Part IV

Equilibrium theory

Chapter 11

Stability of economic equilibrium

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Abstract: We consider a standard Walrasian pure exchange economy. From an economic point of view, equilibrium in such a model only makes sense if none of the participants can benefit from throwing out some of the commodities. Equilibria with this property are called *stable*. Examples show that an equilibrium need not be stable even if it is unique and the preferences are defined by nice utility functions. In this paper we study stability from an infinitesimal point of view and give sufficient conditions for (local and global) stability.

Key words: Walrasian economy, equilibrium, stability, throwing out of commodities

1 Introduction

The classical problem of existence of equilibrium prices in competitive economy (i.e. prices for which demand equals supply) was solved in the fifties in the classical paper by Arrow and Debreu (cf. Intriligator, 1971; Nikaido, 1997). By now the results of these authors have been considerably generalized. However, the models considered in most of the papers on economic equilibrium are just modifications of the Walrasian model introduced more than a century ago. In this model, the producers' goal is to maximize their incomes and the consumers' goal is to maximize their utilities.

The Walrasian model has been repeatedly criticized in the economic literature, but only recently it has been noticed that even the restrictions imposed on the consumers' behavior are not natural within the framework of the model. Inspired by the classical result of Samuelson on the advantages provided by foreign trade, Gale (1974) constructed an example of a pure exchange economy in which some agents can redistribute their initial endowments between themselves in such a way that the new equilibrium state is better for each of them than the old one (of course, some of the outsiders are bound to suffer). However, it should be noted that cooperation between economic agents contradicts the assumptions of independence of economic agents and competitiveness of Walrasian economy.

Aumann and Peleg (1974) gave an example of a pure exchange economy in which some of the participants can improve their position (in equilibrium) by throwing out (or hiding) some of their initial endowments. Economies in which this is possible are called *unstable*, and those in which this is impossible are called *stable*. It is clear that the Walrasian approach makes sense only with respect to stable economies. At the same time, unstable economies not only naturally arise in economic theory, but also reflect certain economic realities.

Examples of this phenomenon are given by overproduction crises when, in order to keep prices high, producers destroy surplus commodities, and by "humanitarian" aid when commodities are transferred to outsiders whose participation in trade is insignificant. Besides these extreme examples, there are numerous others when producers benefit from cutting down production in order to improve their position by inflating prices.

It therefore seems to be desirable to develop a mathematical theory of stable and unstable economies. Recently, first steps on the way of developing such a theory were made by several authors (cf. e.g. Efimov and Shapovalov, 1979; Polterovich and Spivak, 1978). In the present paper we consider this problem from the point of view of differentiable manifolds.

We consider the following economic setup. There are m economic agents with demand functions $f_1(p, w_1), \dots, f_m(p, w_m)$ (where p is the price vector and w_k is the income of the k-th participant) with initial endowments $\omega_1, \dots, \omega_m$ (where ω_k is a nonnegative l-vector and l is the number of commodities). In this paper we consider the case that the equilibrium price vector is unique (modulo a nonzero scalar factor). However our methods also allow to treat the general case as well.

Let $p(\omega)$ be the equilibrium price vector corresponding to initial endowments ω. Then, as a result of trade, the k-th participant acquires a commodity vector $f_k(p(\omega), \langle \omega_k, p(\omega) \rangle)$ $(k = 1, \dots, m)$. Suppose that the

i-th participant threw out a part of his initial endowment, and let $\omega' = \{\omega_1, \ldots, \omega'_i, \ldots, \omega_m\}$, $\omega'_i \leq \omega_i$ be the new initial endowment. Then stability or instability of ω is determined by the relation between the utilities of $f_i(p(\omega), \langle \omega_i, p(\omega) \rangle)$ and $f_i(p(\omega'), \langle \omega'_i, p(\omega') \rangle)$. The present paper is devoted to a study of the behavior of these utilities under variation of initial endowments.

Throwing out commodities does not always seem realistic. Often it is more natural to assume that commodities are hidden and being stored. However, the lack of good dynamic models describing several cycles of production and consumption forces us to limit ourselves to the case of throwing out. From the point of view of economic theory this can be justified by assuming that we consider only perishable commodities (or commodities whose storage is very costly).

We proceed with briefly describing the organization of the paper. In the first section we generalize results of Balasko (1975), Debreu (1970), and Dierker (1974) and give a characterization of regular economies convenient for our purposes. In the second section we introduce the notion of infinitesimal stability and prove several criteria for infinitesimal stability. Using these criteria, we prove infinitesimal stability of economies with normal demand and equilibrium gross substitutability as well as economies close to equilibrium. Special attention is devoted to boundary economies (i.e. to the case when not all of the economic agents have all the commodities present in their initial endowments) since this case often occurs in examples and applications. In the third section we prove the main economic results of this paper. In particular, it is shown that in the case of normal demand the economic agents can only lose on discarding (positive amounts of) all the commodities (cf. 4.7) and that economies close to equilibrium are stable (cf. 4.6). It is also shown that generic infinitesimal stability implies stability (cf. 4.4, 4.9 and 4.10), and that economies with normal demand and equilibrium gross substitutability are stable (cf. 4.11).

It should be noted that our Theorem 4.11 overlaps with the main theorem 11 from Polterovich and Spivak (1978), although none of these theorems follows from the other one. Furthermore, Polterovich and Spivak study the notion of *coalition stability* (commodities can be redistributed between members of some coalition; cf. also Efimov and Shapolov, 1979). Our methods also allow to consider coalition stability.

2 Regular economies

2.1 We consider a pure exchange market with m economic agents and l commodities. Denote by $\mathbb{R}^k_+$ (resp. $\overline{\mathbb{R}}^k_+$) the open positive (resp. closed nonnegative) orthant of the k-dimensional Euclidean vector space $\mathbb{R}^k$, and let $f_i(p, w_i)$, where $p \in \mathbb{R}^l_+$ is a price vector and $w_i \in \mathbb{R}^1_+$ is the income of the i-th participant ($i = 1, \ldots, m$), be the demand function of the i-th participant. [1]

We assume that $f_i : \mathbb{R}^l_+ \times \mathbb{R}^1_+ \to \mathbb{R}^l_+$, $f_i(\mathbb{R}^l_+ \times \{0\}) = 0$ and f_i is generated by a preference relation on $\overline{\mathbb{R}}^l_+$ defined by a strictly monotone strictly quasiconcave twice continuously differentiable utility function $u_i : \overline{\mathbb{R}}^l_+ \to \mathbb{R}^1_+$:

$$f_i(p, w_i) = \{x \mid u_i(x) = \max u_i(y),\ y \in \overline{\mathbb{R}}^l_+, \langle y,\ p\rangle = w_i\}. \tag{2.1}$$

Thus f_i is a well defined (single valued) continuously differentiable (cf. Debreu, 1972) function satisfying the Walras relation

$$\langle p, f_i(p, w_i)\rangle = w_i, \quad i = 1, \ldots, m. \tag{2.2}$$

Furthermore, from (2.1) it follows that f_i is homogeneous of degree zero, i.e. demand does not change under price rescaling:

$$f_i(\alpha p, \alpha w_i) = f_i(p, w_i), \quad \alpha \in \mathbb{R}^l_+,\ i = 1, \ldots, m. \tag{2.3}$$

2.2 Let $f : \mathbb{R}^l_+ \times (\overline{\mathbb{R}}^1_+)^m \to \overline{\mathbb{R}}^l_+$,

$$f(p; w_1, \ldots, w_m) = \sum_{i=1}^{m} f_i(p, w_i) \tag{2.4}$$

be the aggregate demand function of all the economic agents. In what follows we need to impose a certain condition on the behavior of f near the boundary of the positive price orthant. Here follows such a condition which seems most convenient for our purposes.

$$\begin{array}{l} \text{Let } p_n \in \mathbb{R}^l_+,\ w_n = ((w_1)_n, \ldots, (w_m)_n) \in \mathbb{R}^m_+,\ n = 1, 2, \ldots; \\ p_n \to \bar{p} \in (\partial\overline{\mathbb{R}}^l_+ \setminus \{0\}),\ w_n \to \overline{w} \in \mathbb{R}^m_+ \setminus \{0\}. \\ \text{Then } \parallel f(p_n; w_n) \parallel \to \infty. \end{array} \tag{A}$$

[1] Throughout the paper subscripts refer to agents, whereas superscripts refer to components, e.g. f_i^j is the demand of consumer i for commodity j.

2.3 Let $\omega_i \in \overline{\mathbb{R}}^l_+$ be the vector of initial endowments of the i-th participant ($i = 1, \ldots, m$). For given demand functions, the economy is defined by the initial endowments, thus the collection of all economies is

$$\overline{\Omega} = \{\omega \mid \omega_i \geq 0, \quad i = 1, \ldots, m; \ \sum_{i=1}^{m} \omega_i > 0\}. \tag{2.5}$$

Notice that we assume that for any economy $\omega = (\omega_1, \ldots, \omega_m) \in \overline{\Omega}$ all the commodities are represented in the market.

We say that $p \in \mathbb{R}^l_+$ is an *equilibrium price vector* for the initial endowments $\omega \in \overline{\Omega}$ if

$$f(p; \langle \omega_1, p \rangle, \ldots, \langle \omega_m, p \rangle) = \sum_{i=1}^{m} \omega_i. \tag{2.6}$$

It is clear that if p is an equilibrium price vector, then, for $\alpha \in \mathbb{R}^1_+$, αp is also an equilibrium price vector.

Using methods similar to those employed in Balasko (1975), Debreu (1970) and Dierker (1974), one can show that, under our assumptions (which are somewhat weaker than in the papers above), for each allocation of initial resources $\omega \in \overline{\Omega}$ there exists an equilibrium price vector. Observe that when $\omega \in (\overline{\mathbb{R}}^l_+)^m \setminus \overline{\Omega}$, $\omega \neq 0$ (i.e. some, but not all commodities are not available in the economy), then there is no equilibrium.

