Chapter I), we see that

\[
\dim Y_{L_u} \leq (s - \varepsilon - 1) - n - 1 = n - \delta - \varepsilon = (k_0 - 1)\delta = \dim Y_u. \tag{1.4.3}
\]

From (1.4.2) and (1.4.3) it follows that \( Y_u \) is a component of \( Y_{L_u} \). Varying \( u \), we see that for a generic point \( u' \in S^{k_0-1}Y_u \)

\[
Y_{u'} = Y_u \tag{1.4.4}
\]

\((Y_u \subset Y_{L_u} = Y_{L_u} \) in view of Proposition 1.4 in Chapter V). From (1.4.4) it follows that for a generic collection of \( k \) points

\[
(y_0, \ldots, y_{k-1}), \quad y_i \in Y_u, \quad 0 \leq i \leq k - 1, \quad k \leq k_0
\]

and a generic point \( v \in (y_0, \ldots, y_{k-1}) \) we have \( Y_v \subset Y_u \). In particular, for \( k = 2 \) from this it follows that

\[
\delta(Y_u) \geq \delta(X) = \delta. \tag{1.4.5}
\]

Since \( u \) is a generic point of \( S^{k_0-1}X \),

\[
\delta(Y_u) \leq \delta \tag{1.4.6}
\]

Combining (1.4.5) and (1.4.6) we conclude that

\[
\delta(Y_u) = \delta \tag{1.4.7}
\]

which proves the first of the equalities (1.4.1).

The fact that

\[
P_u = \{ u' \in S^{k_0-1}X \mid T_{S^{k_0-1}X,u'} = L_u \}
\]
is a linear subspace immediately follows from the reflexivity theorem of C. Segre (cf. e.g. [49; 50]). From Proposition 1.4(b) in Chapter V it follows that
\[ Y_u \subseteq S^{k_0-1}Y_u \subseteq \mathbb{P}_u. \]

On the other hand, assertion a) of the same proposition shows that for a generic point \( u' \in \mathbb{P}_u \)
\[ Y_{u'} \subseteq Y_{L_u}. \quad (1.4.8) \]
Since \( Y_u \) is irreducible, from (1.4.3) and (1.4.8) it follows that
\[ S^{k_0-1}Y_u = \mathbb{P}_u. \quad (1.4.9) \]
It remains to show that \( k_0(Y_u) = k_0 - 1 \) (this is the second of the inequalities (1.4.1)) and \( \dim \mathbb{P}_u = f(k_0 - 1) \). In view of Proposition 1.2, to do this it suffices to verify that
\[ S^kY_u \not= \mathbb{P}_u, \quad k < k_0 - 1. \quad (1.4.10) \]
But (1.4.10) immediately follows from the fact that \( u \) is a generic point of \( S^{k_0-1}X \), so that \( u \in \mathbb{P}_u \setminus S^kX \) for \( k < k_0 \). Thus Theorem 1.4 holds for \( k = k_0 - 1 \) and therefore for all \( 2 \leq k \leq k_0 - 1 \). We observe that the same argument shows that equalities (1.4.1) also hold for \( k = 1 \), so that for a generic point \( z \in SX \) the variety \( Y_z \) is nonsingular,
\[ \dim Y_z = \delta(Y_z) = \delta, \quad SY_z = \mathbb{P}_z = \{ z' \in SX \mid T_{SX,z'} = T_{SX,z} \} \quad (1.4.11) \]
(from (1.4.11) it immediately follows that \( Y_z \) is a hypersurface).

Now we consider the case \( k = k_0 \). To prove (1.4.7) it suffices to verify that for a generic point \( u \in S^{k_0}X = \mathbb{P}^N \), a generic pair of points \( x, y \in Y_u \), and a generic point \( z \in \langle x, y \rangle \) we have \( Y_z \subseteq Y_u \). Consider the morphism
\[ \varphi^{1,k_0}: S_{SX,S^{k_0-2}}X \to \mathbb{P}^N \]
(cf. §1 of Chapter V) and put
\[ Y_u^1 = p^{1,k_0-2}_1 \left( \varphi^{1,k_0-2}(u) \right). \]
From 1.2(c) and formula (1.5.4) in Chapter V it follows that
\[ \dim Y_u^1 = (s_1 + s_{k_0-2} + 1) - s_{k_0} \]
\[ = (2n + 2 - \delta) + s_{k_0-2} - (s_{k_0-1} + n + 1 - \delta_{k_0}) \]
\[ = (2n + 2 - \delta) - [(s_{k_0-2} + n + 1 - \delta_{k_0-1}) + (n + 1 - \delta_{k_0})] \]
\[ = \delta_{k_0-1} + \delta_{k_0} - \delta = 2(k_0 - 1)\delta. \quad (1.4.12) \]
From (1.4.12) it follows that
\[ \dim Y_u^1 + 2\delta = 2 \dim Y_u, \]
and hence there exists a point \( \tilde{z} \in Y_u^1 \) such that \( x, y \in Y_{\tilde{z}} \subseteq Y_u \). Furthermore, it is clear that \( \tilde{z} \) is a generic point of \( \mathbb{P}_{\tilde{z}} \), and from (1.4.11) it follows that \( Y_{\tilde{z}} = Y_{\tilde{z}} \subseteq Y_u \) which implies (1.4.7). From the already established part of Theorem 1.4 it follows that
\[ k_0(Y_u) = k_0(X) = k_0 = \frac{\dim Y_u}{\delta}. \]
Hence Proposition 1.2 ((b) \( \Rightarrow \) (a) for \( n \equiv 0 \) (mod \( \delta \))) shows that \( Y_u \) is a Scorza variety. \( \square \)
1.5. Corollary. Let $X^n \subset \mathbb{P}^{M(n,\delta)}$, $\delta = \delta(X)$ be a Scorza variety. Then for a generic point $z \in SX$ the variety $Y_z$ is a nonsingular $\delta$-dimensional quadric.

Proof. In the proof of Theorem 1.4 it was shown that for a generic point $z \in SX$ the variety $Y_z$ is a nonsingular hypersurface in $\mathbb{P}_z = SY_z = \mathbb{P}^{\delta+1}$ (cf. (1.4.11)). Since $X$ is not a hypersurface, a generic secant intersects $X$ at exactly two points (cf. [34, Chapter IV, §3; 39, 2.5; 64, §7 B]) which proves Corollary 1.5 (in view of Proposition 2.1 in Chapter IV, Corollary 1.5 also follows from Corollary 1.6 below). □

1.6. Corollary. Let $X^n \subset \mathbb{P}^{M(n,\delta)}$, $\delta = \delta(X)$ be a Scorza variety. Then $\delta = 1, 2, 4, \text{ or } 8$, and if $u \in S^2X$ is a generic point, then $Y_u$ is a Severi variety (cf. Definition 1.2 in Chapter IV).

Proof. From Theorem 1.4 it follows that $Y_u$ is an extremal variety,

$$\dim Y_u = 2\delta, \quad \delta(Y_u) = \delta, \quad \dim \langle Y_u \rangle = f(2) = 3\delta + 2. \quad (1.6.1)$$

From Remark 2.6 in Chapter V it follows that $Y_u$ is a Severi variety. In view of Theorem 3.10 in Chapter IV, from (1.6.1) it follows that $\delta = \delta(Y_u)$ can assume only four values indicated in the statement of the corollary. □

Corollary 1.6 shows that for classification of Scorza varieties it suffices to consider four cases, viz. $\delta = 1, 2, 4, 8$. This will be done in the next four sections.
VI. SCORZA VARIETIES

2. Scorza varieties with $\delta = 1$

The goal of the present section is to give a proof of the following main result.

2.1. Theorem. Let $X \subset \mathbb{P}^N$, $n \geq 2$ be a nonsingular nondegenerate variety such that $s_X \leq 2n$ (by Theorem 1.4 of Chapter I, this condition holds if and only if $SX = TX$). Then $N \leq \frac{n(n+3)}{2}$ and equality holds if and only if $X = v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ is the Veronese variety.

In particular, $M(n, 1) = \frac{n(n+3)}{2}$ and the Veronese variety is the only Scorza variety of dimension $n$ with $\delta = 1$.

Proof. The inequality $N \leq \frac{n(n+3)}{2}$ was already proven in Corollary 2.12 in Chapter V.

Suppose that $N = f(n) = \frac{n(n+3)}{2}$. From Proposition 2.2 in Chapter V it follows that that $\delta = 1$. We start with the following general result which holds for all $\delta$.

2.2. Lemma. Let $X \subset \mathbb{P}^{M(n, \delta)}$, $\delta = \delta(X) > 0$ be a Scorza variety. Suppose that $n \equiv 0 \pmod{\delta}$ so that $M(n, \delta) = f(k_0)$, $k_0 = \frac{n}{\delta}$, and let $u \in S^{k_0-1}X$ and $z \in SX$ be generic points. Then $s_{k_0-1} = N - 1$ and $(Y^\delta_z \cdot Y^n_u) = 1$.

Proof of Lemma 2.2. From Proposition 1.2 (c) and formulas (1.5.6) and (1.5.7) in Chapter V it follows that

$$n = \delta k_0 = k_0 \delta,$$

$$s_{k_0-1} = s_{k_0} - (n + 1 - \delta k_0) = N - 1.$$  

From (1.4.2) it follows that

$$T(Y_u, X) \subset L_u = T_{S^{k_0-1}X, u}.$$  

(2.2.1)

Since $X$ is a nondegenerate variety, from (2.2.1) and Theorem 1.4 of Chapter I it follows that

$$\dim S(Y_u, X) = \dim Y_u + n + 1 = 2n + 1 - \delta = s.$$  

(2.2.2)

Equality (2.2.2) means that

$$S(Y_u, X) = SX$$  

(2.2.3)

(similarly, one can show that $S(Y_z, S^{k_0-2}X) = S^{k_0-1}X$). From (2.2.3) it follows that

$$Y_z \cap Y_u \neq \emptyset.$$  

(2.2.4)

In view of Corollary 1.5, $Y_z^\delta \subset \mathbb{P}^{\delta+1}_z = SY_z$ is a nonsingular quadric. Since $L_u \not\supset \mathbb{P}_z$, $L_u \cap P_z$ is a hyperplane in $\mathbb{P}_z$ which is tangent to $Y_z$ at all points of $Y_z \cap Y_u$. From this and (2.2.4) it immediately follows that $(Y_z \cdot Y_u) = 1$.

Lemma 2.2 is proved.

We return to the case $\delta = 1$. In this case (2.2.1) means that $L_u \cdot X = rY_u + E_u$, where $r \geq 2$, $Y_u \not\subset \supp E_u$. From Lemma 2.2 it follows that

$$2 = \deg Y_z = (L_u \cdot Y_z) = r + (E_u \cdot Y_z).$$  

(2.2.5)

Formula (2.2.5) shows that

$$r = 2, \quad (E_u \cdot Y_z) = 0.$$  

(2.2.6)
2.3. Lemma. In the assumptions of Theorem 2.1, for a generic point \( u \in S^{n-1}X \) we have \( E_u = 0 \).

Proof of Lemma 2.3. First we observe that if \( E_u \neq 0 \), then \( E_u = E \) is the fixed part of the algebraic system \( L_u \cdot X \), where \( u \) runs through the set of generic points of \( S^{n-1}X \). In fact, if this were not so, then \( E_u \) would contain a generic point \( x \in X \), and if \( x' \) is another generic point of \( X \) and \( z \) is a generic point of the line \( \langle x, x' \rangle \), then \( Y_z \) would intersect the divisor \( E_u \) at a finite set of points. Since \( x \in Y_z \cap E_u \), this set is non-empty contrary to (2.2.6).

Lemma 2.3 now follows from the following result.

2.4. Lemma. Let \( X^n \subset \mathbb{P}^{M(n, \delta)} \), \( \delta = \delta(X) > 0 \) be a Scorza variety. Then the algebraic system of divisors \( H_u = (L_u \cdot X)_{\mathbb{P}^n} \), \( L_u = T_{S^{n-1}X,u} \), \( u \in S^{k_0-1}X \) does not have fundamental points on \( X \).

Proof of Lemma 2.4. Suppose the converse, and let \( y \) be a fundamental point. Then

\[ y \in \bigcap_{u \in S^{k_0-1}X} T_{S^{k_0-1}X,u}, \]

and therefore \( S^{k_0-1}X \) is a cone with vertex \( y \). Hence Lemma 2.4 is a consequence of the following result.

2.5. Lemma. Let \( X^n \subset \mathbb{P}^{M(n, \delta)} \), \( \delta = \delta(X) > 0 \) be a Scorza variety. Then \( S^{k_0-1}X \) is not a cone.

Proof. We prove the lemma by induction. If \( n = 2\delta \), then \( X \) is a Severi variety and Lemma 2.5 follows from results of Chapter IV. Using Theorem 1.4, suppose that for a generic point \( u \in S^{k_0-1}X \) the variety \( S^{k_0-2}Y_u \) is not a cone and the variety \( S^{k_0-1}X \) is a cone with vertex \( v \). Let \( u' \) be a generic point of the line \( \langle v, u \rangle \). Then

\[ u' \in S^{k_0-1}X, \quad L_{u'} = T_{S^{k_0-1}X,u'} = T_{S^{k_0-1}X,u} = L_u \]

and hence \( Y_{u'} = Y_u \) (cf. (1.4.4)). For a generic point \( x \in Y_u \), we consider the curve

\[ C_{v,x,u} = \mathbb{P}^1_{u_1} \left( (p^{0,k_0-2} \times \varphi^{0,k_0-2})^{-1} (x \times \langle v, u \rangle^0) \right) \subset S^{k_0-2}Y_u \]  

(2.5.1)

in the plane

\[ \Pi = \langle v, x, u \rangle \subset S^{k_0-1}Y_u, \]  

(2.5.2)

where \( \langle v, u \rangle^0 \) denotes the set of generic points of the line \( \langle v, u \rangle \). Since \( s_{k_0-2} \leq s_{k_0-1} - 2 \) (cf. formula (1.5.4) in Chapter V), we may assume that

\[ \langle v, u \rangle \cap S^{k_0-2}Y_u = \langle v, u \rangle \cap S^{k_0-2}X = v. \]  

(2.5.3)

From (2.5.1) and (2.5.3) it follows that \( \langle v, u \rangle \cap C_{v,x,u} = v \), and since \( u \) is a generic point of \( \Pi \), \( C_{v,x,u} \) consists of several lines passing through \( v \). Hence for a generic point \( w \in S^{k_0-2} \) we have \( \langle v, w \rangle \subset S^{k_0-2}Y_u \), i.e., contrary to our assumption, \( S^{k_0-2}Y_u \) is a cone with vertex \( v \).

This contradiction proves Lemma 2.5 and therefore also Lemmas 2.4 and 2.3.
From Lemma 2.3 it follows that
\[ L_u \cdot X = 2Y_u, \tag{2.5.4} \]
so that in particular \( Y_u \) is an ample divisor on \( X \). Now it is easy to prove Theorem 2.1 by induction on \( n \) using Theorem 1.4, Corollary 1.6, the classical result of Severi for \( n = 2 \) (cf. [82, n°8] or Theorem 4.7a in Chapter IV), and the well known theorem to the effect that if a nonsingular variety \( X^n, n > 2 \) contains an ample divisor \( Y \simeq \mathbb{P}^{n-1} \), then \( X \simeq \mathbb{P}^n \) (cf. [63]). However we give a direct proof of Theorem 2.1.

From (2.5.4) it follows that the image of the rational map
\[ S^{n-1} \to \text{Pic}^0 X, \quad u \mapsto \text{cl} (Y_u - Y_{u_0}) \]
(where \( u_0 \) is a fixed generic point of \( S^{n-1}X \)) is contained in the set of points of order two on the Picard variety. Since \( S^{n-1}X \) is an irreducible variety, from this it follows that for generic points \( u, u' \in S^{n-1}X \) we have \( Y_u \sim Y_{u'} \) (where \( \sim \) denotes linear equivalence).

Consider the complete linear system of divisors \( H = |Y_u| \) on the variety \( X \). We observe that a general divisor \( H \in \mathcal{H} \) has the form
\[ H = Y_u, \quad u \in S^{n-1}X. \tag{2.5.5} \]
In fact, let \( x_0, \ldots, x_{n-1} \in H \) be a generic collection of \( n \) points, and let \( u \in \langle x_0, \ldots, x_{n-1} \rangle \) be a generic point. Then it is clear that \( u \) is a generic point of \( S^{n-1}X \), and from Proposition 1.4 b) of Chapter V it follows that
\[ L_u = \langle T_{x_0}, x_0, \ldots, T_{x_{n-1}} \rangle. \tag{2.5.6} \]
On the other hand, since \( X \) is an extremal variety, \( X \) is linearly normal, and therefore the divisor \( 2H \sim 2Y_u \) is cut by a hyperplane \( L_H \subset \mathbb{P}^N \), i.e. \( 2H = L_H \cdot X \). Hence \( T(H, X) \subset L_H \), so that from (2.5.6) it follows that
\[ L_H = L_u, \quad 2H = L_H \cdot X = L_u \cdot X = 2Y_u, \]
and \( H = Y_u \) as stated in (2.5.5).

From Lemma 2.4 it follows that the linear system \( \mathcal{H} \) does not have fundamental points and hence defines a morphism
\[ h: X \to \mathbb{P}^{\dim \mathcal{H}}. \tag{2.5.7} \]
From (1.4.3) and (2.5.5) it follows that
\[ \dim \mathcal{H} = \dim (S^{n-1}X)^* = \dim (\underbrace{X \times \cdots \times X}_n) - \dim (\underbrace{Y_u \times \cdots \times Y_u}_n) = n. \tag{2.5.8} \]
Since in view of (2.5.4) the linear system \( \mathcal{H} \) is ample, (2.5.7) and (2.5.8) show that \( h: X \to \mathbb{P}^n \) is a finite covering.
The preimage of the linear system of quadrics in \( \mathbb{P}^n \) with respect to the morphism \( h \) is a linear subsystem of the complete linear system \( |2Y_u| = L_u \cdot X \) on \( X \), and since

\[
N + 1 = h^0(X, \mathcal{O}_X(1)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) = \binom{n+2}{2}
\]
since \( N + 1 = h^0(X, \mathcal{O}_X(1)) \) and \( X = v_2(\mathbb{P}^n) \).

These two linear systems coincide with each other. Hence \( h \) is an isomorphism, \( h^{-1} = v_2 \) and \( X = v_2(\mathbb{P}^n) \). \( \square \)

2.6. Remark. For \( n = 1 \) we have \( k_0 = 1 \), so that each plane curve is extremal, i.e. \( M(1, 1) = m(1, 1) = 2 \).

2.7. Remark. As we already pointed out, Theorem 2.1 was first proved by Severi \([82, \text{ no. 8]} \) (cf. Remark 4.11 in Chapter IV) for \( n = 2 \). Ådlandsvik independently proved a theorem related to Theorem 2.1 to the effect that if \( s_X \leq 2n \) and \( S^{n-1}X \) is not a cone (the last condition is quite hard to verify), then \( X \) is a Veronese variety (cf. \([103]\)).

2.8. Remark. It is clear that the morphism (2.5.7) is also defined by the linear system \( |Y_{u_0} + H| \), where \( u_0 \) is a fixed generic point of \( S^{n-1}X \), i.e. by the linear system of hyperplane sections of \( X \) passing through the linear subspace \( \langle Y_{u_0} \rangle \). Hence \( h \) is induced by projecting from the subspace \( \langle Y_{u_0} \rangle \), where

\[
\dim \langle Y_{u_0} \rangle = N - n - 1 = \frac{(n-1)(n+2)}{2} = f(n-1).
\]

In fact, the map inverse to the Veronese mapping

\[
v_2 : \mathbb{P}^n \to v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}
\]
is defined by projecting from the linear subspace

\[
\langle v_2(\mathbb{P}^{n-1}) \rangle = \mathbb{P}^{\frac{(n-1)(n+2)}{2}} \subset \mathbb{P}^{\frac{n(n+3)}{2}}
\]
(cf. Chapter III, §3).

2.9. Corollary. Let \( X^n \subset \mathbb{P}^r \) be a nonsingular variety, \( r \leq 2n \). Then \( h^0(X, \mathcal{O}_X(1)) \leq \binom{n+3}{2} = \left[ \frac{(n+3)^2}{2} \right] \) with equality holding if and only if \( r = 2n \) and either \( n = 1 \) or \( X \hookrightarrow \mathbb{P}^{2n} \) is the embedding of the projective space \( \mathbb{P}^n \cong X \) defined by a collection of \( 2n + 1 \) quadratic forms \( Q_0, \ldots, Q_{2n} \) on \( \mathbb{P}^n \).
In this section we prove the following result.

3.1. Theorem. Let $X^n \subset \mathbb{P}^N$, $n \geq 4$ be a nonsingular nondegenerate variety. Suppose that $s_X < 2n$. Then

$$N \leq m(m+2) = (m+1)^2 - 1 \quad \text{for } n = 2m \equiv 0 \pmod{2},$$
$$N \leq (m+1)(m+2) - 1 \quad \text{for } n = 2m + 1 \equiv 1 \pmod{2}.$$  

Furthermore, the inequalities turn into equalities if and only if $X = \mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)}$ or $X = \mathbb{P}^m \times \mathbb{P}^{m+1} \subset \mathbb{P}^{m^2 + 3m + 1}$ is a Segre variety.

In particular, $M(n, 2) = n(n + 4) - n \mod 2$ and the Segre variety $\mathbb{P}^{\frac{n-n\mod 2}{2}} \times \mathbb{P}^{\frac{m+n\mod 2}{2}}$ is the only $n$-dimensional Scorza variety with $\delta = 2$.

Proof. The inequality

$$N \leq f\left(\left\lfloor \frac{n}{2} \right\rfloor \right) = \frac{n(n + 4) - n \mod 2}{4}$$

splitting into the pair of inequalities given in the statement of the theorem follows from Proposition 2.2 (ii) and Theorem 2.3 in Chapter V.

Suppose that $N = f\left(\left\lfloor \frac{n}{2} \right\rfloor \right)$. From Proposition 2.2 in Chapter V it follows that $\delta = 2$. Proposition 1.2 shows that $k_0 = \left\lfloor \frac{n}{2} \right\rfloor = m, \quad \delta_k = 2k, \quad 0 \leq k \leq m$.

Suppose first that $n = 2m$. By Lemma 2.2, $s_{m-1} = N-1$, and for a generic point $u$ of the hypersurface $S^{m-1}X \subset \mathbb{P}^N (N = m(m+2))$ the hyperplane $L_u = T_{S^{m-1}X, u}$ is tangent to $X$ along $Y_u$, codim$_X Y_u = 2$. Furthermore, for a generic point $z \in SX$ we have $(Y_u \cdot Y_z) = 1$. Put $H_u = L_u \cdot X$. For a generic point $y \in Y_u$ and a generic point $z \in S_yX$

$$(Y_z \cdot H_u)_X = (Y_z \cdot L_u)_P^N = (Y_z \cdot T_{Y_z, y})_{P^3}$$

(here, according to Corollary 1.5, $Y_z$ is a nonsingular two-dimensional quadric), $\mathbb{P}^3 = (Y_z)$. Thus

$$(Y_z \cdot H_u)_X = l_1^z \cup l_2^z,$$

where $l_1^z, l_2^z$ is a pair of lines intersecting at $y$. If $z'$ is another generic point of $S_yX$ and

$$(Y_{z'} \cdot H_u)_X = (Y_{z'} \cdot H_u)_X,$$  \hspace{1cm} (3.1.1)

then it is clear that $Y_{z'} = Y_z$, so that $z' \in \mathbb{P}^3$. Varying $z$ in the set of general points of the cone $S_yX$, we obtain a $(\dim S_yX - \dim \mathbb{P}^3) = (n - 2)$-dimensional family of pairs of lines $l_1^z, l_2^z \subset H_u$ such that

$$l_1^z \cap Y_u = l_2^z \cap Y_u = (l_1^z \cdot l_2^z)_Y = y.$$
Furthermore, we may assume that the pairs \((l_1^z, l_2^z)\) are ordered so that the lines \(l_1^z\) (resp. \(l_2^z\)), \(z \in S_yX\) form an irreducible family \(\mathcal{F}_1^y\) (resp. \(\mathcal{F}_2^y\)) of lines in \(H_u\) passing through \(y\).