2.4 Let $\overline{W} \subset \overline{\Omega} \times \mathbb{R}^l_+$ be given by

$$\overline{W} = \{(\omega, p) \mid \omega \in \overline{\Omega}, \ p \text{ is an equilibrium price vector at } \omega\},$$

and let $\bar{\pi} : \overline{W} \to \overline{\Omega}$ be the map induced by the projection onto the first factor. Following Debreu, we will call $\overline{W}$ the *Walras correspondence*. The meaning of the word "correspondence" is that over each point $\omega \in \overline{\Omega}$ there lies the set of equilibrium prices at ω. From (2.6) it is clear that $\overline{W}$ is defined in $\overline{\Omega} \times \mathbb{R}^l_+$ by the following l equations:

$$\zeta^j(\omega; p) = f^j(p; \langle \omega_1, p \rangle, \ldots, \langle \omega_m, p \rangle) - \sum_{i=1}^{m} \omega_i^j = 0, \quad j = 1, \ldots, l. \tag{2.7}$$

The function $\zeta = (\zeta^1, \ldots, \zeta^l)$ is called the (aggregate) *excess demand function.*

According to (2.2), the equations (2.7) are not independent, but satisfy *Walras law*, i.e., they are subject to the linear relation

$$\langle p, \zeta(\omega, p) \rangle = 0. \tag{2.8}$$

2.5 It is not very convenient to deal with $\overline{\Omega}$ and $\overline{W}$ since they are not manifolds. Of course, as it is usually done, one could limit oneself to considering their interiors, but this would force one to lay aside many important examples. Therefore we proceed in a different way.

First of all, we observe that a participant without initial endowments cannot buy or sell anything and therefore can be excluded from the economy. Thus one can replace the set of admissible initial endowments by its subset

$$\overline{\Omega}^* = \{\omega \in \overline{\Omega} \mid \omega_i \neq \{0\},\ i = 1, \ldots, m\}. \tag{2.9}$$

Let $\omega \in \overline{\Omega}^*$. Then it is not hard to see that there exists a neighborhood V_ω of the set $\bar{\pi}^{-1}(\omega)$ in $(I\!R^l)^m \times I\!R^l_+$ such that the function $\zeta(\omega; p)$ is defined in V_ω (and satisfies the relation (2.8)). Put

$$V = \bigcup_{\omega \in \overline{\Omega}^*} V_\omega \subset (I\!R^l)^m \times I\!R^l_+,$$

and let W be defined in V by the system of l equations (2.7) satisfying the linear relation (2.8). Let $\pi : W \to (I\!R^l)^m$ be the restriction of the projection $(I\!R^l)^m \times I\!R^l_+ \to (I\!R^l)^m$ on W, and put $\Omega = \pi(W)$. It is clear that

$$\pi^{-1}(\overline{\Omega}^*) = \overline{W}^* = \bar{\pi}^{-1}(\overline{\Omega}^*); \quad \pi|_{\overline{\Omega}^*} = \bar{\pi}|_{\overline{\Omega}^*} \tag{2.10}$$

(here, as usual, $\phi|_X$ denotes the restriction of map ϕ on a subset X).

2.6 Proposition *Let $1 \le i \le m$. Then the functions p^j, $w_i = \langle \omega_i, p \rangle$, ω_r^j, $1 \le j \le l$; $1 \le r \le m$, $r \neq i$ form a system of local (and global) coordinates at each point of W. In particular, W is an $(lm+1)$-dimensional C^1-manifold and π is a C^1-mapping. Moreover, Ω is open in $(I\!R^l)^m$ and if*

$$W \supset W^n = \{(\omega, p) \in W \mid \sum_{j=1}^{l} p^j = 1\}, \quad \pi^n = \pi|_{W^n}, \tag{2.11}$$

then $\pi^n : W^n \to \Omega$ is a surjective proper map of degree 1.

Proof. Consider the map

$$\sigma : V \to Z = I\!R^l_+ \times I\!R^1_+ \times (I\!R^l)^{m-1} \tag{2.12}$$

defined by the functions from the statement of our proposition. For a point $z \in Z$, $z = (p; w_i; \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_m)$ we put

$$\begin{aligned} &\tau(z) = (\omega_1, \ldots, \omega_{i-1}, \omega_i(z), \omega_{i+1}, \ldots, \omega_m; p), \\ &\tau : Z \to I\!R^{lm} \times I\!R^l_+, \end{aligned} \tag{2.13}$$

where

$$\omega_i^j(z) =$$

$$f^j(p; \langle\omega_1, p\rangle, \ldots, \langle\omega_{i-1}, p\rangle, w_i, \langle\omega_{i+1}, p\rangle, \ldots, \langle\omega_m, p\rangle) - \sum_{r \neq i} \omega_r^j. \tag{2.14}$$

It is clear that

$$\tau \circ \sigma|_W = \mathrm{id}_W : W \to W, \tag{2.15}$$

where, as usual, $\mathrm{id}_W : W \to W$ denotes the identity map. From (2.14), (2.15) and the definition of W it follows that $\sigma(W)$ is open in Z and that $\sigma|_W : W \to \sigma(W)$ is a C^1-diffeomorphism. This proves the first part of Proposition 2.6.

Next we observe that there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \Omega & & \sigma(W) \arrow[ll, "\rho"] \end{tikzcd} \tag{2.16}$$

where

$$\rho(z) = (\omega_1, \ldots, \omega_{i-1}, \omega_i(z), \omega_{i+1}, \ldots, \omega_m). \tag{2.17}$$

Since, in view of (2.1), the maps f_i are open and since a composition of open maps is itself open, we see that $\Omega = \rho(\sigma(W))$ is open in $(\mathbb{R}^l)^m$. The remaining claims of Proposition 1.6 are proved in Balasko (1975, §5). $\square$

2.7 We recall that an economy with initial endowments ω is called *regular* if one of the following equivalent conditions holds (cf. Balasko, 1975; Debreu, 1970):

a) In a neighborhood of ω, π^n is an unramified covering (i.e. a proper map which is a diffeomorphism in a neighborhood of each point $z \in (\pi^n)^{-1}(\omega)$; since from 2.6 we already know that π^n is proper, this condition holds iff for each $z \in (\pi^n)^{-1}(\omega)$ the differential $(d\pi^n)_z$ is an isomorphism of tangent spaces);

b) In a neighborhood of ω, the map π is a trivial bundle. In other words, for each $z \in \pi^{-1}(\omega)$ the differential $(d\pi)_z$ is surjective.

In what follows we need explicit formulae for the differential $d\pi$. We fix a number i, $1 \le i \le m$ and compute the matrix of the differential $d\pi$ taking p^j, w_i, and ω_r^j as local coordinates in W (cf. 1.6) and ω_i^j, ω_r^j ($j = 1, \dots, l$, $r = 1, \dots, i-1, i+1, \dots, m$) as local coordinates in Ω. Differentiating (2.14), we immediately see that

$$d\pi = \begin{pmatrix} S^i & * \\ 0 & \mathrm{id}_{l(m-1)} \end{pmatrix}, \tag{2.18}$$

where $d\pi$ is a matrix of order $lm \times (lm+1)$, $*$ is a matrix of order $l \times l(m-1)$, $\mathrm{id}_{l(m-1)}$ is the identity matrix of order $l \times (m-1)$, and S^i is the $l \times (l+1)$-matrix for which

$$S^i_{jt} = \begin{cases} \frac{\partial f^j}{\partial p^t} + \sum_{r \ne i} \frac{\partial f_r^j}{\partial w_r} \cdot \omega_r^t = \frac{\partial \zeta^j}{\partial p^t} - \frac{\partial f_i^j}{\partial w_i} \cdot \omega_i^t, \quad 1 \le j, t \le l; \\ \frac{\partial f_i^j}{\partial w_i}, \ 1 \le j \le l, \ t = l+1. \end{cases} \tag{2.19}$$

2.8 Proposition *The following conditions are equivalent:*

a) $\omega \in \Omega$ *is a regular economy;*

b) $\mathrm{rk}(S^i)^{l \times (l+1)} = l$, *i.e. the rows of* S^i *are linearly independent;*

c) $\mathrm{rk}((\frac{\partial \zeta}{\partial p})^{l \times l} - (\frac{\partial f_i}{\partial w_i})^{l \times 1} \cdot \omega_i^{1 \times l}) = l$;

d) $\mathrm{rk}(\frac{\partial \zeta}{\partial p})^{l \times l} = l - 1$.

Proof. a) $\Leftrightarrow$ b) immediately follows from (2.18).
b) $\Leftrightarrow$ c). We observe that by (2.19)

$$S^i = \left(\left(\frac{\partial \zeta}{\partial p} - \frac{\partial f_i}{\partial w_i} \cdot \omega_i \right)^{l \times l} ; \frac{\partial f_i}{\partial w_i} \right)^{l \times (l+1)}. \tag{2.20}$$

Since ζ is a homogeneous function of degree zero of p (cf. (2.3)), the Euler formula yields

$$\frac{\partial \zeta}{\partial p} \cdot p = 0. \tag{2.21}$$

From (2.20) and (2.21) it follows that

$$S^i = \left(\frac{\partial \zeta}{\partial p} - \frac{\partial f_i}{\partial w_i} \cdot \omega_i \right)^{l \times l} \cdot \left(\mathrm{id}_l ; -\frac{p}{w_i} \right)^{l \times (l+1)}, \tag{2.22}$$

where id_l is the identity $l \times l$-matrix and $w_i = \langle \omega_i, p \rangle$.
Relation (2.22) shows that

$$\mathrm{rk} S^i = \mathrm{rk}\left(\frac{\partial \zeta}{\partial p} - \frac{\partial f_i}{\partial w_i} \cdot \omega_i\right), \tag{2.23}$$

which proves that conditions b) and c) are equivalent.

b) $\Leftrightarrow$ d). From (2.20) it follows that

$$\mathrm{rk} S^i = \mathrm{rk}\left(\frac{\partial \zeta}{\partial p}; \frac{\partial f_i}{\partial w_i}\right). \tag{2.24}$$