For generic lines \(l_1 \in \mathcal{F}_1^y, l_2 \in \mathcal{F}_2^y\) we have
\[
l_1 \cap l_2 = (Y_z \cdot H_u)_X,
\]
where \(z\) is a generic point of the plane \((l_1, l_2)\). Hence the dimension of the families \(\mathcal{F}_1^y\) and \(\mathcal{F}_2^y\) is equal to \(\frac{1}{2}(n-2) = m-1\), and the closure of the subset of the divisor \(H_u\) swept out by the lines \(l_1^z\) (resp. \(l_2^z\)) is an irreducible \(m\)-dimensional cone \(C_1^y\) (resp. \(C_2^y\)) with vertex \(y\).

Varying \(y\) in the set of generic points of \(Y_u\), we obtain two irreducible subvarieties
\[
C_1^u = \bigcap_{y \in Y_u} C_1^y, \quad C_2^u = \bigcap_{y \in Y_u} C_2^y.
\]
Since a general point \(y \in Y_u\) is contained in an \((m-1)\)-dimensional family of lines from \(\mathcal{F}^y\), a general point \(x \in C_i^u\) is also contained in at most \((m-1)\)-dimensional subset of lines from the family \(\mathcal{F}_i = \bigcap_{y \in Y_u} \mathcal{F}_i^y\) \((i = 1, 2)\). Hence
\[
\dim C_i^u \geq \dim Y_u + m - (m - 1) = 2m - 1 = n - 1,
\]
and therefore \(C_i^u\) is an irreducible component of the divisor \(H_u = L_u \cdot X\) and
\[
C_i^u \cdot Y_z = l_i^z \quad (i = 1, 2).
\]
Thus
\[
C_1^u \neq C_2^u, \quad C_1^u \cap C_2^u \supset Y_u.
\]

From the above considerations it follows that \(H_u\) is a connected divisor whose components pairwise intersect with each other along cycles of codimension two in \(X\) lying in \(\text{Sing} \ H_u\) and
\[
(C_1^u + C_2^u \cdot Y_z) = l_1^z + l_2^z = (L_u \cdot Y_z)_Y = (H_u \cdot Y_z)_X. \quad (3.1.2)
\]

Let \(E, F\) be an arbitrary pair of irreducible components of \(H_u\). Then \(E \cap F \subseteq \text{Sing} \ H_u = Y_{L_u}\), i.e.
\[
T(E \cap F, X) \subset L_u. \quad (3.1.3)
\]
On the other hand, from the nondegeneracy of \(X\) it follows that
\[
S(E \cap F, X) \nsubseteq L_u. \quad (3.1.4)
\]
In view of (3.1.3) and (3.1.4), Theorem 1.4 from Chapter I implies that
\[
\dim S(E \cap F, X) = \dim E \cap F + n + 1 = 2n - 1 = \dim SX.
\]
Hence \(S(E \cap F, X) = SX\), and therefore, for a generic point \(z \in SX\),
\[
0 < \text{card} \left( Y_z \cdot (E \cap F) \right) < \infty \quad (3.1.5)
\]
VI. SCORZA VARIETIES

(\text{compare with the proof of Lemma 2.2}). From (3.1.1), (3.1.2), and (3.1.5) it follows that
\[ H_u = C_1^u + C_2^u, \quad (C_1^u \cdot C_2^u)_X = Y_u. \]  
\hspace{1.5cm} (3.1.6)

Since \( s = 2n - 1 \), the variety \( X \) can be isomorphically projected to \( \mathbb{P}^{2n-1} \). From Barth’s theorem [6] it follows that
\[ h^1(X, \mathcal{O}_X) = \frac{1}{2} h^1(X, \mathbb{C}) = 0, \quad H^1(X, \mathcal{O}_X^*) \hookrightarrow H^2(X, \mathbb{Z}), \]
and the Picard variety \( \text{Pic}^0 X \) is trivial. Hence for general points \( u, u' \in S^{m-1} X \) we have
\[ C_1^u \equiv C_1^{u'} (i = 1, 2). \]

Consider the complete linear systems \( L_i = |C_1^u| (i = 1, 2) \) on the variety \( X \). We observe that a general divisor \( D \in L_i \) has the form \( D = C_1^u \), where \( u \) is a suitable general point of \( S^{m-1} X \). In fact, from the linear normality of Scorza varieties it follows that for a generic point \( u' \in S^{m-1} X \) we have
\[ D + C_1^{u'} = (L_u \cdot X), \]
\[ L_u = \langle TX, x_0, \ldots, TX, x_{m-1} \rangle \subset L. \]

Hence
\[ D = C_1^u, \quad C_1^{u'} = C_1^{u'}, \quad i = 1, 2, \]
as required.

Since the dimension of the algebraic family of divisors \( H_u, u \in S^{m-1} X \) is equal to \( m \dim X - m \dim Y = n \), the dimensions \( d_i \) of the linear systems \( L_i \) \((i = 1, 2)\) satisfy the relation
\[ d_1 + d_2 = n = 2m. \]  
\hspace{1.5cm} (3.1.7)

Suppose for example that \( d_1 \leq d_2 \). Fixing a generic point \( u \in S^{m-1} X \) and varying a generic point \( u' \in S^{m-1} X \), we obtain a \( d_2 \)-dimensional family of \((n-2)\)-dimensional cycles
\[ Y_{u''} = C_1^u \cap C_2^u \subset C_1^u. \]
Furthermore, \( u'' \in S^{m-1} C_1^u \), and therefore
\[ d_2 \leq m \dim C_1^u - m \dim Y_{u''} = m. \]  
\hspace{1.5cm} (3.1.8)

From (3.1.7) and (3.1.8) it follows that
\[ d_1 = d_2 = m. \]  
\hspace{1.5cm} (3.1.9)

From Lemma 2.4 it follows that the linear systems \( L_1 \) and \( L_2 \) do not have fundamental points. In view of (3.1.9), the linear system \( L_i \) defines a morphism
\[ h_i : X \rightarrow \mathbb{P}^m \quad (i = 1, 2). \]
Put 
\[ h = h_1 \times h_2 : X \to \mathbb{P}^m \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{m(m+2)}. \]

The preimage of the complete linear system of hyperplane sections of \( \mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)} \) (the Segre embedding) with respect to the morphism \( h \) is a linear subsystem of the complete linear system of hyperplane sections of \( X \), and since 
\[ N + 1 = h^0(X, \mathcal{O}_X(1)) = h^0(\mathbb{P}^m \times \mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1) \otimes \mathcal{O}_{\mathbb{P}^m}(1)) = (m + 1)^2, \]
these two systems coincide with each other. Hence \( h \) is an isomorphism and 
\[ X \cap \mathbb{P}^N = \mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)} \]
is a Segre variety.

Consider now the case \( n = 2m + 1 \). Let \( u \) be a generic point of \( \mathbb{P}^N \), \( N = m^2 + 3m + 1 \). By Theorem 1.4, \( Y_u \) is a 2\( m \)-dimensional Scorza variety, \( m \geq 2 \), 
\[ (Y_u) = S^m_u = \mathbb{P}_u \]
is a linear subspace of \( \mathbb{P}^N \) of dimension \( m(m + 2) \), and 
\[ Y_u = X \cap \mathbb{P}_u. \]

Consider the set of hyperplanes passing through the subspace \( \mathbb{P}_u \), and let \( \mathcal{L}'_1 \) be the corresponding linear system of hyperplane sections of \( X \). Then 
\[ \dim \mathcal{L}'_1 = (m^2 + 3m + 1) - (m^2 + 2m) - 1 = m, \]
and for each hyperplane \( L \supset \mathbb{P}_u \)
\[ (L \cdot X)_{\mathbb{P}^N} = Y_u + C_L, \]
where \( C_L \) is a divisor on \( X \). Put \( \mathcal{L}_1 = |C_L| \). From the linear normality of \( X \) it follows that the linear system \( \mathcal{L}_1 \) on \( X \) is complete.

As in the case \( n = 2m \), we can apply Barth’s theorem [6] to verify that \( \mathrm{Pic}^0 X = 0 \), so that for generic points \( u, u' \in \mathbb{P}^N \) we have \( Y_{u'} \sim Y_u \). In particular, from this it follows that the linear system \( \mathcal{L}_1 \) does not depend on the choice of point \( u \in \mathbb{P}^N \). Lemma 2.4 shows that 
\[ \bigcap_{u \in \mathbb{P}^N} Y_u = \emptyset. \quad (3.1.10) \]

Hence the linear system \( \mathcal{L}_1 \) does not have fundamental points and defines a morphism \( h_1 : X \to \mathbb{P}^m \).

We denote by \( \mathcal{L}_2 \) the complete linear system \( |Y_u| \). In view of (3.1.10), \( \mathcal{L}_2 \) also does not have fundamental points. We denote by \( h_2 \) the morphism corresponding to the linear system \( \mathcal{L}_2 \).

We observe that the dimension of the system of divisors \( Y_u, u \in \mathbb{P}^N \) is equal to 
\[ (m + 1) \dim X - (m + 1) \dim Y_u = m + 1. \]
VI. SCORZA VARIETIES

From the linear normality of $X$ and the relation

$$N + 1 = h^0(X, \mathcal{O}_X(1)) = h^0(\mathbb{P}^m \times \mathbb{P}^{m+1}, \mathcal{O}_{\mathbb{P}^m}(1) \otimes \mathcal{O}_{\mathbb{P}^{m+1}}(1)) = (m + 1)(m + 2)$$

it follows that a general divisor $D \in L_2$ has the form $D = Y_u$ for some point $u \in \mathbb{P}^N$, and the morphism

$$h = h_1 \times h_2 : X \to \mathbb{P}^m \times \mathbb{P}^{m+1} \to \mathbb{P}^{m^2+3m+1}$$

is an isomorphism. In other words,

$$X \subset \mathbb{P}^N = \mathbb{P}^m \times \mathbb{P}^{m+1} \subset \mathbb{P}^{m^2+3m+1}$$

is a Segre variety. \(\square\)

3.2. Remark. For $\delta = 2$, $n \leq 3$ we have $k_0 = 1$, and each variety $X^n \subset \mathbb{P}^{2n-1}$ is extremal. In other words,

$$M(3, 2) = m(3, 2) = 5, \quad M(2, 2) = m(2, 2) = 3.$$ 

3.3. Remark. As we already observed in Chapter IV (cf. Remark 4.11), for $n = 4$, $X$ is a Severi variety, and in this case Theorem 3.1 was proved by Scorza [77; 78] and Fujita and Roberts [25]. Adlansvik independently proved a theorem related to Theorem 3.1 to the effect that if $n = 2m$, $s_X \leq 2n - 1$, and $S^{m-1}X$ is not a cone (the last condition is quite hard to verify), then $X$ is the Segre variety $\mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m^2+3m+1}$ (cf. [103]).

3.4. Remark. As it was shown in §3 of Chapter III, the Segre variety $\mathbb{P}^k \times \mathbb{P}^l \subset \mathbb{P}^{(k+1)(l+1)-1}$ is the image of the projective space $\mathbb{P}^{k+l}$ under the rational map

$$\mathbb{P}^{k+l} \dashrightarrow \mathbb{P}^{(k+1)(l+1)-1}$$

(3.4.1)

defined by the linear system of quadrics in $\mathbb{P}^{k+l}$ passing through a union of two nonintersecting linear subspaces $\mathbb{P}^{k-1} \subset \mathbb{P}^{k+l}$ and $\mathbb{P}^{l-1} \subset \mathbb{P}^{k+l}$. The rational map (3.4.1) is a birational isomorphism, and the inverse rational map

$$\mathbb{P}^k \times \mathbb{P}^l \dashrightarrow \mathbb{P}^{k+l}$$

is defined by projecting from the linear subspace $\mathbb{P}^{k+l-1}$ spanned by the subvariety

$$\mathbb{P}^{k-1} \times \mathbb{P}^{l-1} \subset \mathbb{P}^k \times \mathbb{P}^l.$$ 

3.5. Corollary. Let $X^n \subset \mathbb{P}^r$ be a nonsingular variety, $r \leq 2n - 1$. Then $h^0(X, \mathcal{O}_X(1)) \leq \frac{(n+2)^2}{4}$, and equality holds if and only if $r = 2n - 1$ and either $n \leq 3$ or $X^n \hookrightarrow \mathbb{P}^{2n-1}$ is an isomorphic embedding of the Segre variety $\mathbb{P}^{\frac{k^2}{2}} \times \mathbb{P}^{\frac{l^2}{2}} \simeq X$ by means of the mapping $(Q_0 : \cdots : Q_{2n-1})$ defined by a collection of $2n$ forms $Q_0, \ldots, Q_{2n-1}$ of bidegree $(1,1)$ with respect to coordinates of factors.
4. Scorza varieties with $\delta = 4$

The goal of the present section is to give a proof of the following main result.

4.1. Theorem. Let $X^n \subset \mathbb{P}^N$, $n \geq 8$ be a nonsingular nondegenerate variety. Suppose that $s_X < 2n - 2$. Then

$$N \leq \frac{n(n+6) + \varepsilon(2 - \varepsilon)}{8}, \quad \varepsilon = n \mod 4.$$ 

Furthermore, equality holds if and only if $n$ is even and $X = G \left( \frac{n}{2} + 1, 1 \right) \subset \mathbb{P}^\frac{n(n+6)}{2}$ is the Grassmann variety (under the Plücker embedding).

In particular, for $n \equiv 0 \pmod{2}$ we have $M(n, 4) = n(n+6)/2$, and the Grassmann variety $G \left( \frac{n}{2} + 1, 1 \right)$ is the only $n$-dimensional Scorza variety with $\delta = 4$.

Proof. The inequality $N \leq f \left( \left[ \frac{n}{4} \right] \right)$ which is equivalent to the inequality given in the statement of the theorem follows from Proposition 2.2 (ii) and Theorem 2.3 in Chapter V.

Suppose that $N = f \left( \left[ \frac{n}{4} \right] \right)$. From Proposition 1.2 it follows that

$$k_0 = \left[ \frac{n}{4} \right], \quad \delta_k = 4k, \quad 0 \leq k \leq \left[ \frac{n}{4} \right].$$

Suppose first that $n \equiv 0 \pmod{4}$, i.e. $n = 4k_0$. We use the following result generalizing Lemma 2.2.

4.2. Lemma. Let $X^n \subset \mathbb{P}^{M(n,\delta)}$, $\delta = \delta(X) > 0$ be a Scorza variety. Suppose that $n \equiv 0 \pmod{\delta}$ (so that, in accordance with Proposition 1.2, $n = k_0\delta$), let $a, b$ be natural numbers such that $a + b = k_0$, and let $v \in S^a X$ and $w \in S^b X$ be generic points. Then $(Y_v \cdot Y_w)X = 1$.

Proof of Lemma 4.2. First we show that $Y_v \cap Y_w \neq \emptyset$. To do this it suffices to verify that for a generic point $w \in S^b X$ we have

$$S(Y_w, S^{a-1}X) = S^a X.$$ (4.2.1)

We verify equality (4.2.1) by induction on $a$. For $a = 1$, (4.2.1) reduces to formula (2.2.3) proved in Lemma 2.2. Assuming (4.2.1), we show that

$$S(Y_w, S^{a}X) = S^{a+1} X$$

for a generic point $w' \in S^{b-1}X$. Let $x$ be a generic point of $X$, and let $w$ be a generic point of $(w', x) \subset S^b X$. Then from (4.2.1), Theorem 1.4, and Lemma 2.2 it follows that

$$S(Y_w, S^a X) = S \left( Y_w, S(Y_w, S^{a-1}X) \right)$$

$$= S \left( S(Y_w, Y_w), S^{a-1}X \right) = S(Y_w, S^{a-1}X)$$

$$= S \left( Y_w, S(Y_w, S^{a-1}X) \right) = S(Y_w, S^{a}X)$$

$$= S \left( S(Y_w, S^{a-1}X), X \right) = S(S^a X, X) = S^{a+1} X.$$ (4.2.2)
as required.

In view of Proposition 1.2 c) and formula (1.5.4) from Chapter V

\[ s_a = a_{a-1} + n + 1 - a\delta = a_{a-1} + k_0\delta + 1 - a\delta = s_{a-1} + b\delta + 1 = s_{a-1} + \text{dim} \ Y_w + 1, \]

and from (4.2.1) it follows that for generic points \( v \in S^a X, w \in S^b X \) the varieties \( Y_v \) and \( Y_w \) intersect at finitely many points.

To prove Lemma 4.2 it remains to verify that the morphism

\[ \varphi : S_{Y_v, S^a X} \to S^{a+1} X \]

is birational. This is again proved by induction on \( a \), and, in view of the chain of equalities (4.2.2), it suffices to show that in the commutative diagram of rational maps

\[ (4.2.3) \]

generic fibers of the morphism in the bottom are connected. But this is really so since otherwise for a generic point \( v' \in S^{a+1} X \) the intersection \( (Y_{v'} \cdot Y_w)_X \) would consist of several distinct \( \delta \)-dimensional quadrics of the form \( Y_z, z \in S X \) (cf. Corollary 1.5), where by Lemma 2.2

\( (Y_v \cdot (Y_{v'} \cdot Y_w)) = 1 \)

contrary to the induction assumption according to which

\[ (Y_v \cdot (Y_{v'} \cdot Y_w)_{X}) = (Y_v \cdot Y_w)_X = 1. \]

□

We need several results valid for an arbitrary Scorza variety \( X^n \subset \mathbb{P}^{M(n,\delta)} \), \( \delta = \delta(X) > 0 \) such that \( n \equiv 0 \, (\text{mod} \, \delta) \) (recall that by Lemma 2.2 under these assumptions \( S^{k_0-1} X \) is a hypersurface in \( \mathbb{P}^N \)).

4.3. Lemma. Let \( X^n \subset \mathbb{P}^{M(n,\delta)}, \delta = \delta(X) \) be a Scorza variety. Suppose that \( n \equiv 0 \, (\text{mod} \, \delta) \). Let \( 0 \leq a \leq k_0 - 1 \), and let \( v \) be a generic point of \( S^a X \). Then \( \text{deg} \, S^{k_0-1} X = k_0 + 1, \text{mult}_v S^{k_0-1} X = k_0 - a \).

Proof. We argue by induction. Suppose that Lemma 4.3 holds for \( k'_0 < k_0 \). For \( a = k_0 - 1 \) the assertion of the lemma is obvious. Let \( 0 \leq a \leq k_0 - 1, b = k_0 - a \), and let \( v \) be a generic point of \( S^a X \) and \( w \) a generic point of \( S^b X \). By Lemma 4.2,

\[ Y_v \cap Y_w = x \in X. \]

By Theorem 1.4, \( Y_v \) and \( Y_w \) are Scorza varieties, and from the induction assumption it follows that

\[ (x, v) \cap S^{a-1} X = \{x, v'\}, \quad (x, w) \cap S^{b-1} X = \{x, w'\}, \quad (4.3.1) \]
where \( v' \) and \( w' \) are generic points of the varieties \( S^{a-1}X \) and \( S^{b-1}X \) respectively.

Put
\[
u = \langle v, w \rangle \cap \langle v', w' \rangle \in S^{k_0-1}X \quad (4.3.2)
\]
(cf. fig. 1).

Then \( u \) is a generic point of \( S^{k_0-1}X \), and therefore
\[
\text{mult}_v S^{k_0-1}X + \text{mult}_w S^{k_0-1}X + 1 \leq d, \quad (4.3.3)
\]
where
\[
d = \deg S^{k_0-1}X.
\]

We claim that (4.3.3) is actually an equality, i.e.
\[
\text{mult}_v S^{k_0-1}X + \text{mult}_w S^{k_0-1}X + 1 = d. \quad (4.3.4)
\]

To show this it suffices to verify that
\[
\langle v, w \rangle \cap S^{k_0-1}X = v \cup w \cup u.
\]

Suppose that this is not so, and let
\[
u' \in ((v, w) \cap S^{k_0-1}X) \setminus (v \cup w \cup u). \quad (4.3.5)
\]

Set
\[
u'' = \langle v', w' \rangle \cap \langle x, u' \rangle.
\]

Then \( u'' \in S^{k_0-1}X \), and from the genericity assumptions it follows that \( \langle x, u'' \rangle \not\in S^{k_0-1}X \) and
\[
\text{mult}_x S^{k_0-1}X \leq d - 2. \quad (4.3.6)
\]
Thus to prove (4.3.4) it suffices to show that (4.3.6) is impossible, i.e.

\[ \text{mult}_v S^{k_0-1}X = d - 1 \]  
(4.3.7)

we recall that for all points \( v \in S^{k_0-1}X \) we have

\[ \text{mult}_v S^{k_0-1}X < d \]  
(4.3.8)

since by Lemma 2.5 the variety \( S^{k_0-1}X \) is not a cone).

We begin with proving (4.3.4) in the case \( a = 1 \). We need to verify that for a generic point \( v \in SX \)

\[ \text{mult}_v S^{k_0-1}X = d - 2. \]  
(4.3.9)

In fact, by (4.3.1) and (4.3.2), for a generic point \( w \in S^{k_0-1}X \) we get a canonically defined point \( u \in \langle v, w \rangle \cap S^{k_0-1}X \) (cf. fig.1). If \( \text{mult}_v S^{k_0-1}X < d - 2 \), then for a general pair of points \( (v, w) \in SX \times S^{k_0-1}X \) the line \( \langle v, w \rangle \) intersects \( S^{k_0-1}X \) transversely in at least two more points \( u \) and \( u' \) (cf. (4.3.5)). However, since the point \( u \) is chosen canonically, projecting \( S^{k_0-1}X \) from the point \( v \) onto \( \mathbb{P}^{N-1} \) we get a contradiction with the fact that \( S^{k_0-1}X \) is an irreducible hypersurface and the monodromy permutes the points \( u \) and \( u' \).

Now (4.3.9) is proved for a generic point \( v \in SX \). Actually from (4.3.2) and (4.3.8) it follows that (4.3.9) holds for an arbitrary point \( v \in SX \setminus X \). On the other hand, for \( x \in X \)

\[ d - 2 = \text{mult}_v S^{k_0-1}X \leq \text{mult}_x S^{k_0-1}X \leq d - 1. \]

Hence to prove (4.3.7) it suffices to show that

\[ \text{mult}_x S^{k_0-1}X > \text{mult}_v S^{k_0-1}X. \]

Suppose that this is not so, and let \( x \) and \( y \) be generic points of \( X \), \( u \) a generic point of \( S^{k_0-1}X \), and \( \Pi = \langle x, y, u \rangle \) the plane spanned by \( x \), \( y \), and \( u \). Then

\[ \Pi \cap S^{k_0-1}X = \langle x, y \rangle \cup C, \]

where \( C \subset \Pi \) is a curve passing through \( u \), and from (4.3.9) it follows that \( \text{deg} C = 2 \). It is clear that for \( z \in \langle x, y \rangle \cap C \)

\[ \text{mult}_z S^{k_0-1}X = d - 1, \]  
(4.3.10)

and from (4.3.9) it follows that \( z \in X \). Since \( x \) and \( y \) are generic points of \( X \), from (4.3.10) and our assumption it follows that \( z \neq x, y \), i.e., contrary to the trisecant lemma (cf. [39, 2.5; 34, Chapter IV, §3; 64, §7B]), the chord \( \langle x, y \rangle \) intersects \( X \) in at least three points). The resulting contradiction proves (4.3.7) and hence (4.3.4) (we remark that (4.3.7) is a special case of (4.3.4) for \( a = 0 \)).