Differentiating (2.8) with respect to p, we get

$$p \cdot \frac{\partial \zeta}{\partial p} + \zeta = 0. \tag{2.25}$$

Since p is an equilibrium price vector for the initial endowments ω, one has

$$p \cdot \frac{\partial \zeta}{\partial p} = 0. \tag{2.26}$$

On the other hand, differentiating (2.2) with respect to w_i, we get the so called Engel aggregation condition (cf. Intriligator, 1971, (7.4.38)):

$$\left\langle p, \frac{\partial f_i}{\partial w_i} \right\rangle = 1. \tag{2.27}$$

From (2.26) and (2.27) it is clear that the vector $\frac{\partial f_i}{\partial w_i}$ does not lie in the linear span of the vectors $\frac{\partial \zeta}{\partial p^j}$, $j = 1, \ldots, l$. Therefore

$$\mathrm{rk} S^i = \mathrm{rk}\left(\frac{\partial \zeta}{\partial p}\right) + 1, \tag{2.28}$$

which proves the equivalence of conditions b) and d). This concludes the proof. $\square$

2.9 Let $\omega \in \overline{\Omega}^*$, and let $1 \leq i \leq m$. Put

$$\begin{array}{lcl} \Omega_i & = & \{\omega' \in \Omega \mid \omega'_r = \omega_r,\ r \neq i\}, \\ \overline{\Omega}_i & = & \{\omega' \in \overline{\Omega} \mid \omega'_r = \omega_r,\ r \neq i\}. \end{array} \tag{2.29}$$

Without loss of generality we may assume that Ω_i is a connected manifold of dimension l. Put $W_i = \pi^{-1}(\Omega_i)$, and let $\pi_i = \pi|_{W_i} : W_i \to \Omega_i$. $\overline{W}_i$ and $\bar{\pi}_i$ are defined in a similar way. From 2.6 it is easy to deduce that W_i is a connected C^1-manifold of dimension $l+1$ and π_i is a surjective map. It is clear that

$$d\pi_i = S^i, \tag{2.30}$$

where $p^j; w_i$ are chosen as local coordinates in W_i, ω_i^j $(j = 1, \dots, l)$ are chosen as local coordinates in Ω_i, and S^i is the $l \times (l+1)$-matrix defined in (2.19).

From this and 2.8 it follows that $\omega \in \Omega_i \subset \Omega$ is a regular value of π_i iff ω is a regular value of π, i.e. the corresponding economy is regular.

We observe that if ω is a regular economy, then, in some neighborhood $U_i \ni \omega$, W_i splits into several nonintersecting branches:

$$\pi_i^{-1}(U_i) = V_{i1} \cup \ldots \cup V_{is}; \quad V_{ir} \cap V_{it} = \emptyset, \ r \neq t, \tag{2.31}$$

where

$$V_{ik} = V_{ik}^n \cdot I\!R_+^1, \quad k = 1, \dots, s \tag{2.32}$$

(cf. (2.11)) and the map

$$\pi_i|_{V_{ik}^n} : V_{ik}^n \to U_i, \quad k = 1, \dots, s \tag{2.33}$$

is a C^1-diffeomorphism.

2.10 Remark The assumption that Ω is obtained by extending $\overline{\Omega}^*$ is not essential, and the above results easily generalize to more general domains. However economists usually consider orthants (or Edgeworth boxes), and the conditions for infinitesimal stability on the boundary assume a nicer form in this case (cf. §2).

3 Infinitesimal stability

3.1 Let $\omega \in \overline{\Omega}^*$ be a regular economy. Let p be an equilibrium price vector in ω, and let

$$(\omega, p) \in V_{it}, \tag{3.1}$$

where $1 \leq t = t(p) \leq s$ (cf. (2.31)). For an arbitrary $\omega' \in U_i$ we define the indirect utility of the i-th participant as

$$v_i^p(\omega') = u_i(f_i(p(\omega'), \langle \omega_i', p(\omega') \rangle)), \tag{3.2}$$

where $p(\omega')$ is the only (modulo a positive factor) equilibrium price vector for the initial endowments ω' such that

$$(\omega', p(\omega')) \in V_{it} \tag{3.3}$$

(here we used the fact that f_i is homogeneous of degree zero). We observe that $p(\omega')$ and hence $v_i^p(\omega')$ is a function of class C^1.

3.2 Definition

(a) The equilibrium distribution corresponding to regular initial endowments $\omega \in \overline{\Omega}^*$ and equilibrium price vector p is called *infinitesimally stable* (resp. *weakly infinitesimally stable*) with respect to the i-th participant if

$$\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) > 0 \tag{3.4}$$

(resp.

$$\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) \geq 0) \tag{3.5}$$

for all j for which $\omega_i^j > 0$. In case (3.4) we write (ω, p) i.s. (i) and in case (3.5) (ω, p) w.i.s. (i).

(b) The equilibrium distribution corresponding to regular initial endowments $\omega \in \overline{\Omega}^*$ and an equilibrium price vector p is called *absolutely infinitesimally unstable* (resp. *strongly absolutely infinitesimally unstable*) with respect to the i-th participant if

$$\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) \leq 0 \tag{3.6}$$

(resp.

$$\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) < 0) \tag{3.7}$$

for all j for which $\omega_i^j > 0$. In case (3.6) we write (ω, p) a.i.us. (i), in case (3.7) (ω, p) s.a.i.us. (i).

In the case when $s = 1$ in (2.31) we say that ω has one of the properties defined in (a) or (b) above if the only equilibrium distribution corresponding to ω has this property.

We notice that in order for Definition 3.2 to make sense it suffices that the differential $d\pi_i$ be surjective at the point $(p, \langle \omega_i, p \rangle)$. The results below hold also in this more general case, but for our purposes it suffices to consider regular economies (a less trivial generalization of Definition 3.2 will be given in 3.15).

3.3 Remark Economically speaking, condition (3.4) (resp. condition (3.7)) implies that the indirect utility of participant i at equilibrium price p is increasing (resp. decreasing) in the initial endowments of i. We observe that the properties of infinitesimal stability and strong absolute infinitesimal instability are open. More precisely, let $\omega \in \overline{\Omega}^*$, $\omega_i > 0$ be a regular economy such that (ω, p) i.s. (i) (resp. (ω, p) s.a.i.us. (i)) for each equilibrium price vector p at ω, and let $\omega' \in \overline{\Omega}^*$ be an allocation sufficiently close to ω. Then (ω', p') i.s. (i) (resp. (ω', p') s.a.i.us. (i)) for each equilibrium price vector p' at ω'.

3.4 To compute the gradient of the function v_i^p it is convenient to pass to the local coordinates introduced in 2.9. We have:

$$\begin{aligned} v_i(p, w_i) &\stackrel{\text{def}}{=} u_i(f_i(p, w_i)), && (3.8) \\ \frac{\partial v_i}{\partial p}(p, w_i) &= u_i'(f_i(p, w_i)) \cdot \frac{\partial f_i}{\partial p}(p, w_i), && (3.9) \\ \frac{\partial v_i}{\partial w_i}(p, w_i) &= u_i'(f_i(p, w_i)) \cdot \frac{\partial f_i}{\partial w_i}(p, w_i), && (3.10) \end{aligned}$$

and from (2.1) it follows that

$$u_i'(f_i(p, w_i)) = \text{grad } u_i(f_i(p, w_i)) = \lambda p, \tag{3.11}$$

where $\lambda = \lambda(p, w_i)$ is a positive scalar.

Differentiating by p the Walras relation (2.2), we get

$$p^{1\times l} \cdot \left(\frac{\partial f_i}{\partial p} \right)^{l\times l} + (f_i)^{1\times l} = 0 \tag{3.12}$$

(this is the so called *Cournot aggregation condition*; cf. Intriligator, 1971, (7.4.41)).

Combining (3.11), (3.12), and (2.27), we see that

$$dv_i = -\lambda \sum_{j=1}^{l} f_i^j dp^j + \lambda dw_i. \tag{3.13}$$

Let

$$\vartheta = \sum_{j=1}^{l} -\alpha_j \frac{\partial}{\partial p^j} + \beta \frac{\partial}{\partial w_i}. \tag{3.14}$$

From (3.13) it follows that

$$\frac{\partial v_i}{\partial \vartheta} > 0 \text{ (resp. } \geq 0) \;\Leftrightarrow\; \langle f_i, \alpha \rangle + \beta > 0 \text{ (resp. } \geq 0). \tag{3.15}$$

3.5 We recall the definition of normal demand.

Definition A demand function $f_i(p, w_i)$ is called *normal* (resp. *strictly normal*) if

$$\frac{\partial f_i}{\partial w_i} \geq 0 \tag{3.16}$$

(resp.

$$\frac{\partial f_i}{\partial w_i} > 0) \tag{3.17}$$

for all $p \in I\!R^l_+$, $w_i \in \overline{I\!R}^1_+$.

Normality of demand means the absence of commodities that have little value for the i-th participant (cf. Intriligator, 1971, (7.4.29)); for example it excludes Giffen commodities.