Next we show that for generic points \( v \in S^aX \), \( w' \in S^{k_0-a-1}X = S^{b-1}X \)

\[ \text{mult}_v S^{k_0-1}X + \text{mult}_{w'} S^{k_0-1}X = d. \]  
(4.3.11)
In fact, it is clear that
\[ d - 1 = \text{mult}_v S^{k_0 - 1}X + \text{mult}_w S^{k_0 - 1}X \leq \text{mult}_v S^{k_0 - 1}X + \text{mult}_w S^{k_0 - 1}X \leq d. \]
Suppose that
\[ \langle v, w' \rangle \cap S^{k_0 - 1}X \ni u, \quad u \neq v, w'. \tag{4.3.12} \]
Then for each point \( x \in Y_v \cap Y_w \) (by Lemma 4.2, \( Y_v \cap Y_w \neq \emptyset \), \( \dim Y_v \cap Y_w \geq (a-1)\delta \)) and a generic point \( w \in \langle x, w' \rangle \) we have
\[ w \in S^b X, \quad \langle v, w \rangle \not\subset S^{k_0 - 1}X, \quad \langle v, w \rangle \cap S^{k_0 - 1}X \ni \tilde{u} = \langle v, w \rangle \cap \langle x, u \rangle \]
(cf. fig. 2, where \( u \in \langle x, u' \rangle, \quad v \in \langle x, v' \rangle, \quad u' \in S^{k_0 - 2}X, \quad v' \in S^{a - 1}X \)).
On the other hand,
\[ \langle v, w \rangle \cap S^{k_0 - 1}X \ni \tilde{u} = \langle v, w \rangle \cap \langle v', w' \rangle \]
(cf. (4.3.2)). From the genericity assumptions it follows that \( \tilde{u} \neq \tilde{u} \), i.e.
\[ \text{mult}_v S^{k_0 - 1}X + \text{mult}_w S^{k_0 - 1}X \leq d - 2 \]
contrary to (4.3.4). Thus assumption (4.3.12) leads to a contradiction. Hence
\[ \langle v, w' \rangle \cap S^{k_0 - 1}X = v \cup w \]
which yields (4.3.11).
From the system of equations (4.3.4) and (4.3.11) it follows that if \( v \in S^a X, \quad v' \in S^{a - 1}X \) are generic points, \( 1 \leq a \leq k_0 - 1 \), then
\[ \text{mult}_{v'} S^{k_0 - 1}X = \text{mult}_v S^{k_0 - 1}X + 1. \tag{4.3.13} \]
Since for a generic point \( u \in S^{k_0 - 1}X \) we have \( \text{mult}_u S^{k_0 - 1}X = 1 \), successive application of (4.3.13) shows that for a generic point \( v \in S^a X \)
\[ \text{mult}_v S^{k_0 - 1}X = k_0 - a, \quad 0 \leq a \leq k_0 - 1. \tag{4.3.14} \]
Combining (4.3.4) and (4.3.14), we get
\[ d = (k_0 - a) + (k_0 - b) + 1 = k_0 + 1. \tag{4.3.15} \]
\[ \Box \]
4.4. Lemma. Let $X^n \subset \mathbb{P}^{M(n, \delta)}$, $\delta = \delta(X) > 0$ be a Scorza variety. Suppose that $n \equiv 0 \pmod{\delta}$, and let $u$ be a generic point of the hypersurface $S^{k_0-1}X$. Then the projection $\pi_u: X \dashrightarrow \mathbb{P}^n$ with center at the subspace $\mathbb{P}^{N-n-1} = \langle Y_u \rangle = S^{k_0-1}Y_u$ is a birational isomorphism. More precisely, let $H_u = X \cap L_u$, where $L_u = T_{S^{k_0-1}X,u}$. Then $\pi_u \mid X \setminus H_u$ is an isomorphism.

Proof. For a point $x \in X$ we set $P_{u,x} = \langle x, P_u \rangle$. We observe that for $x \notin H_u$ we have $P_{u,x} \not\subset S^{k_0-1}X$. In fact, if $P_{u,x} \subset S^{k_0-1}X$, then

$$x \in X \cap T(P_u, S^{k_0-1}X) \subset L_u \cap X = H_u.$$ 

It is clear that the hypersurface $S^{k_0-1}X \cap P_{u,x} \subset P_{u,x}$ contains the linear subspace $P_u$ and the cone $S(x, S^{k_0-2}Y_u)$ as its components. From Lemma 4.3 it follows that

$$1 + \operatorname{deg} S^{k_0-2}Y_u = \operatorname{deg} (S^{k_0-1}X \cap P_{u,x}) \leq \operatorname{deg} S^{k_0-1}X = k_0 + 1. \quad (4.4.1)$$

On the other hand, in view of Theorem 1.4, Lemma 4.3 can also be applied to the Scorza variety $Y_u$, $\dim Y_u = (k_0 - 1)\delta$, $k_0(Y_u) = k_0 - 1 \geq 2$ (the case $k_0 = 2$ was already considered in Chapter IV), so that

$$\operatorname{deg} (S^{k_0-2}Y_u) = k_0. \quad (4.4.2)$$

From (4.4.2) it follows that the inequality (4.4.1) is actually an equality, and therefore

$$S^{k_0-1}X \cap P_{u,x} = P_u \cap S(x, S^{k_0-2}Y_u). \quad (4.4.3)$$

Since

$$S(y, S^{k_0-2}Y_u) \subset S^{k_0-1}X \cap P_{u,x},$$

for each point $y \in (P_{u,x} \setminus P_u) \cap X$, from (4.4.3) it follows that

$$X \cap P_{u,x} = (X \cap P_u) \cup x = Y_u \cup x, \quad (4.4.4)$$

i.e. $\pi_u^{-1}(\pi_u(x)) = x$, which implies our claim. \(\square\)

We turn to a more detailed description of the hyperplane section $H_u = L_u \cdot X$ (we recall that from Corollary 1.15a) in Chapter I it follows that if $\delta > 1$, then $H_u = L_u \cap X$ is a reduced variety). From Lemma 2.2 it follows that for a generic point $z \in SX \cap Y_u$ consists of a single point $y$. Furthermore,

$$(H_u \cdot Y_z)_X = (L_u \cdot Y_z)_{PN} = (T_{Y_z,y} \cdot Y_z)_{P_z}$$

is a cone with vertex $y$ over a nonsingular $(\delta - 2)$-dimensional quadric.

Let $y \in Y_u$ be a generic point, and let $C_y = \bigcup_{z} (H_u \cdot Y_z)_X$, where $z$ runs through the set of general points of the cone $S(y, X)$. Then $C_y$ is a cone with vertex $y$ whose base is an irreducible variety $B_y$ such that

$$2 \dim B_y - 2(\delta - 2) = n - \delta$$
4. SCORZA VARIETIES WITH $\delta = 4$

(here $n - \delta = [(n + 1) - (\delta + 1)]$ is the dimension of the family of quadrics $Y_z$ passing through $y$), i.e.

$$\dim B_y = \frac{n + \delta - 4}{2}, \quad \dim C_y = \frac{n + \delta - 2}{2}.$$ 

Varying $y$ in the set of general points of $Y_u$, we obtain an irreducible variety $C^u = \bigcup_y C_y \subset H_u$. Furthermore, if $x_1, x_2$ is a general pair of points of $C^u$ and $z$ is a general point of $(x_1, x_2)$, then $Y_z \cap L_u$ is the cone over a nonsingular $(\delta - 2)$-dimensional quadric with vertex at the (unique) intersection point $y \in Y_z \cap Y_u$. In other words, $B^u_{x_i} \cap B^u_{x_2} = y$, where

$$B^u_{x_i} = \{y \in Y_u \, | \, \langle x_i, y \rangle \subset X\}, \quad i = 1, 2. \quad (4.4.5)$$

From this it follows that

$$\dim B^u_{x_i} = \frac{\dim Y_u}{2}, \quad (4.4.6)$$

i.e. a generic point $x \in C^u$ is contained in a $(\frac{n-\delta}{2})$-dimensional family of lines of the above type. Hence

$$\dim C^u = (n - \delta) + \frac{n + \delta - 2}{2} - \frac{n - \delta}{2} = n - 1,$$

i.e. $C^u$ consists of components of $H_u$. But from Corollary 1.15 b) in Chapter I it follows that for $\delta > 2$ the hyperplane section $H_u$ is irreducible. Taking into account that each line in $X$ passing through $y$ is contained in $T_{X,y}$, we can sum up the preceding discussion in the following lemma.

4.5. Lemma. In the conditions of Lemma 4.4 suppose in addition that $\delta > 2$. Then $H_u = X \cap L_u = X \cap T(Y_u, X) = C^u$ coincides with the closure of a union of all lines lying in $X$ and intersecting $Y_u$.

4.6. We return to the case $\delta = 4, n \equiv 0 \pmod{4}$. Let $x$ be a generic point of $H_u$. Put $D_x = \bigcup_z Y_z$, where $z$ runs through the set of general points of the cone $S(x, Y_u)$. Since for a generic point $y \in Y_u$ and a generic point $z \in \langle x, y \rangle$

$$Y_z \cap L_u \supset (T_{Y_z,y} \cap Y_z)$$

and

$$Y_z \cap L_u \ni x, \quad x \notin T_{Y_z,y} \cap Y_z,$$

we see that $Y_z \subset L_u$ and therefore

$$Y_u \subset D_x \subset H_u, \quad (4.6.1)$$

where from the definition of $D_x$ and the proof of Lemma 4.5 it follows that both inclusions are strict. Since

$$\dim D_x = (n - 4) + 4 - \dim Y_z \cap Y_u = \dim H_u - (\dim Y_z \cap Y_u - 1),$$
we conclude that \( \dim Y_z \cap Y_u = 2 \) and \( D_z \) is a divisor on \( H_u \) (arguing as in the proof of Proposition 3.2 in Chapter IV). It is easy to show that \( Y_z \cap Y_u = \mathbb{P}^2 \).

If \( x' \in D_x \) is another generic point, then \( D_{x'} = D_x \). In fact, let \( a \in D_x \) be a generic point, and let \( x' \) be a generic point of the line \( \langle a, x' \rangle \). Then \( Y_{x'} \) is a nonsingular four-dimensional quadric,

\[
x' \in Y_{x'}, \quad Y_{x'} \cap Y_u \neq \emptyset, \quad a \in Y_{x'} \subset D_x.
\]

Thus, varying \( x \in H_u \), we obtain a one-dimensional family of divisors \( D_x \subset H_u \). From the proof of Lemma 4.5 it follows that the base of this family is an image of the line \( \langle x, x' \rangle \), where \( x, x' \) is a general pair of points of \( H_u \), so that the family \( \{D_x\} \) is rational.

If \( x_0, \ldots, x_i, 0 \leq i \leq k_0 - 1 \) is a general collection of points of \( D_x \) and \( u \) is a generic point of the \( i \)-dimensional linear subspace \( \langle x_0, \ldots, x_i \rangle \), then

\[
Y_u \subset D_x.
\]

In particular, if \( u' \) is a generic point of \( S^{k_0-1}D_x \), then \( Y_{u'} \subset D_x \subset H_{u'} \), so that

\[
D_x \subset \bigcap_{u' \in S^{k_0-1}D_x} H_{u'}.
\]

From (4.6.3) and formulas (1.5.4) and (1.5.8) in Chapter V it follows that

\[
\dim S^{k_0-1}D_x = \dim S^{k_0-2}D_x + [(n - 1) - (n - 4)]
\]
\[
= \dim S^{k_0-3}D_x + [(n - 1) - (n - 4)] + [(n - 1) - (n - 8)]
\]
\[
= \cdots = (4k_0 - 2) + 3 + 7 + \cdots + [3 + 4(k_0 - 2)] = 2k_0^2 + k_0 - 1.
\]

By Theorem 1.4, the hyperplane \( L_{u'} \) is tangent to \( S^{k_0-1}X \) along the linear subspace

\[
P_{u'} = S^{k_0-1}Y_{u'}, \quad \dim P_{u'} = N - n - 1 = 2k_0^2 - k_0 - 1.
\]

Hence, varying \( u' \in S^{k_0-1}D_x \), we obtain a \([2k_0^2 + k_0 - 1] = (2k_0^2 - k_0 - 1) \] \( 2k_0 \)-dimensional family of hyperplane sections \( H_{u'} \) of the variety \( X \) passing through the \((n - 2)\)-dimensional subvariety \( D_x \subset H_u \). It is clear that this family cuts a \((2k_0 - 1)\)-dimensional family of hyperplane sections of the variety \( H_u \) containing \( D_x \) as an irreducible component.

By Barth’s theorem (cf. [6; 33]), \( H^1(X, \mathcal{O}_X) = 0 \), and the exact sequence

\[
0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(1)) \to H^0(H_u, \mathcal{O}_{H_u}(1)) \to H^1(X, \mathcal{O}_X(1))
\]
shows that the variety \( H_u \subset \mathbb{P}^{N-1} \) is linearly normal (the Scorza variety \( X \subset \mathbb{P}^N \) is linearly normal by definition).

Thus we obtain two linear systems \( I = |D_x| \) and \( II = |(H_u \cdot H_u) - D_x| \) on the variety \( H_u \) whose fundamental subset coincides with \( Y_u \). We have already shown that

\[
\dim I \geq 1, \quad \dim II \geq 2k_0 - 1.
\]

Since \( H_u \) is linearly normal,

\[
\dim H^0(H_u, \mathcal{O}_{H_u}(1 - Y_u)) = \dim \pi_u(H_u) = n - 1 = 4k_0 - 1
\]

(4.6.5)

(here \( |\mathcal{O}_{H_u}(1 - Y_u)| \) is the linear system of hyperplane sections of \( H_u \) passing through \( Y_u \)). From (4.6.5) and (4.6.6) it follows that

\[
\dim I = 1, \quad \dim II = 2k_0 - 1.
\]

Furthermore, the linear system \( I \) maps \( H_u \) onto \( \mathbb{P}^1 \), the linear system \( II \) maps \( H_u \) onto \( \mathbb{P}^{2k_0-1} \), and the projection \( \pi_u \) with center in \( P_u \) defined by the linear system \( |\mathcal{O}(1 - Y_u)| \) maps \( H_u \) onto \( \mathbb{P}^1 \times \mathbb{P}^{2k_0-1} \). In the proof of Lemma 4.5 we actually showed that the fibers of \( \pi_u \) are cones

\[
\pi_u^{-1}(\pi_u(x)) = P_{u,x} \cap X = C_x^u
\]

with vertex \( x \in H_u \setminus Y_u \) and base \( B_x^u \),

\[
\dim B_x^u = \frac{\dim Y_u}{2} = 2k_0 - 2, \quad \dim C_x^u = 2k_0 - 1.
\]

Furthermore, arguing as in the proof of Proposition 3.2 in Chapter IV or using induction on \( n \), it is easy to verify that \( B_x^u = C_x^u \cap Y_u \) is a linear subspace and therefore the fibers of the projection \( \pi_u \) are \((2k_0 - 1)\)-dimensional linear subspaces.

Summing up the above discussion, we get the following result.

4.7. Lemma. In the conditions of Lemma 4.5 suppose in addition that \( \delta = 4 \). Then \( \pi_u(H_u) \subset \pi_u(L_u) \subset \pi_u(\mathbb{P}^N) = \mathbb{P}^1 \times \mathbb{P}^{\frac{N}{2}-1} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \) (the Segre embedding), and a generic fiber of the map \( \pi_u \) is \( \mathbb{P}^1 \times \mathbb{P}^{\frac{N}{2}-1} \) is a linear subspace of \( \mathbb{P}^{\frac{N}{2}-1} \).

4.8. Consider the map

\[
\sigma_u = \pi_u^{-1} : \mathbb{P}^n \rightarrow \mathbb{P}^{\frac{n(n+6)}{8}}, \quad \sigma_u(\mathbb{P}^n) = X.
\]

From Lemma 4.7 it follows that \( \sigma_u \) is defined by a linear system of hypersurfaces in \( \mathbb{P}^n \) passing through \( \mathbb{P}^1 \times \mathbb{P}^{\frac{N}{2}-1} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n \). Since \( \mathbb{P}^1 \times \mathbb{P}^{\frac{N}{2}-1} \) is defined in \( \mathbb{P}^n \) by \( \left( (n + 1) + \left( \frac{5}{2} \right) \right) = \frac{n^2 + 6n + 8}{8} = N + 1 \) quadratic equations, from the linear normality of \( X \) it follows that \( \sigma_u \) is defined by the linear system of quadrics in \( \mathbb{P}^n \) passing through \( \mathbb{P}^1 \times \mathbb{P}^{\frac{N}{2}-1} \).

Theorem 4.1 in the case \( n \equiv 0 \mod 4 \) now follows from the characterization of Grassmannians (cf. [81] and also §3 of Chapter III).
4.9. Lemma. Let $X^n \subset \mathbb{P}^{M(n,4)}$, $\delta(X) = 4$ be a Scorza variety. Then $n \neq 1$ (mod 4).

Proof. Suppose the converse, and let $u$ be a generic point of $\mathbb{P}^N = S^{k_0}X$,

$$N = M(n,4) = f(k_0) = \frac{n^2 + 6n + 1}{8}.$$ 

By Theorem 1.4, $Y_u = X \cap \mathbb{P}_u$ is a Scorza variety of dimension $4k_0 = n - 1$ in the linear subspace $\mathbb{P}_u = \langle Y_u \rangle \subset \mathbb{P}^N$, $\dim \mathbb{P}_u = \frac{n^2 + 6n - 5}{8} < N - 1$. Hence for a general hyperplane $L \supset \mathbb{P}_u$ we have $L \cdot X > Y_u$ which contradicts the Barth-Larsen theorem according to which $H_{2a-2}(X,\mathbb{Z})$ and Pic $X$ are infinite cyclic groups generated by the classes of hyperplane section (cf. [54; 65]). This contradiction proves Lemma 4.9. □

4.10. We turn to the case $n \equiv 2$ (mod 4). Let

$$X^n \subset \mathbb{P}^N, \quad n \equiv 2 \pmod{4}, \quad N = \frac{n(n+6)}{8} \quad (4.10.1)$$ 

be a Scorza variety. According to Theorem 1.4, if $u \in \mathbb{P}^N = S^{k_0}X$ is a generic point, then $Y_u = X \cap \mathbb{P}_u$ is a Scorza variety of dimension $4k_0 = n - 2$ in the linear subspace $\mathbb{P}_u = \langle Y_u \rangle \subset \mathbb{P}^N$,

$$\dim \mathbb{P}_u = \frac{(n-2)(n-4)}{8} = N - \frac{n}{2} - 1. \quad (4.10.2)$$

Let $u' \in \mathbb{P}^N$ be another generic point. Then

$$Y_{u'} = X \cap \mathbb{P}_{u'}, \quad Y_u \cap Y_{u'} = Y_u \cap \mathbb{P}_{u'}, \quad (4.10.3)$$

and since $\dim Y_u + \dim \mathbb{P}_{u'} = (n-2) + (N - \frac{n}{2} - 1) > N$, we see that $Y_u \cap Y_{u'} \neq \varnothing$. From Theorem 1.4 it immediately follows that

$$\delta(Y_u \cap Y_{u'}) = \delta(Y_u) = \delta(X) = 4. \quad (4.10.4)$$

Furthermore, for a general pair of points $u, u' \in \mathbb{P}^N$ we have

$$\dim Y_u \cap Y_{u'} = n - 4. \quad (4.10.5)$$

In fact, if we had

$$\dim Y_u \cap Y_{u'} = \dim Y_u \cap \mathbb{P}_{u'} = n - 3 = \dim Y_u - 1,$$

then $Y_u \cap Y_{u'}$ would coincide with a hyperplane section of $Y_u$ which contradicts (4.10.4).

From Theorem 1.4 and the already proven part of Theorem 4.1 it follows that

$$Y_u = G(2k_0 + 1, 1). \quad (4.10.6)$$
From (4.10.3), (4.10.4), (4.10.5), and (4.10.6) it is easy to deduce that
\[ Y = Y_u \cap Y_u' = G(2k_0, 1) \] (4.10.7)
is the Schubert cycle in \( Y = G(2k_0 + 1, 1) \) parametrizing the lines contained in the hyperplane \( \mathbb{P}^{2k_0} \subset \mathbb{P}^{2k_0+1} \). It also easily follows that
\[ P_u \cap P_{u'} = \langle Y \rangle = \mathbb{P}_Y, \quad \dim \mathbb{P}_Y = \frac{(n-4)(n+2)}{8} = N - n - 1. \]

Put\[ L = \langle P_u, P_{u'} \rangle, \quad \dim L = 2(N - \frac{n}{2} - 1) - (N - n - 1) = N - 1. \]

From the definition it follows that the hyperplane \( L \) is tangent to \( X \) along the subvariety \( Y \), i.e. \( T(Y, X) \subset L \). Denote by \( H \) the hyperplane section \( X \cap L \) and by \( \pi_Y : X \rightarrow \mathbb{P}^n \) the projection with center at \( \mathbb{P}_Y \).

Varying a generic point \( u'' \in \mathbb{P}^N \), we obtain an \( (N - \dim \mathbb{P}_{u''}) = \left( \frac{n}{2} - 1 \right) - \)dimensional rational family of subvarieties \( Y_{u''} \subset X \). Furthermore, if
\[ Y_u \cap Y_{u''} = Y_u \cap Y_u' = Y, \] (4.10.8)
then \( \langle P_u, P_{u''} \rangle = \langle P_u, P_{u'} \rangle = L \).

In this case
\[ Y_{u''}^{n-2} \subset H^{n-1}, \] (4.10.9)
and it is clear that a generic point of \( H \) is contained only in a finite number of subvarieties \( Y_{u''} \) satisfying condition (4.10.8). Since on the Grassmannian \( G(2k_0 + 1, 1) \) there is a \( \dim \mathbb{P}^{2k_0+1} = (2k_0 + 1) = \frac{n}{2} \) -dimensional family of Grassmannians \( G(2k_0 + 1) \), from (4.10.7) it follows that a general variety \( Y = Y_u \cap Y_u' = G(2k_0, 1) \) is contained in a one-dimensional rational family of \( Y_{u''} \) and
\[ H = \bigcup_{Y_{u''} \cap Y_u = Y} Y_{u''} \] (4.10.10)
(cf. (4.10.9)). A word-for-word repetition of the arguments used in the proof of Lemma 4.7 (with the linear system \( |D_x|, Y_u \subset D_x \subset H_u \) replaced by the linear system \( |Y_{u''}|, Y \subset Y_{u''} \subset H \)) shows that
\[ \pi_Y \big|_H : H \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^n \subset \mathbb{P}^n = \pi_Y(X) \] (4.10.11)
is a rational fiber bundle with fiber \( \mathbb{P}^2 \).

We claim that for a generic point \( x \in X \setminus H \)
\[ \mathbb{P}_{Y,x} \cap X = Y \cup x, \] (4.10.12)
where \( \mathbb{P}_{Y,x} = \langle \mathbb{P}_Y, x \rangle \). Suppose that this is not so, and let
\[ y \in \mathbb{P}_{Y,x} \cap X, \quad y \in Y, \quad y \neq x. \] (4.10.13)
Put \( z = \langle x, y \rangle \cap \mathbb{P}_Y \), and let \( v \) be a generic point of \( \mathbb{P}_Y \) and \( v' \in \langle z, v \rangle \) a generic point of the line \( \langle z, v \rangle \). Put

\[ \tilde{u} = \langle x, v \rangle \cap \langle y, v' \rangle \]

(cf. fig. 3).

It is easy to see that \( \tilde{u} \) is a generic point of \( \mathbb{P}_N \). Since \( v, v' \in \mathbb{P}_Y = S^{h_0-1}_Y \subset S^{h_0-1}_X \),

\[ x, y \in Y_{\tilde{u}} \supset Y_v, Y_v'. \]  

(4.10.14)

From (4.10.7) it follows that if \( \mathbb{P}_w = S^{h_0-1}_Y \), then \( \bigcap_{w \in \mathbb{P}_v} \mathbb{P}_w = \emptyset \), and so we may assume that

\[ Y_{v'} \neq Y_v. \]  

(4.10.15)

On the other hand,

\[ Y_{\tilde{u}} \cap Y = Y_{\tilde{u}} \cap Y_u \cap Y_{u'} = (Y_{\tilde{u}} \cap Y_u) \cap (Y_{u'} \cap Y_u). \]  

(4.10.16)

Since \( x \notin H \), \( Y_{\tilde{u}} \supset Y \) and as was shown above \( Y_{\tilde{u}} \cap Y_u \) and \( Y = Y_{v'} \cap Y_u \) are two distinct subgrassmannians of the form \( G(\frac{n}{2}-1,1) \) in \( Y_u = G(\frac{n}{2},1) \). From (4.10.16) it follows that \( Y_{\tilde{u}} \cap Y \) is a Grassmann variety of the form \( G(\frac{n}{2}-2,1) \).