3.6 Theorem *Suppose that the i-th economic agent has normal demand and positive initial endowments* (*i.e.* $\omega_i > 0$)*. Then there are no equilibrium allocations that are absolutely infinitesimally unstable with respect to the i-th participant.*

Proof. According to (2.19),

$$d\pi \left(\frac{\partial}{\partial w_i} \right) = \sum_{j=1}^{l} \frac{\partial f_i^j}{\partial w_i} \frac{\partial}{\partial \omega_i^j}. \tag{3.18}$$

On the other hand, from (3.15) it follows that

$$\frac{\partial v_i}{\partial w_i} = \sum_{j=1}^{l} \frac{\partial f_i^j}{\partial w_i} \cdot \frac{\partial v_i^p}{\partial \omega_i^j} > 0. \tag{3.19}$$

Since the demand is normal (cf. (3.16)), there exists a commodity $1 \le j \le l$ for which

$$\frac{\partial v_i^p}{\partial \omega_i^j} > 0, \tag{3.20}$$

which contradicts (3.6) (because $\omega_i > 0$). This proves the theorem. $\square$

3.7 Next we prove a criterion for infinitesimal stability.

Theorem *Let $\omega \in \overline{\Omega}^*$ be a regular economy, and let p be an equilibrium price vector at ω. Then (ω, p) is i.s. (i) iff the system of equations*

$$(V^i(\omega,p))^{l\times l} \cdot x^{l\times 1} = (f_i(p, w_i))^{l\times 1}, \tag{3.21}$$

where

$$\begin{aligned} (V^i_{j,t}(\omega,p)) &= -\frac{\partial \zeta^t}{\partial p^j}(\omega,p) + \frac{\partial f_i^t}{\partial w_i}(p, \langle \omega_i, p\rangle) \cdot \omega_i^j \\ &= -\frac{\partial f^t}{\partial p^j}(p; \langle \omega_1, p\rangle, \ldots, \langle \omega_m, p\rangle) \\ &\quad -\sum_{r\neq i} \frac{\partial f_r^t}{\partial w_r}(p; \langle \omega_r, p\rangle) \cdot \omega_r^j, \quad j,t = 1,\ldots,l, \end{aligned} \tag{3.22}$$

has a solution $x = (x^1, \ldots, x^l)$ such that

$$x^j > 0 \text{ for all } j \text{ for which } \omega_i^j > 0. \tag{3.23}$$

Proof. Let

$$\frac{\partial}{\partial \omega_i^j}(\omega) = \sum_{r=1}^{l} -\alpha_{rj}\frac{\partial}{\partial p^r} + \beta_j \frac{\partial}{\partial w_i}, \quad j = 1,\ldots,l. \tag{3.24}$$

In view of (3.15),

$$\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) > 0 \;\Leftrightarrow\; \langle f_i(p, w_i), \alpha_j\rangle + \beta_j > 0. \tag{3.25}$$

Taking into consideration (2.19), (2.30), and (3.25), we see that $\frac{\partial v_i^p}{\partial \omega_i^j}(\omega) > 0$ iff the system of equations and inequalities

$$\begin{cases} (V^i(\omega,p))' \cdot (\alpha_j)^{l\times 1} + \beta_j \left(\frac{\partial f_i}{\partial w_i}(p,\omega_i)\right)^{l\times 1} = (\delta_j)^{l\times 1}; \\ \langle f_i(p,w_i), \alpha_j\rangle + \beta_j > 0, \end{cases} \tag{3.26}$$

where $(V^i(\omega,p))'$ is the transpose of $V^i(\omega,p)$ and $(\delta_j)^k = \delta_{jk}$, $k = 1,\dots,l$ (here δ_{jk} is the Kronecker delta), has a solution.

From (3.22) and (2.21) it follows that

$$(V^i(\omega,p))' \cdot p = w_i \cdot \frac{\partial f_i}{\partial w_i}(p,w_i). \tag{3.27}$$

Put

$$\gamma_j = \alpha_j + \frac{\beta_j}{w_i} \cdot p. \tag{3.28}$$

Combining (3.26), (3.28) and (2.2), we see that

$$\begin{cases} (V^i(\omega,p))' \cdot \gamma_j = (V^i(\omega,p))' \cdot \alpha_j + \frac{\beta_j}{w_i} \cdot (V^i(\omega,p))' \cdot p = \delta_j; \\ \langle f_i(p,w_i), \gamma_j \rangle = \langle f_i(p,w_i), \alpha_j \rangle + \frac{\beta_j}{w_i} \langle f_i(p,w_i), p \rangle \\ \qquad = \langle f_i(p,\omega_i), \alpha_j \rangle + \beta_j > 0. \end{cases} \tag{3.29}$$

Thus the system (3.26) has a solution α_j iff the system (3.29) has a solution γ_j (this also immediately follows from the fact that one can introduce a norm for which $w_i \equiv 1$).

Since ω is regular, Proposition 2.8 shows that the matrix $(V^i(\omega,p))'$ is invertible, so that the system (3.29) has a unique solution

$$\gamma_j = ((V^i(\omega,p))')^{-1}\delta_j, \quad j = 1,\dots,l. \tag{3.30}$$

Thus, for infinitesimal stability it is necessary and sufficient that for all j for which $\omega_i^j > 0$ there is an inequality

$$[f_i'(p,w_i)]^{1\times l} \cdot [((V^i(\omega,p))')^{-1} \cdot \delta_j]^{l\times 1} > 0. \tag{3.31}$$

But $f_i'(p,w_i) \cdot ((V^i(\omega,p))')^{-1}$ is the j-th coordinate of the row vector

$$x' = \left(f_i'(p,w_i)^{1\times l} \cdot ((V^i(\omega,p))' \right)^{-1} \tag{3.32}$$

such that x is the unique (since V^i is invertible) solution of the system of equations (3.21). This proves the theorem. □

3.8 An almost word for word repetition of the proof of Theorem 3.7 yields the following result.

Theorem *Let $\omega \in \overline{\Omega}^*$ be a regular economy and let p be an equilibrium price vector at ω.*

a) (ω, p) *is w.i.s.* (i) *iff the system of equations* (3.21) *has a solution* x *such that*

$$x^j \geq 0 \textit{ for all } j \textit{ for which } \omega_i^j > 0. \tag{3.33}$$

b) (ω, p) *is a.i.us.* (i) (*resp. s.a.i.us.* (i)) *iff the system of equations* (3.21) *has a solution* x *such that*

$$x^j \leq 0 \textit{ for all } j \textit{ for which } \omega_i^j > 0. \tag{3.34}$$

(*resp.*

$$x^j < 0 \textit{ for all } j \textit{ for which } \omega_i^j > 0). \tag{3.35}$$

3.9 For some applications it is more convenient to restate the criteria given in Theorems 3.7 and 3.8 in terms of excess demand functions.

Let

$$\zeta_i(\omega, p) = f_i(p, \langle \omega_i, p \rangle) - \omega_i \tag{3.36}$$

be the excess demand function of the i-th participant.

Theorem *Let* $\omega \in \overline{\Omega}^*$ *be a regular economy, and let* p *be an equilibrium price vector at* ω*. Then* (ω, p) *is i.s.* (i) (*resp.* (ω, p) *is w.i.s.* (i)*, resp.* (ω, p) *is a.i.us.* (i)*, resp.* (ω, p) *is s.a.i.us.* (i)) *iff the system of equations*

$$\begin{cases} x^{1\times l}\left(\frac{\partial\zeta}{\partial p}(\omega, p)\right)^{l\times l} + (\zeta_i(\omega, p))^{1\times l} = 0; \\ \left\langle x, \frac{\partial f_i}{\partial w_i}(p, \langle \omega_i, p \rangle)\right\rangle = 1 \end{cases} \tag{3.37}$$

has a solution satisfying (3.23) (*resp.* (3.33)*, resp.* (3.34)*, resp.* (3.35)).

We observe that, in view of (2.21), the first l equations of (3.37) are linearly dependent, and thus one may assume that (3.37) is a system of l equations with l unknowns (from 2.8 it follows that the rank of this system of equations is equal to l).

Proof. In view of Theorems 3.7 and 3.8, it suffices to show that the systems (3.37) and (3.21) are equivalent to each other. We have

$$V^i(\omega, p) = \left(\left(-\frac{\partial \zeta}{\partial p}(\omega, p)\right)^{l\times l} + \left(\frac{\partial f_i}{\partial w_i}(p, \langle \omega_i, p \rangle)\right)^{l\times 1} \omega_i^{1\times l}\right)'. \tag{3.38}$$

Suppose first that x satisfies (3.37). Then

$$\begin{aligned} V^i x' &= -\left(\frac{\partial\zeta}{\partial p}\right)' x' + \left\langle \frac{\partial f_i}{\partial w_i}, x \right\rangle \omega_i \\ &= -\zeta_i(\omega, p) + \omega_i = (f_i(p, \langle \omega_i, p\rangle))^{l\times 1}. \end{aligned} \tag{3.39}$$

Conversely, if x satisfies (3.21), then

$$\left(\frac{\partial\zeta}{\partial p}\right)' x + f_i(p, \langle\omega_i, p\rangle) - \left\langle x, \frac{\partial f_i}{\partial w_i}(p, \langle\omega_i, p\rangle)\right\rangle \omega_i = 0. \tag{3.40}$$

Since, in view of (2.21),

$$p \perp \mathrm{Im}\left(\frac{\partial\zeta}{\partial p}(\omega, p)\right)', \tag{3.41}$$

i.e. the image of the linear operator $\frac{\partial\zeta}{\partial p}(\omega, p)'$ lies in the orthogonal complement to the vector p, one has

$$\langle\omega_i, p\rangle \cdot \left\langle x, \frac{\partial f_i}{\partial w_i}(p, \langle\omega_i, p\rangle)\right\rangle - \langle f_i(p, \langle\omega_i, p\rangle), p\rangle = 0, \tag{3.42}$$

i.e.

$$w_i \left\langle x, \frac{\partial f_i}{\partial w_i}(p, \langle\omega_i, p\rangle)\right\rangle = w_i, \quad \left\langle x, \frac{\partial f_i}{\partial w_i}(p, \langle\omega_i, p\rangle)\right\rangle = 1. \tag{3.43}$$

Since (3.37) is obtained by combining (3.40) and (3.43), this completes the proof. □

3.10 Theorem *Suppose that a regular economy $\omega \in \overline{\Omega}^*$ is an equilibrium allocation for the i-th participant for an equilibrium price vector p, i.e.*

$$\zeta_i(\omega, p) = \zeta(\omega, p) = 0. \tag{3.44}$$

Then (ω, p) is i.s. (i).

Proof. In view of (2.26) and (2.27), to prove the theorem it is sufficient to put $x = p$ in Theorem 3.9 (we recall that under our assumptions $p > 0$). □

Theorem 3.10 says that the equilibrium is i.s. (i) provided that at the equilibrium participant i does not trade, i.e. his consumption equals his initial endowment. One can be more precise if this condition holds for all participants.

3.11 Corollary *Suppose that an allocation ω is Pareto optimal. Then ω is infinitesimally stable with respect to each of the participants (cf. also 3.12).*

3.12 Remark It is well known (cf. e.g. Balasko, 1975) that any Pareto optimal allocation ω is regular and admits a unique (modulo a positive factor) equilibrium price vector. Furthermore, from 2.1 it follows that $\omega_i > 0, i = 1, \ldots, m$. Therefore from Remark 3.3 and Corollary 3.11 it follows that all allocations sufficiently close to ω are infinitesimally stable with respect to each of the participants.

3.13 Corollary *Suppose that all economic agents are sufficiently close to each other in the sense that $\omega_i - \omega_k$ are small vectors and $f_i - f_k$ are small functions* (*with respect to a norm in the space of continuously differentiable functions on suitable compacts*) *for all $1 \leq i, k \leq m$. Then ω is a regular economy, there exists a unique* (*modulo a positive factor*) *equilibrium price vector at ω, and ω is infinitesimally stable with respect to all the participants.*

Thus infinitesimally unstable economies arise only if the economic inequality between the agents is sufficiently high.

3.14 Since in the case when $\omega \in \partial\overline{\Omega}^*$ not all the partial derivatives are involved in Definition 3.2, it is natural to slightly generalize Definition 3.2 and Theorems 3.7–3.9 by considering some economies that fail to be regular.

Definition Let $\omega \in \overline{\Omega}^*$, and let p be an equilibrium price vector at ω. The couple (ω, p) will be called *regular* with respect to the i-th participant and j-th commodity ((ω, p) reg. (i/j)) if

$$\frac{\partial}{\partial \omega_i^j} = d\pi_i(\vartheta), \tag{3.45}$$

where $\frac{\partial}{\partial \omega_i^j}$ is a vector from the standard basis of the tangent space to the linear space $\mathbb{R}^{lm}$ and ϑ is a tangent vector to W_i at the point $(p, \langle \omega_i, p \rangle)$.

We observe that if ω is a regular economy, then for each equilibrium price vector p at ω, each participant $1 \leq i \leq m$, and each commodity $1 \leq j \leq l$ the couple (ω, p) is reg. (i/j). Conversely, from 2.9 it follows that if for each equilibrium price vector p at ω there exists a participant $1 \leq i \leq m$ such that (ω, p) is reg. $i/(1, \ldots, l)$, then ω is a regular economy.

3.15 Definition Let (ω, p) be reg. (i/j). The equilibrium corresponding to ω and p is called *infinitesimally stable* (resp. *infinitesimally unstable,* resp. *strongly infinitesimally unstable*) with respect to the i-th participant and j-th commodity if for each ϑ satisfying (3.45) one has

$$\frac{\partial v_i^p}{\partial \vartheta}(p, \langle \omega_i, p \rangle) > 0 \tag{3.46}$$

(resp.

$$\frac{\partial v_i^p}{\partial \vartheta}(p, \langle \omega_i, p \rangle) \geq 0, \tag{3.47}$$

resp.

$$\frac{\partial v_i^p}{\partial \vartheta}(p, \langle \omega_i, p \rangle) \leq 0, \tag{3.48}$$

resp.

$$\frac{\partial v_i^p}{\partial \vartheta}(p, \langle \omega_i, p \rangle) < 0). \tag{3.49}$$

In the case (3.46) (resp. (3.47), resp. (3.48), resp. (3.49)) we will use abbreviation (ω, p) i.s. (i/j) (resp. (ω, p) w.i.s. (i/j), resp. (ω, p) i.us. (i/j), resp. (ω, p) s.i.us. (i/j)).

We observe that if $\omega \in \overline{\Omega}^*$ is a regular economy, then (ω, p) is i.s. $i \Leftrightarrow (\omega, p)$ is i.s. $(i/1, \dots, l)$; (ω, p) is w.i.s. $(i) \Leftrightarrow (\omega, p)$ is w.i.s. $(i/1, \dots, l)$; (ω, p) is a.i.us. $(i) \Leftrightarrow (\omega, p)$ is i.us. $(i/1, \dots, l)$; (ω, p) is s.a.i.us. $(i) \Leftrightarrow (\omega, p)$ is s.i.us. $(i/1, \dots, l)$.

For the sake of brevity, in what follows we will consider only *infinitesimal stability* with respect to the i-th participant and j-th commodity. However the reader will easily restate Theorems 3.16–3.18 for the other notions introduced in 3.15 (the corresponding theorems are in the same relation to Theorems 3.16–3.18 as Theorem 3.8 is to Theorem 3.7).

3.16 Theorem *Suppose that (ω, p) is reg. (i/j). Then the following conditions are equivalent:*

a) (ω, p) is i.s. (i/j);

b) The system of equations (3.21) has a solution x for which $x^j > 0$;

c) $x^j > 0$ for each solution x of the system (3.21).

Proof. Arguing as in the proof of Theorem 3.7, we see that (ω, p) is i.s. (i/j) iff each solution of the system of linear equations

$$(V^i(\omega, p))' \cdot \gamma_j = \delta_j \tag{3.50}$$

satisfies the inequality

$$\langle f_i(p, w_i), \gamma_j \rangle > 0 \tag{3.51}$$

(cf. (3.15), (3.29)).

a) $\Rightarrow$ b). We observe that for each solution γ_j of the system (3.50) (solutions exist in view of (3.45)) and each $\xi \in \operatorname{Ker}(V^i(\omega,p))'$ the vector $\gamma_j+\xi$ is also a solution of (3.50) (here and in what follows $\operatorname{Ker}A$ denotes the kernel of linear operator A, i.e. the linear subspace of vectors on which A vanishes). From this it follows that if the vector $f_i(p,w_i)$ is not orthogonal to $\operatorname{Ker}(V^i(\omega,p))'$, then (3.50) has a solution that does not satisfy (3.51). Thus from a) it follows that

$$f_i(p,w_i) \perp \operatorname{Ker}(V^i(\omega,p))'. \tag{3.52}$$

It is well known (and easy to show) that (3.52) is equivalent to the condition

$$f_i(p,w_i) \in \operatorname{Im}V^i(\omega,p), \tag{3.53}$$

so that a) implies that the system of equations (3.21) has a solution.

Let

$$V^i(\omega,p)\cdot x = f_i(p,w_i), \quad x \in I\!\!R^l. \tag{3.54}$$

Then from (3.51) it follows that

$$\begin{aligned} 0 \quad < \quad & \langle f_i(p,w_i),\gamma_j\rangle = \langle V^i(\omega,p)\cdot x,\gamma_j\rangle \\ & = \langle x,(V^i(\omega,p))'\cdot\gamma_j\rangle = \langle x,\delta_j\rangle = x^j. \end{aligned} \tag{3.55}$$

The implication c) $\Rightarrow$ b) is obvious.

To prove the implication b) $\Rightarrow$ a) it suffices to read the chain of equalities (3.55) from the right to the left. This concludes the proof. □

3.17 Theorem *Suppose that (ω,p) is reg. (i/j). Then the following conditions are equivalent:*

a) (ω,p) is i.s. (i/j);

b) The system of equations (3.37) has a solution x for which $x^j>0$;

c) $x^j>0$ for each solution x of the system (3.37).

Proof. Theorem 3.17 is deduced from Theorem 3.16 in the same way as Theorem 3.9 was deduced from Theorem 3.7. □

3.18 Definitions 3.14 and 3.15 allow to generalize the notion of infinitesimal stability to the case of not necessarily regular boundary economies.

Definition Let $\omega \in \overline{\Omega}^*$, $1 \leq i \leq m$, and put

$$J = \{j \mid \omega_i^j > 0\}. \tag{3.56}$$

Let p be an equilibrium price vector at ω. Suppose that (ω, p) is reg. (i/J), i.e. (ω, p) is reg. (i/j) for all $j \in J$. We say that the equilibrium corresponding to ω and p is *infinitesimally stable* with respect to the i-th participant ((ω, p) is i.s. (i)) if (ω, p) is i.s. (i/J), i.e. (ω, p) is i.s. (i/j) for all $j \in J$.

From Theorems 3.16 and 3.17 it follows that Theorems 3.7 and 3.9 hold also for this more general definition of infinitesimal stability.

3.19 We turn to a study of infinitesimal stability in systems with equilibrium gross substitutability.

Definition An economy ω is said to have the property of (weak) *equilibrium gross substitutability* if

$$\frac{\partial f^j}{\partial p^k}(p; \langle\omega_1, p\rangle, \ldots, \langle\omega_m, p\rangle) \geq 0, \quad j \neq k \tag{3.57}$$

for each *equilibrium* price vector p at ω.

We observe that Definition 3.19 seems less restrictive than the usual definition of the (global weak) gross substitutability in which it is required that the inequalities (3.57) hold for *all* p. On the other hand, usually gross substitutability is considered for *excess* demand functions:

$$\frac{\partial \zeta^j}{\partial p^k}(\omega, p) \geq 0, \quad j \neq k, \tag{3.58}$$

which, at least in the case when all the participants have normal demand, yields weaker restrictions on the preference relations.

3.20 Theorem *Let $\omega \in \overline{\Omega}^*$ be an economy with equilibrium gross substitutability, and let $1 \leq i \leq m$. Suppose that all participants except, possibly, the i-th one, have normal demand (cf. 3.5) and that one of the following conditions holds:*

a) $\omega_i > 0$;

b) The i-th participant has strictly normal demand;

c) For each equilibrium price vector p at ω the matrix $(p; \langle\omega_1, p\rangle, \ldots, \langle\omega_m, p\rangle)$ is indecomposable.

(We remark that it suffices to require the demand to be normal or strictly normal only for the equilibrium prices.) Then

A) ω is a regular economy;

B) There exists a unique (up to a constant factor) equilibrium price vector p at ω;

C) ω (or, which is the same, the equilibrium allocation corresponding to (ω, p)) is infinitesimally stable with respect to the i-th participant.

Proof. From our assumptions it follows that

$$v_{r,s} \leq 0, \quad r \neq s \tag{3.59}$$

(cf. (3.22)).