From (4.10.14) it follows that

\[ Y_{\tilde{u}} \cap Y \supset Y_{v'}, Y_{v'}. \]  

(4.10.17)

where in view of the already proven case of Theorem 4.1 each of the varieties \( Y^{n-6}_{\tilde{u}}, Y^{n-6}_{v'} \) is also a Grassmann variety of the form \( G(\frac{n}{2} - 2, 1) (n - 6 \equiv 0 \text{ (mod 4)}) \). Thus assumption (4.10.13) leads to a contradiction ((4.10.17) is incompatible with (4.10.15)) which proves (4.10.12).

Thus \( \pi_Y \big|_{X \setminus H} \) is a birational isomorphism. Consider the inverse map

\[ \sigma_Y = \pi_Y^{-1} : \mathbb{P}^n \dasharrow \mathbb{P}^{\frac{n(n+6)}{2}}, \quad \sigma_Y(\mathbb{P}^n) = X. \]
From the already proven case of Theorem 4.1 it follows that for a generic point $u \in \mathbb{P}^N$

$$\pi_Y(Y_u) = \mathbb{P}^n \subset \mathbb{P}^n,$$

and if $\mathbb{P}^{n-1} = \langle \pi_Y(H) \rangle$, then

$$\mathbb{P}^u \cap \mathbb{P}^{n-1} = \langle \mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 2} \rangle,$$

$$\mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 2} = \pi_Y(Y_u \cap H) \subset \pi_Y(H) = \mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 1}.$$  

Suppose that $\sigma_Y$ is defined by $\frac{n^2 + 6n + 8}{8}$ forms $G_0, \ldots, G_{\frac{n(n+6)}{2}}$, deg $G_i = d$, $i = 0, \ldots, \frac{n(n+6)}{2}$. As it was shown in 4.8, after canceling the greatest common divisor $G_i |_{Y_u}$ become quadratic forms. Varying generic point $u \in \mathbb{P}^N$, we see that $d = 2$.

Since $\mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 1}$ is defined in $\mathbb{P}^n$ by $(n + 1) + \left(\frac{3}{2}\right) = \frac{n^2 + 6n + 8}{8} = N + 1$ quadratic equations and

$$G_i |_{\mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 1}} = 0, \quad i = 0, \ldots, \frac{n(n+6)}{8},$$

from this it follows that $\sigma_Y$ is defined by the linear system of quadrics in $\mathbb{P}^n$ passing through $\mathbb{P}^1 \times \mathbb{P}^{\frac{3}{2} - 1}$.

Theorem 4.1 in the case $n \equiv 2$ (mod 4) now follows from the characterization of Grassmann varieties (cf. [81] and §3 of Chapter III).

It remains to consider the case $n \equiv 3$ (mod 4).

4.11. Lemma. Let $X^m \subset \mathbb{P}^{M(n, 4)}$, $\delta(X) = 4$ be a Scorza variety. Then $n \not\equiv 3$ (mod 4).

Proof. Suppose that the lemma does not hold, and let $u$ be a generic point of $\mathbb{P}^N = S^{k_0} X$,

$$k_0 = \frac{n - 3}{4}, \quad N = M(n, 4) = f(k_0) = \frac{n^2 + 6n - 3}{8},$$

and $u'$ a generic point of $S(Y_u, S^{k_0 - 1} X)$. Then $u'$ is a generic point of $\mathbb{P}^N$, and (by Theorem 1.4) $Y_u$ and $Y_{u'}$ are Scorza varieties of dimension $n - 3$. From the already proven case of Theorem 4.1 it follows that $Y_u$ and $Y_{u'}$ are projectively isomorphic to the Grassmann variety of lines in $\mathbb{P}^{\frac{3}{2}}$. Denote by $Y$ the intersection $Y_u \cap Y_{u'}$. It is clear that

$$\dim Y \geq n - 6, \quad \delta(Y) = 4, \quad (4.11.1)$$

and if $z \in SY \setminus Y$, then $Y_z \subset Y$. Geometrically this means that if the subvariety $Y \subset G(\frac{n+1}{2}, 1)$ contains a pair of points $\alpha_1, \alpha_2$ corresponding to non-coplanar lines $l_1, l_2 \subset \mathbb{P}^{\frac{n+1}{2}}$, then for each line $l \subset (l_1, l_2)$ the subvariety $Y$ contains a point $\alpha \in G(\frac{n+1}{2}, 1)$ corresponding to this line. From this it follows that

$$Y = G(m, 1), \quad (4.11.2)$$

where $\mathbb{P}^m \subset \mathbb{P}^{\frac{n+1}{2}}$ is a linear subspace. It is clear that the only subvariety $Y \subset G(\frac{n+1}{2}, 1)$ satisfying conditions (4.11.1) and (4.11.2) is the Grassmann subvariety $Y = G(\frac{n+3}{2}, 1)$.
We observe that if $u' \in \angle x, v \rangle$, where $x \in Y, v \in S^{k_0-1}X$ are generic points and $z$ is a generic point of $Y_u' = p_0^{1,k_0-3}\left((\varphi^{1,k_0-3})^{-1}(v)\right)$, then $z$ is a generic point of $S X$,

\[
\{Y_z \hookrightarrow Y_v \hookrightarrow Y_u\} = \left\{G(3,1) \hookrightarrow G\left(\frac{n-5}{2},1\right) \hookrightarrow G\left(\frac{n-1}{2},1\right)\right\},
\]

\[
\{Y \hookrightarrow Y_{u'}\} = \left\{G\left(\frac{n-3}{2},1\right) \hookrightarrow G\left(\frac{n-1}{2},1\right)\right\},
\]

and therefore for a generic pair of points $u \in S^{k_0-1}X, z \in SX$

\[
Y_u \cap Y_z = G(2,1) = \mathbb{P}^2.
\] (4.11.3)

But from (4.11.3) it follows that

\[
S(Y_u, X) = SX, \quad \dim \varphi^{-1}_X(z) = 2
\]

while we already know that for a generic point $z \in SX$

\[
\dim \varphi^{-1}_X(z) = \dim Y_u + n + 1 - \dim SX = 1.
\]

$\square$

Now all the four cases in Theorem 4.1 are verified, and the proof of the theorem is complete. $\square$

4.12. Remark. For $\delta = 4, n \leq 7$ we have $k_0 = 1$, and each variety $X^n \subset \mathbb{P}^{2n-3}$ is extremal. In other words,

\[
M(7,4) = m(7,4) = 11, \quad M(6,4) = m(6,4) = 9,
\]

\[
M(5,4) = m(5,4) = 7, \quad M(4,4) = m(4,4) = 5.
\]

4.13. Corollary. Let $X^n \subset \mathbb{P}^r$ be a nonsingular variety, $r \leq 2n - 3$. Then $h^0(X, O_X(1)) \leq \left\lfloor \frac{(n+3)^2}{8} \right\rfloor$ with equality holding if and only if $r = 2n - 3$ and either $n \leq 7$ or $n \equiv 0 \pmod{2}$ and $X \hookrightarrow \mathbb{P}^r$ is the embedding of the Grassmann variety $G\left(\frac{n}{2} + 1,1\right) \simeq X$ defined by a collection $(Q_0 : \cdots : Q_{2n-3})$ of $2n - 2$ linear forms of the Plücker coordinates.
5. End of classification of Scorza varieties

It remains to classify Scorza varieties with $\delta = 8$.

5.1. Lemma. Let $X^n \subset \mathbb{P}^{M(n,8)}$, $\delta(X) = 8$ be a Scorza variety. Then $k_0 = \left[\frac{n}{8}\right] = 2$.

Proof. Suppose that $k_0 \geq 3$, and let $v$ be a generic point of $S^3 X$. From Theorem 1.4 it follows that $Y^{24}$ is a Scorza variety with $\delta(Y) = 8$. Hence to prove Lemma 5.1 it suffices to verify that a 24-dimensional variety with $\delta = 8$ does not exist.

In fact, if $X$ were such a variety, then we would have an ascending chain of secant varieties $X^{24} \subset (S^2 X)^{41} \subset S^3 X = \mathbb{P}^{51}$.

Let $u$ be a generic point of $S^2 X$. Put $L_u = T_{S^2 X,u}$, $H_u = (L_u \cdot X) \cap \mathbb{P}^{51} = L_u \cap X$.

Let $x \in H_u$ be a generic point. In the proof of Lemma 4.5 it was shown that $B_u^x = \{ y \in Y_u \mid \langle x, y \rangle \subset X \}$ is a $\dim Y_u - 8$-dimensional subvariety of $Y_u$ (cf. (4.4.5), (4.4.6)). As in 4.6, we see that $B_u^x = SB_u^x$, and therefore $B_u^x$ is a linear subspace (cf. the proof of Lemma 3.6 in Chapter IV). Hence in order to prove Lemma 5.1 it suffices to verify that the Severi variety $Y_u = E^{16} \subset \mathbb{P}^{24}$ does not contain eight-dimensional linear subspaces.

We claim that $E^{16}$ actually does not contain even six-dimensional linear subspaces (we recall that for $z \in SE \setminus E$ we have $\pi_z^{-1}(\pi_z(x)) = \mathbb{P}^5$ for each point $x \in H_z = L_z \cap X$, $x \not\in Y_z$, so that $E$ contains five-dimensional linear subspaces; cf. also [13]). In fact, if $E^{16} \supset \mathbb{P}^6 \supset x$, then

$$T_{E,x} \cap E \supset \mathbb{P}^6.$$  \hfill (5.1.1)

In view of the results of §2 of Chapter III (cf. also Chapter IV, 4.2 c), 4.3), $T_{E,x} \cap E$ is a cone with vertex $x$ and base $S^{10}$, where $S^{10} \subset \mathbb{P}^{15}$ is the spinor variety corresponding to the orbit of highest weight vector of the spinor representation of the group Spin$_{10}$. Hence from (5.1.1) it follows that $S^{10} \supset \mathbb{P}^5$. But for an arbitrary point $y \in \mathbb{P}^5 \subset S^{10}$

$$T_{S,y} \cap S \supset \mathbb{P}^5$$  \hfill (5.1.2)

is a cone with vertex $y$ and base $G(4,1) \subset \mathbb{P}^9$ (cf. §2 in Chapter III). Hence from (5.1.2) it follows that $G(4,1) \supset \mathbb{P}^4$. But for an arbitrary point $\alpha \in \mathbb{P}^4 \subset G(4,1)$

$$T_{G(4,1),\alpha} \cap G(4,1) \supset \mathbb{P}^4$$  \hfill (5.1.3)

is a cone with vertex $\alpha$ and base $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (cf. again §2 of Chapter III). Hence (5.1.3) would imply that $\mathbb{P}^1 \times \mathbb{P}^2 \supset \mathbb{P}^3$ which is clearly impossible. This completes the proof of Lemma 5.1. \hfill \Box

From Lemma 5.1 it follows that the dimension $n$ of a Scorza variety $X^n$ with $\delta(X) = 8$ satisfies the inequalities

$$16 \leq n \leq 23.$$
5.2. Lemma. Let $X^n$ be a Scorza variety such that $\delta(X) = 8$ and $k_0(X) = 2$. Then $n = 16$ and $X = E$ is a Severi variety.

Proof. Suppose that $17 \leq n \leq 23$. Then $N = M(n, 8) = f(2) = 26 + 3\varepsilon$, $\varepsilon = n \mod 8$, and for a generic point $u \in S^2X = \mathbb{P}^N$

$$\{Y_{u}^{16} \subset \mathbb{P}^{26}\} = \{E^{16} \subset \mathbb{P}^{26}\}$$

is a sixteen-dimensional Severi variety (cf. Theorem 1.4). Let $u'$ be a generic point of $S(Y_u, SX)$. Then $u'$ is a generic point of $\mathbb{P}^N$, and $Y_{u'}$ is a sixteen-dimensional Severi variety.