In case a), from (2.27) it follows that

$$V^i \cdot p = \omega_i > 0. \tag{3.60}$$

In case b), (3.27) yields

$$p \cdot V^i = w_i \cdot \frac{\partial f_i}{\partial w_i} > 0. \tag{3.61}$$

In both cases V^i is a matrix with dominant diagonal (cf. Nikaido, 1979, (21.1)), and in particular

$$\det V^i \quad > 0, \tag{3.62}$$

$$(V^i)^{-1} \quad \geq 0. \tag{3.63}$$

(cf. Intriligator, 1971, §6.2).

We claim that the conditions (3.62) and (3.63) are also satisfied in case c). In fact, from our assumptions it follows that the matrix V^i is also indecomposable. From the Perron-Frobenius theorem it follows that if V^i is not invertible, then there exists a vector $x > 0$ such that

$$x \cdot V^i = 0. \tag{3.64}$$

But since $\omega_i \neq 0$, (3.64) contradicts (3.60). Thus V^i is invertible and the properties (3.62) and (3.63) follow from the Perron-Frobenius theorem.

Since V^i is invertible, assertion A) follows from 2.8. Assertion B) follows from (3.62) in view of Proposition 2.6, and assertion C) follows from (3.63) in view of Theorem 3.7. This concludes the proof. □

We observe that if all the participants have normal demand, then from the Perron-Frobenius theorem it immediately follows that the nonzero eigenvalues of the matrix $\left(\frac{\partial \zeta^j}{\partial p^k}\right)$ have negative real parts. Hence the equilibrium corresponding to (ω, p) is locally stable with respect to tâtonnement.

3.21 Our immediate goal consists in strengthening Theorem 3.20 extending it to boundary economies. First we prove the following result.

Theorem. *Let $\omega \in \overline{\Omega}^*$ be an economy with equilibrium gross substitutability, and let J be the subset of the set $\{1, \ldots, l\}$ defined in* (3.56). *Suppose further that all the participants have normal demand. Then, for each $1 \leq i \leq m$ and each equilibrium price vector p at ω, (ω, p) is reg. (i/J).*

Proof. In view of (3.29) and 3.14, it suffices to show that for $j \in J$

$$\delta_j \in \mathrm{Im}(V^i)'. \tag{3.65}$$

As we already observed in 3.16, (3.65) is equivalent to the condition

$$\delta_j \perp \mathrm{Ker} V^i. \tag{3.66}$$

In other words, it suffices to show that for each $x \in \mathrm{Ker} V^i$ one has

$$x^j = 0, \quad j \in J. \tag{3.67}$$

By the Perron-Frobenius theorem, there exists a nonnegative vector $\gamma_1 \in \mathrm{Ker} V^i$. We put

$$K_1 = \{k \mid \gamma_1^k > 0\}, \quad 1 \leq k \leq l. \tag{3.68}$$

We observe that the matrix V^i is decomposable. More precisely,

$$V^i_{r,s} = 0, \quad r \notin K_1, \ s \in K_1. \tag{3.69}$$

In fact, for $r \notin K_1$ one has

$$(V^i \cdot \gamma_1)^r = \sum_{s=1}^{l} V^i_{r,s} \cdot \gamma_1^s = 0. \tag{3.70}$$

From (3.70) it follows that

$$V^i_{r,r} \cdot \gamma_1^r = \sum_{s \neq r} (-V^i_{r,s}) \cdot \gamma_1^s. \tag{3.71}$$

Since $r \notin K_1$, one has $\gamma_1^r = 0$. Hence from (3.59) it follows that if $\gamma_1^s > 0$, then $V_{r,s}^i = 0$, which yields (3.69). Re-indexing, if necessary, the commodities, one can represent V^i in the form

$$V^i = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}, \tag{3.72}$$

where the square matrix A_1 corresponds to the commodities from K_1, the square matrix D_1 corresponds to the commodities from $\hat{K}_1$, and

$$K_1 \cap \hat{K}_1 = \emptyset, \quad K \cup \hat{K}_1 = \{1, \ldots, l\}. \tag{3.73}$$

From (3.27) it follows that the semi-positive vector $\frac{\partial f_i}{\partial w_i}$ lies in $\mathrm{Im}(V^i)' = (\mathrm{Ker} V^i)^{\perp}$. Hence

$$\frac{\partial f_i^s}{\partial w_i} = 0, \quad s \in K_1. \tag{3.74}$$

From (3.27) and (3.74) we conclude that (in obvious notations)

$$p^{K_1} \cdot A_1 = 0. \tag{3.75}$$

On the other hand, since the matrix B_1 is non-positive, one has

$$A_1 \cdot p^{K_1} = \omega_i^{K_1} - B_1 \cdot p^{\hat{K}_1} \geq \omega_i^{K_1}. \tag{3.76}$$

From (3.75) and (3.76) it follows that

$$0 = (p^{K_1} \cdot A_1) \cdot p^{K_1} = p^{K_1} \cdot (A_1 \cdot p^{K_1}) \geq \langle p^{K_1}, \omega_i^{K_1} \rangle. \tag{3.77}$$

Since $p^{K_1} > 0$, (3.77) implies that

$$\omega_i^{K_1} = B_1 \cdot p^{\hat{K}_1} = 0, \tag{3.78}$$

i.e.

$$J \cap K_1 = \emptyset.$$

Next we observe that D_1 is a matrix of the same type as V^i. Suppose that

$$D_1 \cdot \gamma_2 = 0, \quad \gamma_2 \geq 0, \tag{3.79}$$

and put

$$K_2 = \{k \in \hat{K}_1 \mid \gamma_2^k > 0\}. \tag{3.80}$$

Then, after a suitable renumbering of commodities, D_1 can be represented in the form

$$D_1 = \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix}, \tag{3.81}$$

where the matrix A_2 corresponds to the commodities from K_2.

From (3.78) it follows that

$$B_1 = 0, \tag{3.82}$$

and therefore

$$p^{\hat{K}_1} \cdot D_1 = \frac{\partial f_i^{\hat{K}_1}}{\partial w_i} - p^{K_1} \cdot B_1 = \frac{\partial f_i^{\hat{K}_1}}{\partial w_i}. \tag{3.83}$$

As above, we see that

$$p^{K_2} \cdot A_2 = 0, \quad \omega_i^{K_2} = B_2 \cdot p^{\hat{K}_2} = 0, \tag{3.84}$$

so that $J \cap K_2 = \emptyset$.

Proceeding in the same way, we get

$$V^i = \begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_f & \\ 0 & & & D_f \end{pmatrix}, \tag{3.85}$$

where D_f is an indecomposable matrix corresponding to the subset of commodities $\hat{K}_f$, $\hat{K}_f \cup K_f = \hat{K}_{f-1}$, $\hat{K}_f \cap K_f = \emptyset$ and

$$J \subset \hat{K}_f. \tag{3.86}$$

Arguing as in the construction of D_f, we see that D_f is a nonsingular matrix. Let $x \in \mathrm{Ker} V^i$. Then

$$D_f \cdot x^{\hat{K}_f} = 0. \tag{3.87}$$

Hence

$$x^{\hat{K}_f} = 0. \tag{3.88}$$

In view of (3.86), (3.67) now follows from (3.88), which concludes the proof. □

3.22 Since $f_i(p, \langle \omega_i, p \rangle) > 0$, the Perron-Frobenius theorem shows that, under the assumptions of Theorem 3.21, the system (3.21) has a solution if and only if V^i is a non-degenerate matrix. Hence, in the conditions of Theorem 3.21, Theorem 3.16 applies only to regular economies (when it reduces to Theorem 3.7). Thus to consider boundary economies with equilibrium gross substitutability we need to further generalize the definition of infinitesimal stability (other definitions from 3.2 and 3.15 can be generalized in a similar way).

Definition Let $\omega \in \overline{\Omega}^*$, and let p be an equilibrium price vector at ω. Suppose that (ω, p) reg. (i/K) $(1 \leq i \leq m,\ K \subset \{1, \dots, l\})$. Put

$$\Omega_{i/K} = \{\omega' \in \Omega_i \mid (\omega_i')^k = \omega_i^k, k \notin K\}. \tag{3.89}$$

A tangent vector ϑ to W_i at the point (ω, p) will be called (i/K)-*actual* if ϑ is tangent to a smooth curve C in W_i such that

$$\pi_i(C) \subset \Omega_{i/K}. \tag{3.90}$$

The equilibrium corresponding to ω and p is called *actually infinitesimally unstable* with respect to the i-th participant and commodities from K ((ω, p) act.i.s. (i/K)) if for an arbitrary (i/K)-actual tangent vector ϑ to W_i at (ω, p) such that

$$d\pi_i(\vartheta) = \sum_{k \in K} \alpha_k \frac{\partial}{\partial \omega_i^k}, \quad \alpha_k \leq 0,\ \sum_{k \in K} \alpha_k < 0, \tag{3.91}$$

one has that

$$\frac{\partial v_i^p}{\partial \vartheta} < 0. \tag{3.92}$$

When $K = J = \{j \mid \omega_i^j > 0\}$ we say that (ω, p) is actually infinitesimally stable with respect to the i-th participant ((ω, p) act.i.s. (i)).

We observe that if $\omega_i > 0$, then (ω, p) act.i.s. $(i) \Leftrightarrow (\omega, p)$ i.s. (i).

3.23 Theorem *Let $\omega \in \overline{\Omega}^*$, and suppose that for each economy $\omega' \in \Omega_{i/J}$ sufficiently close to ω there exists a unique (modulo a constant factor) equilibrium price vector and there is equilibrium gross substitutability. Suppose further that all the participants have normal demand. Then (ω, p) is actually infinitesimally stable with respect to the i-th participant.*

Proof. Let C be a smooth curve in W_i such that $\pi_i(C) \subset \Omega_{i/J}$, $(\omega, p) \in C$, and the tangent vectors at an arbitrary point $\xi \in C$ satisfy the condition (3.91). Proceeding as in 3.21, for each point $\xi \in C$ we construct a decomposition of the matrix $V^i(\xi)$. It is easy to see that if one chooses the curve C to be sufficiently small, then

$$\hat{K}_f(\xi) \supseteq \hat{K}_f(\omega, p). \tag{3.93}$$

We proceed by descending induction on $\hat{K}_f$. If $\hat{K}_f = \{1, \dots, l\}$, then the economy is regular and it suffices to argue as in the proof of Theorem 3.20 C).