Put $Y = Y_u \cap Y_{u'}$. It is clear that $Y$ is a nonsingular variety,

$$9 = 32 - 23 \leq \dim Y \leq 15, \quad (5.2.1)$$

and if $z \in SY \setminus Y$, then $Y_z \subset Y$ (cf. the proof of Lemma 4.11). From (5.2.1) and Corollary 2.11 in Chapter II it follows that

$$SY = \mathbb{P}^r, \quad r = 2\dim Y - 7 \geq 11,$$

and therefore

$$SE \supset \mathbb{P}^{11}. \quad (5.2.2)$$

But according to the results of §2 of Chapter III and Remark 2.5 in Chapter IV $(SE)^* \simeq E$ and each hyperplane $T_{SE,z} (z \in SE \setminus E)$ is tangent to $SE$ along the linear subspace $\mathbb{P}_z^9 = \langle Y_z \rangle$. Since for $z \in \mathbb{P}^{11} \setminus E$

$$T_{SX,z} \supset \mathbb{P}^{11}, \quad (5.2.3)$$

from (5.2.2) and (5.2.3) it follows that $\mathbb{P}^{11} \subset E$. But in the proof of Lemma 5.1 we verified that the variety $E$ does not contain even six-dimensional linear subspaces. The resulting contradiction proves Lemma 5.2 (the non-existence of Scorza varieties $X^n$ with $17 \leq n \leq 19$ can be more easily deduced from the Barth-Larsen theorem [54], but we preferred to give a more uniform proof). □

Combining the assertions of Lemmas 5.1 and 5.2 and Theorem 4.7 in Chapter IV we obtain the following result.

5.3. Theorem. Let $X^n \subset \mathbb{P}^N$, $n \geq 16$ be a nonsingular nondegenerate variety. Suppose that $s_X < 2n - 6$. Then $N \leq \frac{n(n+10)+6(6-\varepsilon)}{16}$, $\varepsilon = n \mod 4$. Furthermore, equality holds if and only if $X = E \subset \mathbb{P}^{26}$ is a sixteen-dimensional Severi variety corresponding to the orbit of highest weight vector of the simplest representation of the group $E_6$. In other words, $M(16, 8) = 26$ and $M(n, 8) < \frac{n(n+10)+6(6-\varepsilon)}{16}$ for $n > 16$.

5.4. Remark. For $\delta = 8$, $8 \leq n \leq 15$ we have $k_0 = 1$ and each variety $X^n \subset \mathbb{P}^{2n-7}$ is extremal. In other words,

$$M(n, 8) = m(n, 8) = 2n - 7, \quad 8 \leq n \leq 15.$$
5. END OF CLASSIFICATION OF SCORZA VARIETIES

5.5. Corollary. Let $X^n \subset \mathbb{P}^r$ be a nonsingular variety, $r \leq 2n - 7$. Then $h^0(X, O_X(1)) \leq \frac{(n+5)^2}{10}$ with equality holding if and only if $r = 2n - 7$ and either $n \leq 14$ or $n = 16$ and $X \subset \mathbb{P}^2$. An isomorphic projection of the sixteen-dimensional Severi variety $E^{16} \subset \mathbb{P}^2$.

Combining Theorems 2.1, 3.1, 4.1, and 5.3 and taking into account properties of the function $f$ depicted in Fig. 3 in Chapter V, we obtain the following

5.6. Classification theorem. Let $X^n \subset \mathbb{P}^N$ be a nonsingular nondegenerate variety over an algebraically closed field $K$. Then $N \leq \frac{n(n+\delta+2)\varepsilon(\delta-\varepsilon-2)}{2\delta} \leq \frac{(n+\delta+1)^2}{2\delta} - 1$, where $\delta = 2n + 1 - s$, $s = \dim SX$, $\varepsilon = \delta \left\{ \frac{n}{\delta} \right\} = n \mod \delta$. If $\text{char } K = 0$, then $N = \frac{n(n+\delta+2)\varepsilon(\delta-\varepsilon-2)}{2\delta}$ in the following cases:

(i) $n < 2\delta$, $\varepsilon = n - \delta$, $s = N = 2n + 1 - \delta$;
(ii) $\delta = 1$, $N = \frac{n(n+3)}{2}$, $X = v_2(\mathbb{P}^n)$ is the Veronese variety;
(iii) $\delta = 2$, $N = \frac{n(n+4)-n\mod2}{4}$, $X = \mathbb{P}^\frac{n+1}{2} \times \mathbb{P}^\frac{n+1}{2}$ is the Segre variety;
(iv) $\delta = 4$, $n \equiv 0 \mod 2$, $N = \frac{n(n+6)}{8}$, $X = G \left( \frac{n}{2} + 1, 1 \right)$ is the Grassmann variety;
(v) $\delta = 8$, $n = 16$, $N = 26$, $X = E$ is the sixteen-dimensional Severi variety.

The varieties (i)–(iv) are Scorza varieties, and for them $N = \frac{(n+\delta+1)^2}{2\delta} - 1$.

The Scorza variety $X^n$ corresponds to the orbit of highest weight vector of an irreducible representation of a semisimple group $G$ in a vector space $V$ with highest weight $\Lambda$, where

\begin{itemize}
  \item[(i)] $G = SL_{n+1}$, $\Lambda = 2\varphi_1$;
  \item[(ii)] $G = SL_{\frac{n+2}{2}} \times SL_{\frac{n+2}{2}}$, $\Lambda = \varphi_1 \oplus \varphi_2$;
  \item[(iii)] $G = SL_{\frac{n+2}{2}}$, $\Lambda = \varphi_2$;
  \item[(iv)] $G = E_6$, $\Lambda = \varphi_1$
\end{itemize}

(here $\varphi_i$ is the $i$-th fundamental weight).

The Scorza variety $X^n$ is the image of $\mathbb{P}^n$ under the rational map $\sigma : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ defined by the linear system of quadrics passing through the subvariety $A \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$, where

\begin{itemize}
  \item[(i)] $A = \emptyset$;
  \item[(ii)] $A = \mathbb{P}^\frac{n-1}{2} \cup \mathbb{P}^\frac{n-1}{2}$;
  \item[(iii)] $A = \mathbb{P}^1 \times \mathbb{P}^\frac{n-1}{2}$;
  \item[(iv)] $A = S^{10}$.
\end{itemize}

In other words, the Scorza variety $X^n$ is obtained from the Veronese variety $v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ by projecting from the linear span $\langle v_2(A) \rangle$ of the image of the variety $A \subset \mathbb{P}^n$ under the Veronese map $v_2$.

5.7. Remark. As in Remark 4.3 in Chapter IV, it is not hard to verify that for an arbitrary point $x$ of the Scorza variety $X^n \subset \mathbb{P}^N$ the variety $T_{X,x} \cap X$ is a cone with vertex $x$ and base $A$. Under the map $\pi = (\sigma |_X)^{-1}$ each of the cones $T_{X,x} \cap X$ is mapped onto its base. The map $\pi$ is an isomorphism outside of $\sigma(\mathbb{P}^{n-1})$ and $\sigma(\mathbb{P}^{n-1}) = \bigcup_{y \in \text{Sing } \sigma(\mathbb{P}^{n-1})} T_{X,y} \cap X$, $\pi : \sigma(\mathbb{P}^{n-1}) \to A$. 
5.8. Remark. As in Remark 4.6 in Chapter IV, it is not hard to verify that the linear system of quadrics cut in a general linear subspace $\mathbb{P}^{n-1} \subset T_{X,X}$ by the linear system of quadrics passing through the Scorza variety $X$ and defining a rational map $\mathbb{P}^{n-1} \dashrightarrow Y \subset \mathbb{P}^{\delta(n-\delta)}$ (where $Y \subset X^*$ is naturally isomorphic to $\text{Sing}(\sigma(\mathbb{P}^{n-1}))$) is the second fundamental form in the sense of [29] and the subvariety $A \subset \mathbb{P}^{n-1}$ is the fundamental subset of this form.

Combining Corollaries 2.9, 3.5, 4.13, and 5.5 with Theorem 2.10 of Chapter V we obtain the following.

5.9. Theorem. Let $X^n \subset \mathbb{P}^r$ be a nonsingular variety over an algebraically closed field $K$. Then $h^0(X, \mathcal{O}_X(1)) \leq \left\lceil \frac{(4n-r+3)^2}{8(n-r+1)} \right\rceil$. If in addition char $K = 0$ and $r \geq \frac{3n}{2} + 1$, then equality holds if and only if $X$ is an isomorphic projection of a Scorza variety $X$ to a projective space $\mathbb{P}^s$, $s = \dim S \bar{X}$, so that in particular $r = 2n, 2n - 1, 2n - 3$ or $2n - 7$ (if $r < \frac{3n}{2} + 1$, then from Corollary 2.11 in Chapter II it follows that $h^0(X, \mathcal{O}_X(1)) = r + 1$).

5.10. Remark. It is worthwhile to observe that in the most important case when $n \equiv 0 \pmod{\delta}$ classification of Scorza varieties over an algebraically closed field $K$, char $K = 0$ is parallel to classification of Jordan matrix algebras due to Albert (cf. [10; 44, Chapter V]). More precisely, $\mathbb{P}^N = \mathbb{P}(\mathcal{J}_{\frac{n}{2}+1})$ where $\mathcal{J}_{\frac{n}{2}+1}$ is the Jordan algebra of Hermitean matrices of order $\frac{n}{2}+1$ over a composition algebra $\mathcal{A}$, $\dim K \mathcal{J}_{\frac{n}{2}+1} = \frac{n(n+\delta)(n+2)}{2}$ (we recall that for $\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \frac{n}{2} \geq 2$ is an arbitrary integer and $\mathcal{J}_{\frac{n}{2}+1}$ is a special Jordan algebra and for $\mathcal{A} = \mathcal{A}_3$ we have $\frac{n}{2} = 2$ and $\mathcal{J}_3$ is an exceptional Jordan algebra; cf. Theorem 4.8 in Chapter III) and $X$ corresponds to the cone $\{A \in \mathcal{J}_{\frac{n}{2}+1} \mid \text{rk } A \leq 1\}$. More generally, the variety $S^kX$, $0 \leq k \leq k_0 = \frac{n}{2}$ corresponds to the cone $\{A \in \mathcal{J}_{\frac{n}{2}+1} \mid \text{rk } A \leq k + 1\}$.

Arguing as in Theorem 4.9 of Chapter IV, we can restate Remark 5.10 as follows.

5.11. Theorem. A nonsingular nondegenerate variety $X^n \subset \mathbb{P}^N, n \equiv 0 \pmod{\delta}$, $N = f \left(\frac{n}{2}\right), \delta = \delta(X)$ over an algebraically closed field $K$, char $K = 0$ is a Scorza variety if and only if $X$ is the ‘Veronese variety’ of dimension $\frac{n}{2} \geq 2$ over the composition algebra $\mathcal{A}$, $\dim K \mathcal{A} = \delta$, i.e. $X$ is the image of the ‘projective space’ $\mathbb{P}^f(\mathcal{A}) = (\mathcal{A} \frac{n}{2} + 1 \setminus 0) / \mathcal{A}^*$ (where $\mathcal{A}^*$ is the subset of invertible elements of the algebra $\mathcal{A}$) under the map $(x_0 : \cdots : x_l : \cdots)^{\nu_l} \rightarrow (\cdots : x_l x_m : \cdots), 0 \leq l \leq m \leq \frac{n}{2}$ (for $\delta = 8$ we have $n = 16$ since, due to the lack of associativity, for larger $n$ the variety $v_2(\mathbb{P}(\mathcal{A}_3))$ is no longer defined by vanishing of the minors of order two of a Hermitean $\left(\frac{n}{2}+1\right) \times \left(\frac{n}{2}+1\right)$-matrix; this corresponds to $\mathcal{J}_3$ being the only Jordan matrix algebra over $\mathcal{A}_3$).
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151

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