Assuming that the claim of Theorem 3.23 is proved for all $\hat{K}_f \supseteq \hat{K}_f(\omega, p)$, $\hat{K}_f \neq \hat{K}_f(\omega, p)$, we show that (ω, p) act.i.s. (i). We consider the following two cases.

a) For all $\xi \in C$ sufficiently close to (ω, p) one has $\hat{K}_f(\xi) = \hat{K}_f(\omega, p) = K$. From (3.85) it follows that we may assume that

$$p^j(\xi) = p^j, \quad j \notin K. \tag{3.94}$$

Since the matrix D_f is non-degenerate and has all the properties of the matrix V^i, (3.92) immediately follows from Theorem 3.7 (cf. (3.63)).

b) Suppose that there exists a sequence $\xi_n \to (\omega, p)$ such that $\hat{K}_f(\xi_n) \supseteq \hat{K}_f(\omega, p)$, $\hat{K}_f(\xi_n) \neq \hat{K}_f(\omega, p)$. In view of the induction assumption, from the continuity it follows that $\frac{\partial v_i^p}{\partial \vartheta} \leq 0$. Now the strict inequality is established in the same way as in a). This concludes the proof. $\square$

3.24 From the economic point of view, it is interesting to study the behavior of prices under throwing out of goods. However, in order to speak about *absolute* prices, one needs to introduce a suitable normalization. The most convenient normalization is $w_i \equiv 1$, where $1 \leq i \leq m$ is a fixed participant. For this normalization one has the following result.

Theorem *Let* $\omega \in \overline{\Omega}^*$.

a) *Suppose that* $\omega_i > 0$ *and* ω *i.s.* (i) (*resp.* ω *w.i.s.* (i)). *Then there exists a* (*local*) *throwing out for which the prices of all commodities* $j \in J$ *increase* (*do not decrease*).

b) *Suppose that* ω *has one of the properties listed in 3.20 and 3.23, and let* J *be defined by the formula* (3.56). *Then for each* (*local*) *throwing out the prices of all commodities* $j \in J$ *do not decrease* (*in the conditions of Theorem 3.20, the prices of* all *commodities do not decrease*).

Proof. a) In view of 2.7 and 21.8, under our normalization

$$d\pi_i = -V^i. \tag{3.95}$$

But by Theorem 3.7 (resp. 3.8)

$$(V^i)(f_i(p(\omega), \langle \omega, p(\omega)\rangle))_j > 0 \text{ (resp. } \geq 0), \quad j \in J, \tag{3.96}$$

and it suffices to perform throwing out along the (positive) vector $f_i(p(\omega), \langle \omega, p(\omega)\rangle)$.

b) In the conditions of Theorem 3.20 our claim follows from (3.95) and (3.63). In the conditions of Theorem 3.23 one needs to use a more detailed analysis performed in 3.21. □

4 Local and global stability

4.1 Throughout this section we assume that for a fixed initial endowment $\omega \in \overline{\Omega}^*$ and each initial endowment $\omega' \in \overline{\Omega}$, $\omega' \leq \omega$ there exists a unique (modulo a constant factor) equilibrium price vector (this assumption will be lifted elsewhere). We denote by $p(\omega)$ the (normalized) equilibrium price vector corresponding to ω.

Definition An economy ω is called *globally* (resp. *locally*) *stable* with respect to the i-th participant if for each economy $\omega' \in \overline{\Omega}_i$ (resp. for each each economy $\omega' \in \overline{\Omega}_i$ sufficiently close to ω), $\omega' \leq \omega$ one has

$$v_i(\omega') < v_i(\omega), \tag{4.1}$$

where

$$v_i(\omega) = u_i(f_i(p(\omega), \langle \omega_i, p(\omega)\rangle)) \tag{4.2}$$

(cf. (3.2)).

In a similar way one can introduce local and global counterparts of the other definitions from 3.2.

The connection between the notions of infinitesimal, local and global stability is revealed by the following theorems.

4.2. Theorem *If ω is globally stable with respect to the i-th participant, then ω is locally stable with respect to the same participant. Conversely, suppose that for each economy $\omega' \in \overline{\Omega}_i$, $\omega' \leq \omega$ the economy ω' is locally stable with respect to the i-th participant. Then ω is globally stable.*

The proof is obvious.

4.3 Theorem *Suppose that ω reg. (i/J) and $(\omega, p(\omega))$ i.s. (i/J), where $J = \{j \mid \omega_i^j > 0\}$. Then ω is locally stable with respect to the i-th participant.*

The proof is obvious.

4.4 Corollary *Suppose that ω' reg. (i/J) for each $\omega' \in \overline{\Omega}_i$, $\omega' \leq \omega$ and that $(\omega', p(\omega'))$ i.s. (i/J). Then ω is globally stable.*

4.5 Theorem *Let $\omega \in \overline{\Omega}^*$ be an economy which is globally stable with respect to the i-th participant and such that $\omega_i > 0$, $\sum_{k \neq i} \omega_k > 0$. Suppose that ω is regular and infinitesimally stable with respect to the i-th participant. Then each economy $\tilde{\omega}$ sufficiently close to ω is globally stable with respect to the i-th participant.*

Proof. As was observed in 3.3, there exists a neighborhood $U \ni \omega$ such that each economy $\omega' \in U$ satisfies the assumptions of the theorem except, possibly, global stability. Let $U' \subset U$ be a smaller neighborhood of ω. The function v_i attains maximum on the compact $\{\omega' \in \overline{\Omega}_i \mid \omega'_i \leq \omega_i,\, \omega' \notin U'\}$; let $\bar{v}$ be this maximum. Let

$$v_i(\omega) - \bar{v} = \varepsilon > 0. \tag{4.3}$$

Since the function v_i is continuous on the compact

$$K = \{\omega' \in \overline{\Omega} \mid \omega' \in \overline{\Omega}_i(\bar{\omega}), \quad \omega'_i \leq \bar{\omega}_i \ \exists \bar{\omega} \in \bar{U}\}, \tag{4.4}$$

v_i is uniformly continuous on this compact. Choose a constant $\delta > 0$ so that for $x, y \in K$, $\|x - y\| < \delta$ one has

$$\|v_i(x) - v_i(y)\| < \frac{\varepsilon}{2} \tag{4.5}$$

and

$$\delta < \operatorname{dist}(\bar{U}', \partial\bar{U}). \tag{4.6}$$

We claim that for

$$\tilde{\omega} \in U', \quad \|\tilde{\omega} - \omega\| < \delta, \tag{4.7}$$

$\tilde{\omega}$ is a globally stable economy. Suppose the converse, and let $\omega' \in \overline{\Omega}_i(\tilde{\omega})$,

$$\omega'_i \leq \tilde{\omega}_i, \ v_i(\omega') \geq v_i(\tilde{\omega}). \tag{4.8}$$

Since each economy from U is infinitesimally stable with respect to the i-th participant, an economy ω' satisfying (4.8) cannot lie in U.

Consider an economy $\omega'' \in \overline{\Omega}_i(\omega)$ such that

$$(\omega''_i)^j = \max\{(\omega'_i + \omega_i - \tilde{\omega}_i)^j, 0\}, \quad j = 1, \ldots, l. \tag{4.9}$$

Then $\|\omega' - \omega''\| < \delta$, and therefore

$$\|v_i(\omega') - v_i(\omega'')\| < \frac{\varepsilon}{2}, \quad \omega'' \notin U'. \tag{4.10}$$

Since $\omega'' \leq \omega$,

$$v_i(\omega'') < \bar{v} = v_i(\omega) - \varepsilon. \tag{4.11}$$

From (4.10), (4.8), (4.7) and (4.5) it follows that

$$v_i(\omega'') > v_i(\omega'') - \frac{\varepsilon}{2} \geq v_i(\tilde{\omega}) - \frac{\varepsilon}{2} > v_i(\omega) - \varepsilon, \tag{4.12}$$

which contradicts (4.11). This contradiction completes the proof. □

We notice that, under suitable conditions, Theorem 4.5 can be generalized to the case of boundary economies.

4.6 Theorem *Each economy from a neighborhood of the Pareto subset* (*i.e. the set of Pareto optimal allocations*) *is globally stable with respect to each of the participants.*

Proof. Since a Pareto optimal allocation $\omega \in \overline{\Omega}^*$ is regular, Corollary 3.11 and Theorem 4.5 show that it is sufficient to prove global stability with respect to each of the participants of ω itself.

Let $1 \leq i \leq m$, and let $\omega' \in \overline{\Omega}_i$, $\omega'_i \leq \omega_i$. Suppose that

$$v_i(\omega') \geq v_i(\omega). \tag{4.13}$$

Consider an allocation $\bar{\omega}$ such that

$$\begin{cases} \bar{\omega}_k = f_k(p(\omega'), \langle \omega'_k, p(\omega') \rangle), \quad k \neq i, \\ \bar{\omega}_i = f_i(p(\omega'), \langle \omega'_i, p(\omega') \rangle) + (\omega_i - \omega'_i). \end{cases} \tag{4.14}$$

Then

$$\sum_{k=1}^{m} \bar{\omega}_k = (\omega_i - \omega'_i) + \sum_{k=1}^{m} \omega'_k = \omega_i + \sum_{k \neq i} \omega_k = \sum_{k=1}^{m} \omega_k, \tag{4.15}$$

where

$$u_k(\bar{\omega}_k) \geq u_k(\omega'_k) = u_k(\omega_k), \quad k \neq i \tag{4.16}$$

and, in view of strict monotonicity of u_i and condition (4.13),

$$u_i(\bar{\omega}_i) > v_i(\omega') \geq v_i(\omega) = u_i(\omega_i). \tag{4.17}$$

Relations (4.15)–(4.17) show that $\bar{\omega}$ is preferred to ω, contrary to the assumption that ω is Pareto optimal. Hence $v_i(\omega') < v_i(\omega)$ and ω is globally stable, which proves the theorem. □

4.7 Theorem *Let $\omega \in \overline{\Omega}^*$ be a (not necessarily stable) economy. Suppose that the i-th participant has positive initial endowments (i.e. $\omega_i > 0$) and normal demand. Then for each economy $\omega' \in \overline{\Omega}_i$ such that $\omega'_i < \omega_i$ there exists an economy $\omega'' \in \overline{\Omega}_i$ such that $\omega'_i \leq \omega''_i < \omega_i$ and $v_i(\omega') < v_i(\omega'')$. In other words, the i-th participant does not gain from throwing out all commodities.*

Proof. By Theorem 3.6, the function v_i is differentiable at the point ω' along the vector

$$\vartheta = \sum_{j=1}^{l} \frac{\partial f_i^j}{\partial \omega_i}(p(\omega'), \langle \omega', p(\omega') \rangle) \frac{\partial}{\partial \omega_i^j}, \tag{4.18}$$

where

$$\vartheta \geq 0, \quad \frac{\partial v_i}{\partial \vartheta} > 0. \tag{4.19}$$

Consider an economy $\omega''_\varepsilon \in \overline{\Omega}_i$ such that

$$(\omega''_\varepsilon)_i = \omega'_i + \varepsilon \frac{\partial f_i}{\partial w_i}(p(\omega'), \langle \omega', p(\omega') \rangle), \tag{4.20}$$

where ε is a positive number. Since demand is normal, one has $\omega''_\varepsilon \geq \omega'$. Since $\omega'_i < \omega_i$, for small ε one has $(\omega''_\varepsilon)_i < \omega_i$. Finally, from (4.19) it follows that $v_i(\omega''_\varepsilon) > v_i(\omega')$ provided that ε is small enough. This concludes the proof. □

4.8 Theorem *The set of economies $\omega \in \overline{\Omega}^*$ which are weakly globally stable with respect to the i-th participant is closed in $\overline{\Omega}^*$ (we recall that the definition of weak global stability is obtained from the definition of global stability given in 4.1 by replacing strict inequality* (4.1) *by a non-strict one).*

Proof. Suppose that the assertion of the theorem is false. Then there exist an economy $\omega \in \overline{\Omega}^*$ and a sequence of economies $\omega(n)$ weakly globally stable with respect to the i-th participant such that

$$\lim_{n \to \infty} \omega(n) = \omega \tag{4.21}$$

and an economy

$$\omega' \in \overline{\Omega}_i(\omega), \quad \omega'_i \leq \omega_i \tag{4.22}$$

such that

$$v_i(\omega') > v)_i(\omega). \tag{4.23}$$

We define an economy $\omega''(n) \in \overline{\Omega}_i(\omega(n))$ by putting

$$(\omega_i''(n))^j = \max\{(\omega_i' - \omega_i + \omega_i(n))^j, 0\}, \quad j = 1, \ldots, l. \tag{4.24}$$

It is clear that

$$\omega_i''(n) \leq \omega_i(n), \tag{4.25}$$

and by our assumptions

$$v_i(\omega''(n)) \leq v_i(\omega(n)). \tag{4.26}$$

But it is easy to see that

$$\lim_{n\to\infty} \omega''(n) = \omega'. \tag{4.27}$$

From (4.21), (4.26), (4.27) and the continuity of the function v_i it follows that

$$v_i(\omega') \leq v_i(\omega), \tag{4.28}$$

contrary to (4.23). This contradiction completes the proof. □

4.9 We proceed with extending 4.4 to the case of not necessarily regular economies.

Theorem *Let $K \subset \overline{\Omega}$ be a nowhere dense closed subset of measure zero containing the set of all critical economies. Suppose that each economy $\omega \in \overline{\Omega} \setminus K$ is weakly infinitesimally stable with respect to the i-th participant. Then each economy $\omega \in \overline{\Omega}$ is weakly globally stable with respect to the i-th participant.*

Proof. The proof is easy when $f(p; w_1, \ldots, w_m)$ is an analytic function. In general, it is sufficient to verify that if $\omega \in \overline{\Omega}$, $\omega' \in \overline{\Omega}_i$ and $\omega_i' \leq \omega_i$, then $v_i(\omega') \leq v_i(\omega)$.

Let $\alpha = \alpha(\omega, \omega')$ be the constant vector field on $\overline{\Omega}_i$ consisting of vectors of length one whose integral curves are lines parallel to the line joining ω_i with ω_i' with orientation from ω_i to ω_i'. Let ξ be a continuous vector field on $\overline{\Omega}_i$ consisting of vectors of length one collinear at each point to the vector field $\frac{\partial f_i}{\partial w_i}$ (this last vector field is non-vanishing in view of (2.27)). Further, let δ be a positive number. We denote by K_δ a union of balls covering K the sum of whose volumes does not exceed δ. From the existence of partition of unity it follows that there exists a continuous vector field $\beta = \beta(\omega)$ on $\overline{\Omega}_i$ consisting of vectors of length one such that

$$\begin{aligned} \beta(\tilde{\omega}) &= \alpha(\tilde{\omega}), \qquad \tilde{\omega} \notin K_\delta; \\ \beta(\tilde{\omega}) &= \xi(\tilde{\omega}), \qquad \tilde{\omega} \in K. \end{aligned} \tag{4.29}$$

It is worthwhile to observe that, although the function v_i need not be differentiable (at the points of K), in view of (3.18), v_i is differentiable along the vector field β. We choose $\delta = \delta(\varepsilon)$ so small that there exists a point $\tilde{\omega} \in \overline{\Omega}_i$ such that

$$\|\tilde{\omega} - \omega\| < \varepsilon \tag{4.30}$$

and that for the integral curve $C = C(\tilde{\omega}, \delta)$ of the vector field $\beta(\delta)$ passing through $\tilde{\omega}$ one has

$$\mu(C \cap K_\delta) < \varepsilon \tag{4.31}$$

(where μ is the standard measure on the curve C).

Let T be the time required for the current corresponding to the vector field α to run through the interval $[\omega_i, \omega_i']$, and let $\tilde{\omega}' = \tilde{\omega}'(\tilde{\omega}, \delta)$ be the end of the interval of the integral curve $C(\tilde{\omega}, \delta)$ run through during the time T. It is clear that

$$\lim_{\varepsilon \to 0} \tilde{\omega}' = \omega'. \tag{4.32}$$

One has

$$\begin{aligned} v_i(\tilde{\omega}') - v_i(\tilde{\omega}) &= \textstyle\int_0^T (v_i)'_\beta dt \\ &= \textstyle\int_{C(t) \notin K_\delta} (v_i)'_\beta dt + \int_{C(t) \in K_\delta} (v_i)'_\beta dt \\ &= \textstyle\int_{C(t) \notin K_\delta} (v_i)'_\alpha dt + \int_{C(t) \in K_\delta} (v_i)'_\beta dt. \end{aligned}$$

Since the economies outside of K are weakly infinitesimally stable, the first summand in (4) is non-positive. On the other hand,

$$\int_{C(t) \in K_\delta} (v_i)'_\beta dt \le \left| \int_{C(t) \in K_\delta} (v_i)'_\xi dt \right|. \tag{4.33}$$

Since the integrand in the right hand part of (4.33) does not depend on ε, one has

$$v_i(\tilde{\omega}') - v_i(\tilde{\omega}) \le \varepsilon M, \tag{4.34}$$

where M is a constant which does not depend on ε (this argument can be further simplified if the i-th participant has normal demand).

Letting $\varepsilon \to 0$ and using (4.30), (4.32) and the continuity of the function v_i, we see that

$$v_i(\omega') \leq v_i(\omega). \tag{4.35}$$

This proves the theorem. □

4.10 Remark If in the statement of Theorem 4.9 one replaces the assumption of weak infinitesimal stability by the assumption of infinitesimal stability, then from our proof it follows that all economies from $\overline{\Omega} \setminus K$ are globally stable (this result can be extended to the boundary economies by replacing the assumptions of regularity and infinitesimal stability by the assumptions of (i/J)-regularity and (i/J)-infinitesimal stability (cf. 4.4)).

4.11 We turn to a study of systems with gross substitutability.

Theorem *Let $\omega \in \overline{\Omega}^*$, and suppose that for each $\omega' \in \overline{\Omega}$, $\omega' \leq \omega$ one has equilibrium gross substitutability and all the participants have normal demand. Then ω is globally stable with respect to each of the participants.*

Proof. In the case when at least one of the assumptions a)–c) of Theorem 3.20 is satisfied for at least one of the participants, Theorem 4.11 follows from Theorem 3.20 and Corollary 4.4 (we observe that *weak* global stability of ω follows from an obvious variant of Theorem 4.9).

To prove Theorem 4.11 in the general case we use the fact that, according to Theorem 3.23, $(\omega, p(\omega))$ act.i.s. $(1, \dots, m)$. The global stability is deduced from the actual infinitesimal stability in the same way as it is deduced from the infinitesimal stability in Corollary 4.4. □

We remark that the assumption of equilibrium gross substitutability is satisfied, in particular, in the case of systems with global gross substitutability (i.e. in the case when the inequality (3.57) holds for *all* $p \in \mathbb{R}^l_+$).

4.12 Theorem *Let $\omega \in \overline{\Omega}^*$, and suppose that for each $\omega' \in \overline{\Omega}_i$ one has equilibrium gross substitutability and all the participants have normal demand. Then, under the normalization $w_i \equiv 1$, $p^j(\omega') \geq p^j(\omega)$ for all $j \in J$. If, furthermore, ω satisfies one of the conditions of Theorem 3.20, then $p^j(\omega') \geq p^j(\omega)$ for all j.*

Proof. Theorem 4.12 immediately follows from Theorem 3.24. □

Acknowledgement Translation from Russian, pp. 72–106 in "Methods of the theory of extremal problems in economics" (edited by V. L. Levin), Nauka, 1981.

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